A Dynamical Systems Based Hierarchy for Shannon, Metric and Topological Entropy

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Introduction

- Entropy first used by Clausius to indicate spreading of energy, since that time there have been many definitions of entropy.
- Common thread is a dynamical system representation of Helmholtz,

$$S(E, V) = k_B \log \oint p dx.$$

- Where $S(E, V)$ satisfies Helmholtz's theorem,

$$\frac{\partial S(E,V)}{\partial E} = \frac{1}{T}, \quad \frac{\partial S(E,V)}{\partial V} = \frac{P(E,V)}{T(E,V)}.$$

- Result is consistent with first and second laws of thermodynamics for large number of particles with significant differences between Boltzmann and Gibbs for small number of particles. (To be discussed in a future paper)

Goals

- Establish a dynamical systems based hierarchy for three definitions of entropy,
  1. Shannon (Information)
  2. Kolmogorov-Sinai (Metric)
  3. Adler, Konheim & McAndrew (Topological)
- We show that Shannon entropy is a special case of metric entropy and with the imposition of certain properties that metric entropy is a special case of topological entropy.

Shannon $\leftarrow$ **Metric** $\leftarrow$ **Topological**

Represents the exponential rate of growth of distinct orbits.

Definitions

- Discrete Dynamical System is an action of the form,with X nonempty Hausdroff topological space with topology $\mathfrak{T}$ and $\mathbb{N}^* = 0 \cup \mathbb{N}$,
$$F : \mathbb{N}^* \times X \to X.$$
- f is a continuous map, with f^n is the n th iterate ,
$$F(n, x) = f^n(x), f : X \to X.$$
- Assuming X to be compact with $\mathcal{U} = \{U\}$ and $\mathcal{V} = \{V\}$ open coverings of X we have the common refinement also an open covering,
$$\mathcal{U} \vee \mathcal{V} = \{U \cap V : (U, V) \in \mathcal{U} \times \mathcal{V}\}.$$

Topological Entropy (continued)

- Iterating the refinement, for each $n \in \mathbb{N}$, we have

$$\mathcal{U}^n = \mathcal{U} \vee f^{-1}(\mathcal{U}) \vee \cdots \vee f^{-n+1}(\mathcal{U}).$$

- The topological entropy is then defined,

$$h_{\mathfrak{T}}(f) = sup \left\{ h_{\mathfrak{T}}(\mathcal{U}, f) : \mathcal{U} \in OC(X) \right\}.$$

Where $OC(X)$ is the set of all open coverings of X, and

$$h_{\mathfrak{T}}(\mathcal{U}, f) = \lim_{n \to \infty} \frac{\log N(\mathcal{U}^n)}{n} = \inf \left\{ \frac{\log N(\mathcal{U}^n)}{n} : n \in \mathbb{N} \right\}.$$

Exponent rate of growth of typical orbits.

Definitions

- A measurable dynamical system consists of a nonempty set X a σ-algebra of μ- measurable subsets $\mathcal{M}$ of X, and a measurable function $f : X \to X$, with the following properties, for $A \in \mathcal{M}$.
  1. $f^{-1}(A) = \mathcal{M}$
  2. $\mu \circ f^{-1} = \mu$
- A measurable partition of X is a finite pairwise disjoint sequence,

$$\mathcal{P} = \{Q_1, \cdots, Q_m\} \text{ with } X = \cup_{k=1}^{m} Q_k.$$

Metric Entropy (continued)

- The entropy of a measurable partition is,

$$H(\mathcal{P}) = -\sum_{Q\in\mathcal{P}} \mu(Q)\log\mu(Q).$$

- Define a common refinement ,

$$\mathcal{P}\vee\widetilde{\mathcal{P}} = \left\{Q\cup\widetilde{Q} : (Q,\widetilde{Q})\in\mathcal{P}\times\widetilde{\mathcal{P}}\right\}.$$

- Analagousley to topological entropy, we iterate the common refinement,

$$\mathcal{P}^n = \mathcal{P}\vee f^{-1}(\mathcal{P})\vee\cdots\vee f^{-n+1}(\mathcal{P}).$$

Metric Entropy (continued)

- Taking note of the similarity with topological entropy, we define the metric entropy,

$$h_\mu(f) = sup \left\{ H(\mathcal{P}, f) : \mathcal{P} \in \mathfrak{P}(X) \right\},$$

with

$$H(\mathcal{P}, f) = \lim_{n \to \infty} \frac{H(\mathcal{P}^n)}{n}.$$

Shannon Entropy

Measure of the most efficient way of transmitting information.

Definitions

- Shannon Entropy based on transmission of information.
- Define a nonempty finite set of messages or symbols,

$$S = \{s_1, \cdots, s_m\}.$$

- For each s_i define a discrete probability $p(s_i) \geq 0$, with,

$$p(s_1) + \cdots + p(s_m) = 1.$$

- The Shannon entropy is then,

$$H(S) = -\sum_{i=1}^{m} p(s_i) \log p(s_i).$$

Relationship Between Topological and Metric Entropy

A relationship between the two entropies was established by Dinaburg, Goodman, Goodwyn and Misiurewicz using variational techniques referred to as the **variational principle**.

