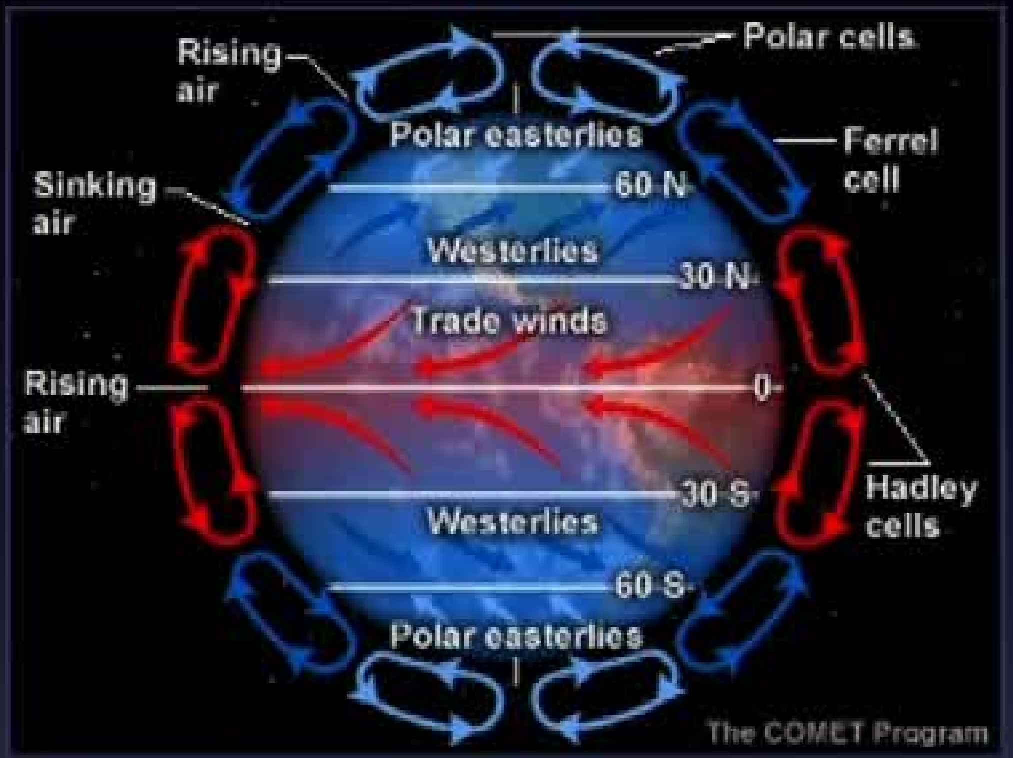
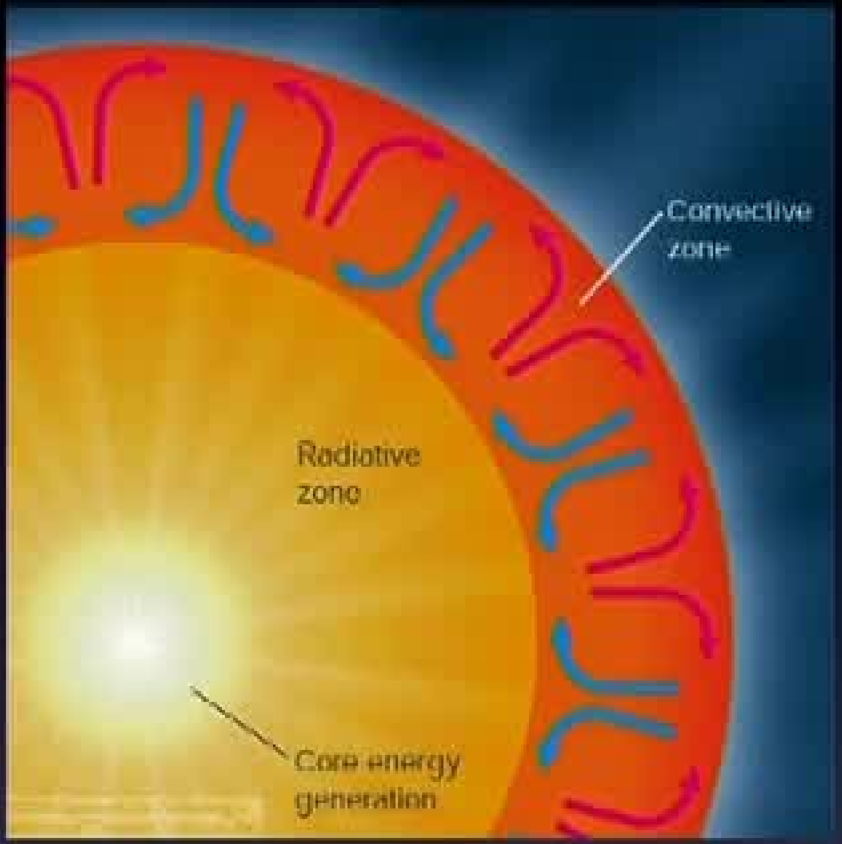
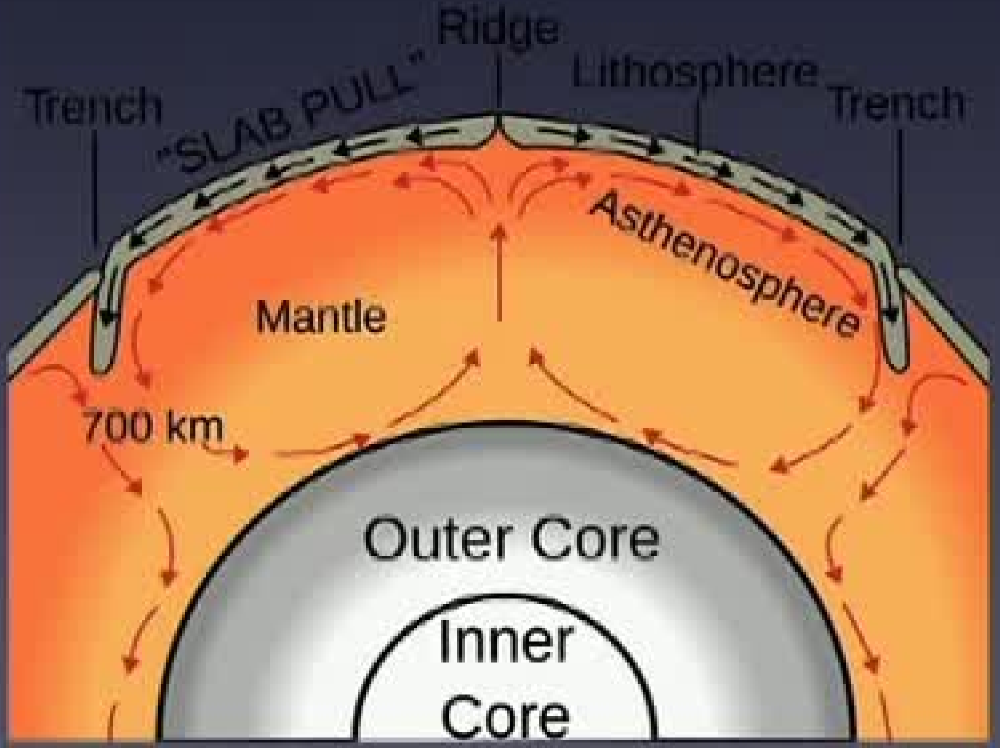
Wall to Wall Optimal Transport