Theorem: If X is a compact metric space with topology $\mathfrak{T}$ and $f : X \to X$ is a continuous map, then

$$h_{\mathfrak{T}}(f) = sup\left\{h_{\mu}(f) : \mu \in \mathfrak{M}(X)\right\}.$$

Where $\mathfrak{M} = \{\mu\}$ and μ is an f-invariant Borel probability measure on the σ-algebra $\mathcal{M}_{\mu}$ of subsets of X.

- We want to construct a topology $\mathfrak{T}$ for the phase space X of the **Discrete Measurable Dynamical System (DMDS)** $\mathfrak{D} = (f, X, \mathcal{M}, \mu)$.
- Make the assumption that we can compactly embed X in a metric space Y with metric d, with a metric topology $\mathfrak{T}_d$.
- Assume μ is a Borel measure, so all open sets are μ measurable.

Definitions

- A finite open cover $\mathcal{U} = \{U_1 \ldots, U_m\}$ of X is minimal with respect to a measurable partition $\mathcal{P} = \{Q_1, \ldots, Q_m\}$ of X if $Q_k \subset U_k$ for every $1 \leq k \leq m$
- A measurable partition $\mathcal{P}$ is an $\alpha - \beta$ partition of X with respect to $\mathfrak{T}_d$ and denoted as $\mathcal{P}(\alpha, \beta)$.
    1. $\mu(Q_k) > 0$
    2. $Q_k$is connected in the topology $\mathfrak{T}_d$
    3. $d(Q_k) < \beta$
    4. There exists at least one point $x \in Q_k$ such that the closed ball $\widetilde{B}_\alpha(x) = \{y \in X : d(x, y) \leq \alpha\}$

Definitions (continued)

- The DMDS $\mathfrak{D}$ is L-type, if the following hold:
  1. $\mu(B_\alpha(x)) > 0$ for $\alpha > 0$.
  2. $\mu(B_\alpha(x)) \to 0$ as $\alpha \to 0$
  3. For every Q_k of an $\alpha - \beta$ partition there is s connected open set $U_k(\epsilon)$
  4. There is an open covering in accordance with the previous definition $\mathcal{U}(\mathcal{P}, \epsilon) = \{U_1(\epsilon), \ldots, U_m(\epsilon)\}$
- The DMDS $\mathfrak{D}$ is T-compatible if there exist a compact metric topology $\mathfrak{T}_d$ on X if the following hold:
  1. f is continuous with respect to $\mathfrak{T}_d$
  2. μ is Borel with respect to $\mathfrak{T}_d$
  3. There exists at least one $\alpha - \beta$ partition of X.
  4. The measurable dynamical system is of L -type for $\mathfrak{T}_d$.

Theorem

Suppose the DMDS is T- compatible with respect to the topology on X and the following hold:

- There exists a sequence of partitions $\{\mathcal{P}(\alpha_n, \beta_n)\}$ such that the $\mathcal{P}(\alpha_n, \beta_n) \prec \mathcal{P}(\alpha_{n+1}, \beta_{n+1})$ $\forall$ $\{n_k\}$ with $\forall$ $n \in \mathbb{N}$ and $\{\alpha_n\}$ and $\{\beta_n\}$ are decreasing and converge to zero.
- The above sequence is such that for every increasing sequence of natural numbers $\{j_k\}$ there exists a dominating sequence of natural numbers $\{n_k\}$ with $n_k > j_k$ for all $k \in \mathbb{N}$ such that $\mathcal{P}^{n_k}(\alpha_k, \beta_k) = \left\{Q_{1(k,n_k)}, \ldots, Q_{m(k,n_k)}\right\}$ and

$$H(\mathcal{P}^{n_k}(\alpha_k, \beta_k)) = \log(m(k, n_k)) - \sigma(k, n_k) \tag{1}$$

for all $k \in \mathbb{N}$ where $\sigma(k, n_k) > 0$ is bounded for all $(k, n_k) \in \mathbb{N} \times \mathbb{N}$.

Outline of Proof

It follows from basic properties of metric entropy that,

$$h_\mu(\mathcal{P}(\alpha_n, \beta_n), f) \uparrow h_\mu(f)$$

Observe that for each of the $\mathcal{P}(\alpha_n, \beta_n)$ there is a decreasing sequence $\{\epsilon_n\}$ and corresponding ϵ_n tight minimal open covers $\mathcal{U}_n$ such that

$$h_{\mathfrak{T}}(\mathcal{U}_n, f) \uparrow h_{\mathfrak{T}}(f)$$

The key element in the proof is that (1) means that each $\mathcal{P}^{n_k}$ is nearly equiprobable and that a sufficiently tight open cover $\mathcal{U}$ of $\mathcal{P}(\alpha_k, \beta_k)$ produces a tight open cover $\mathcal{U}^{n_k}$ of $\mathcal{P}^{n_k}(\alpha_k, \beta_k)$ such that

$$\log N(\mathcal{U}^{n_k}) = \log(m(k, n_k))$$

Outline of Proof (continued)