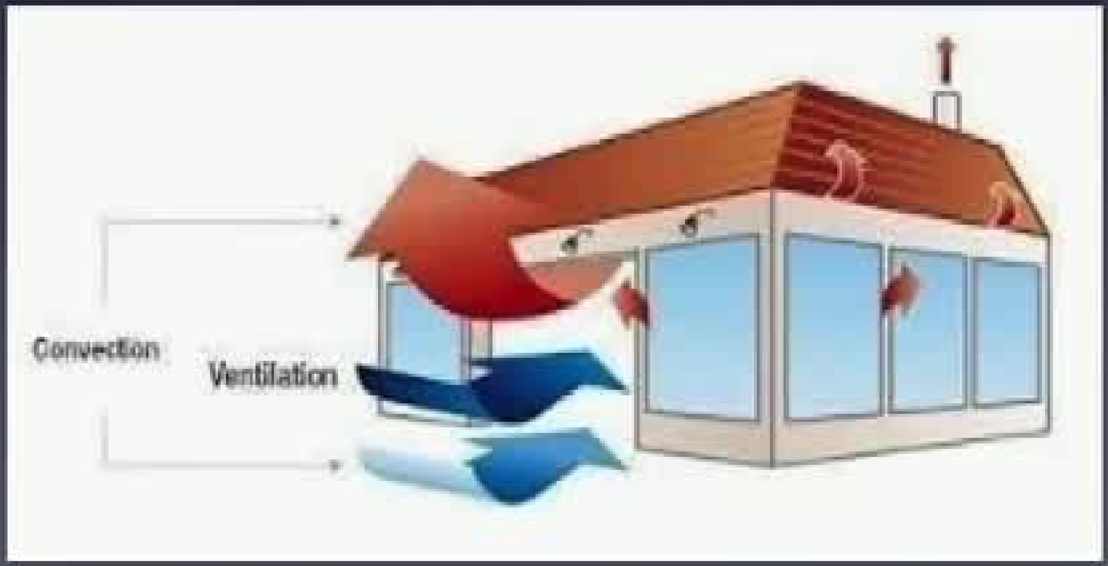
Andre Souza, Ian Tobasco, and The Doering

Thermal Convection

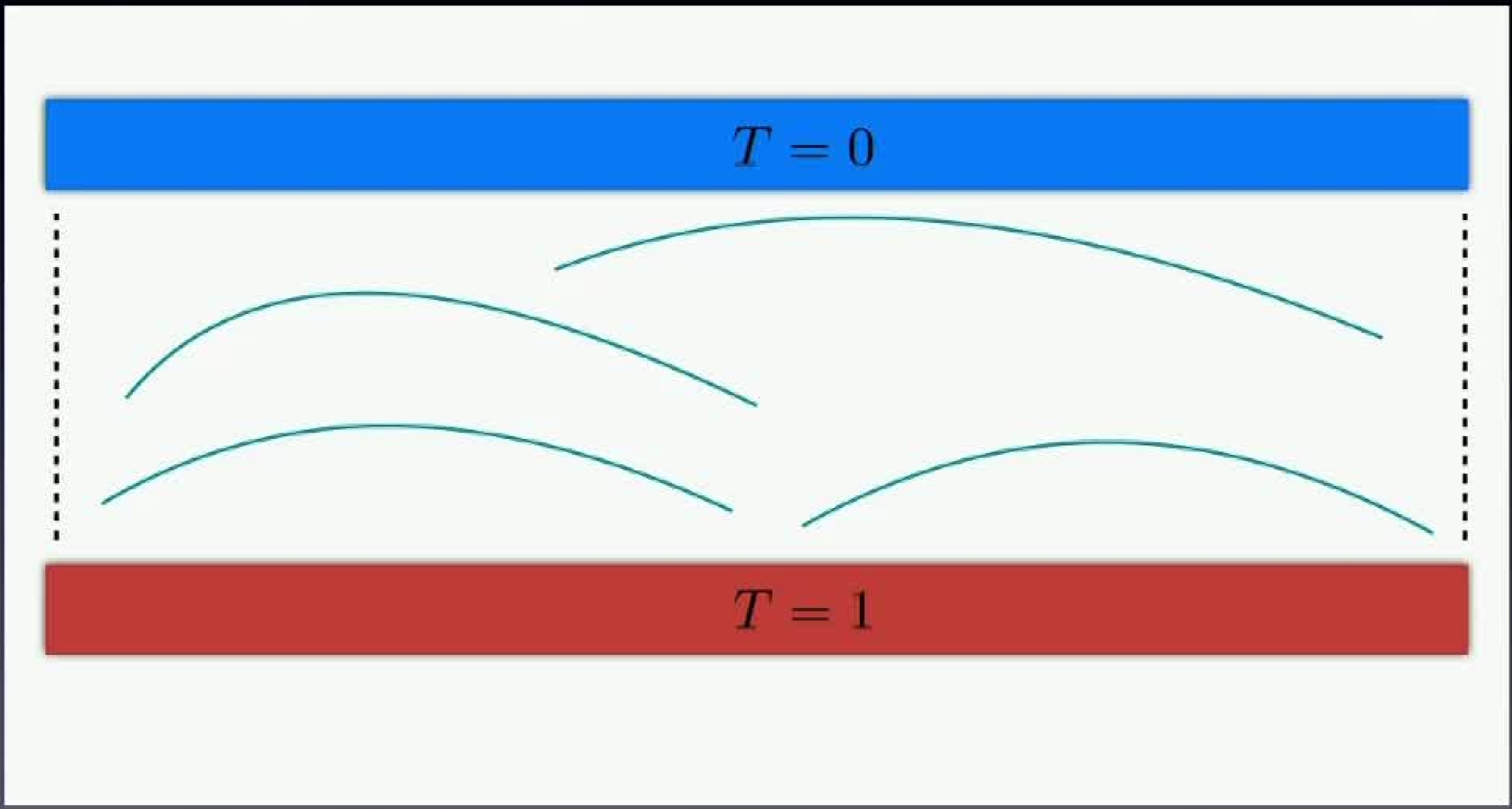








Simplest Context



Physically Relevant Constraints