Hence,

$$\frac{\log N(\mathcal{U}^{n_k})}{n_k} - \frac{H(\mathcal{P}^{n_k}(\alpha_k, \beta_k))}{n_k} = \frac{\sigma(k, n_k)}{n_k} \to 0 \text{ as } \; n_k \to 0$$

But the partitions and coverings can be chosen so that,

$$\frac{\log N(\mathcal{U}^{n_k})}{n_k} \to h_{\mathfrak{T}}(f) \text{ and } \frac{H(\mathcal{P}^{n_k}(\alpha_k, \beta_k))}{n_k} \to h_\mu(f)$$

as $n_k \to 0$.

Shannon Entropy as a Special Case of Metric Entropy

- Formulate the Shannon entropy in the context of a Bernoulli scheme.
- Use the Kolmogorov-Sinai Theorem to show they are equal.
- Define the phase space X as the set of all bi-infinite sequences of symbols, $X = S^{\mathbb{Z}} = \{\varsigma : \varsigma : \mathbb{Z} \to S\}$.
- Define a set function and use a theorem by Kolmogorov that the set function can be extended to a probability measure.
- Show that the set function is invariant under a shift map

Shannon Entropy as a Special Case of Metric Entropy

More explicitly,

$X = \{(\ldots, \varsigma_{-2}, \varsigma_{-1}, \varsigma_0, \varsigma_1, \varsigma_2, \ldots) : \varsigma \in S = \{s_1, s_2, \ldots, s_n\} \, \forall k \in \mathbb{Z}\}$

Define the bijective map (Bernoulli shift) $\mathfrak{s} : X \to X$,

$$\mathfrak{s}(\varsigma)(k) = \varsigma(k+1).$$

We then define a cylinder set $C(F, \psi) = \left\{\varsigma \in X : \varsigma_{|F} = \psi\right\}$ Using this we define a set function $\mu_0 : \mathcal{C} \to \mathbb{R}$ and use a theorem by Kolmogorov that the above function can be uniquely extended to a complete probability measure.

Tent Map

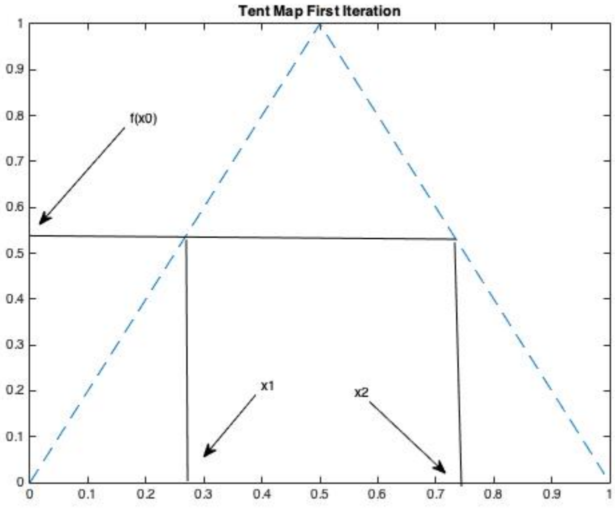
As an example, we look at the tent map $f = \Lambda : [0,1] \to [0,1]$ defined as,

$$\Lambda(x) = \begin{cases} 2x, & 0 \leq x < 1/2 \\ 2(1-x), & 1/2 \leq x \leq 1. \end{cases}$$

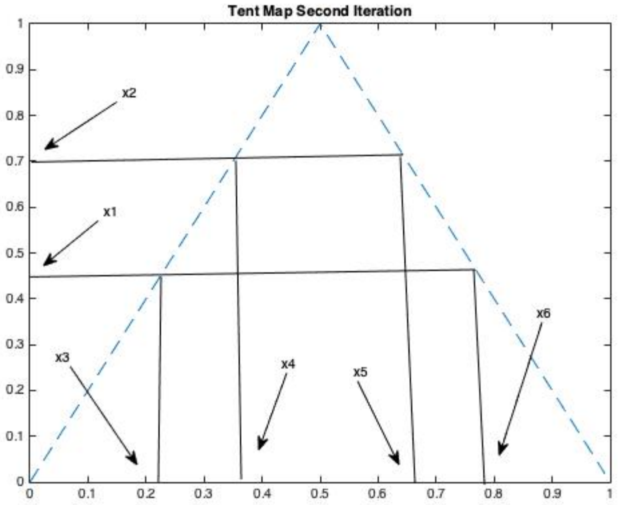
Combining this with the Lebesgue measure and the Euclidean topology $\mathfrak{T}_e$, and successively bisecting the unit interval. We obtain a sequence of partitions satisfying the above and conclude,

$$h_{\mathfrak{T}_e}(\Lambda) = h_\mu(\Lambda) = \log 2.$$

Tent Map First Iteration



Tent Map Second Iteration



Thank you.

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