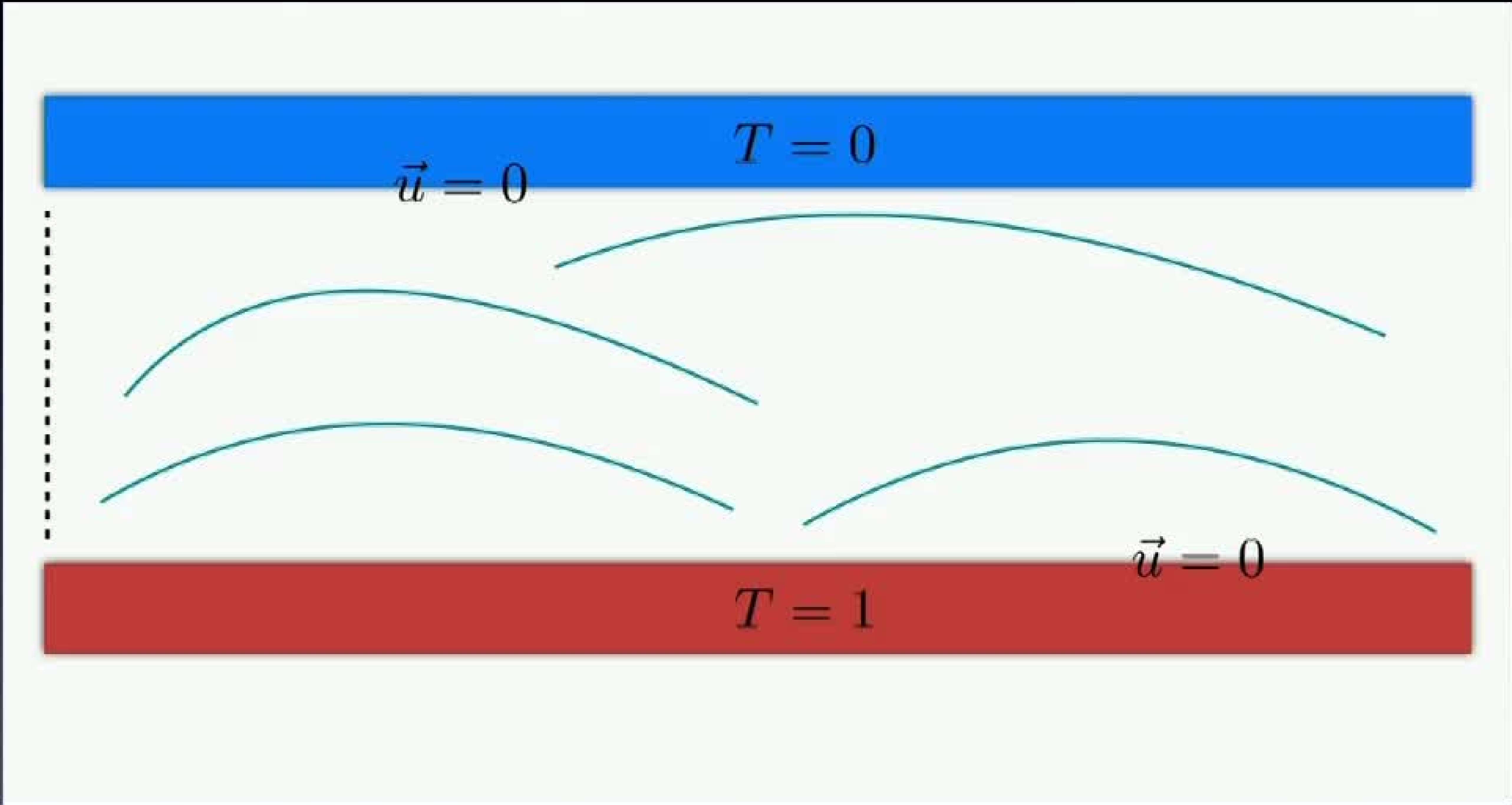
$$\partial_t T + \nabla \cdot (\vec{u}T - \nabla T) = 0$$

$\nabla \cdot \vec{u} = 0$: Incompressibility

Boundary Conditions

$\langle |\nabla \vec{u}|^2 \rangle < \infty$: Finite Enstrophy

Domain / BC



Optimize

$$\sup_{\vec{u}} \; \langle u_3 T \rangle$$

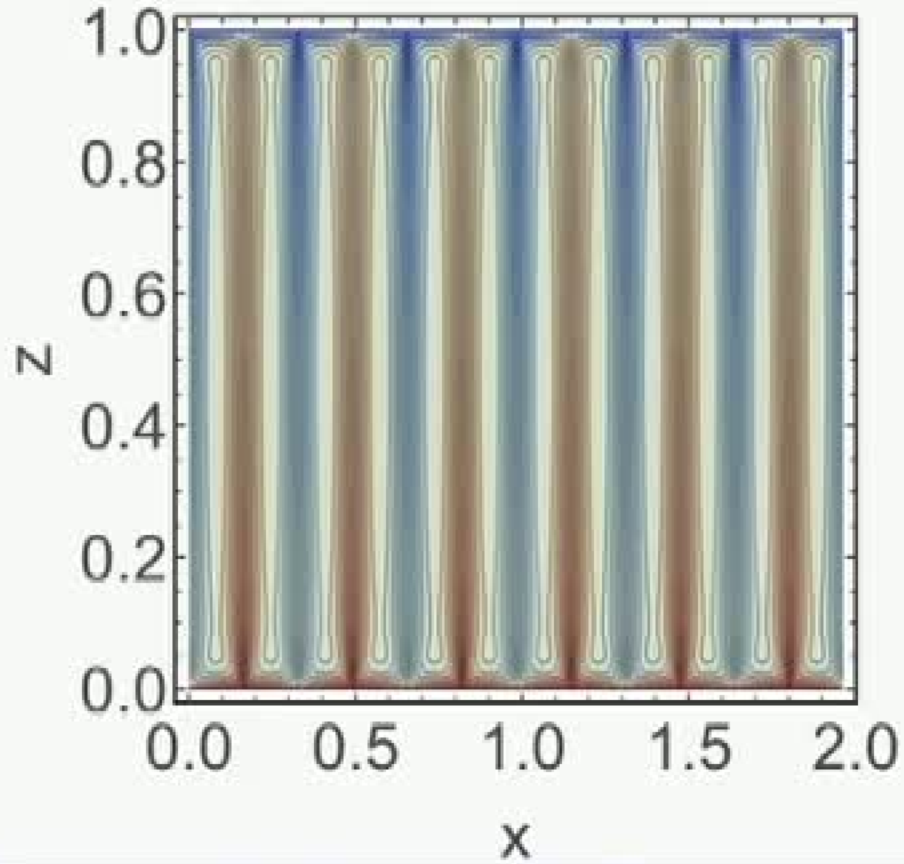
subject to

$$\langle \|\nabla \vec{u}\|^2 \rangle = \mathrm{Pe}^2$$

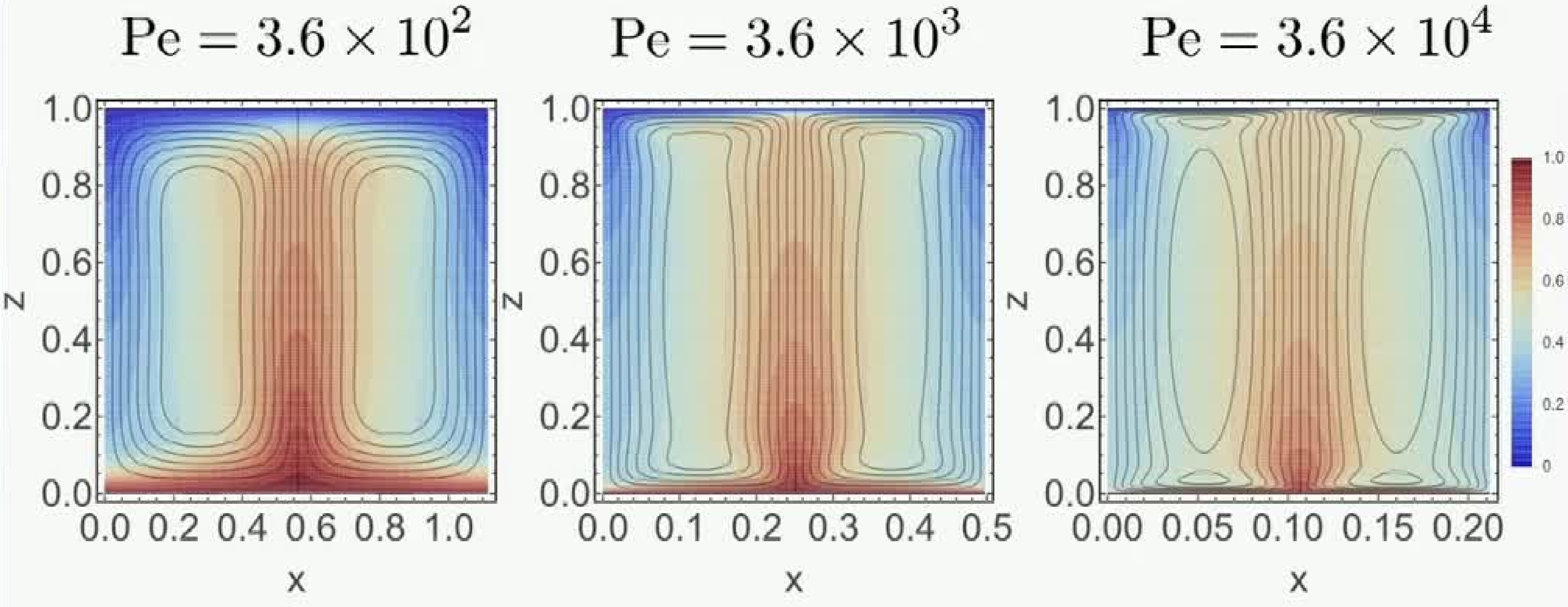
$$\nabla \cdot \vec{u} = 0$$

$$\vec{u} \cdot \nabla T = \Delta T$$

$$\|\nabla\vec{u}:\nabla\vec{u}\|_2 = 1.2\times 10^4$$

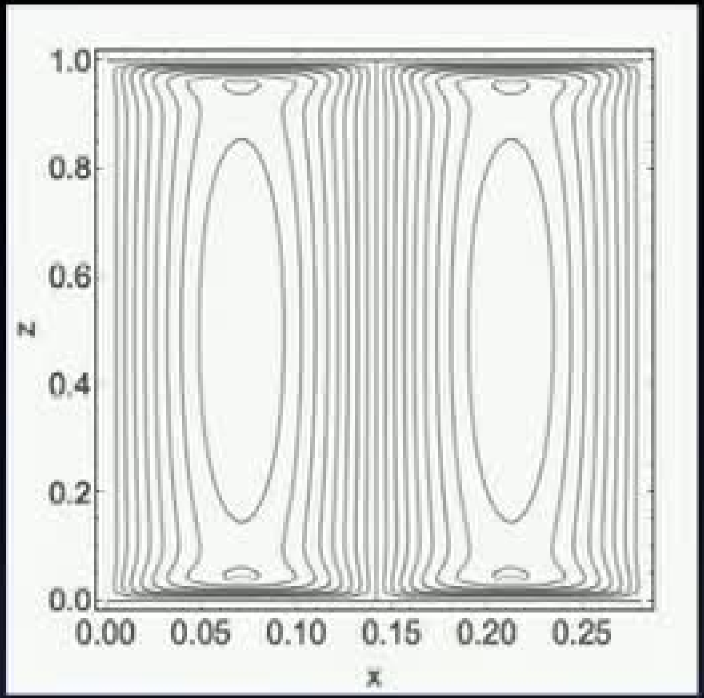


In a single cell



Structure

100%



Structure

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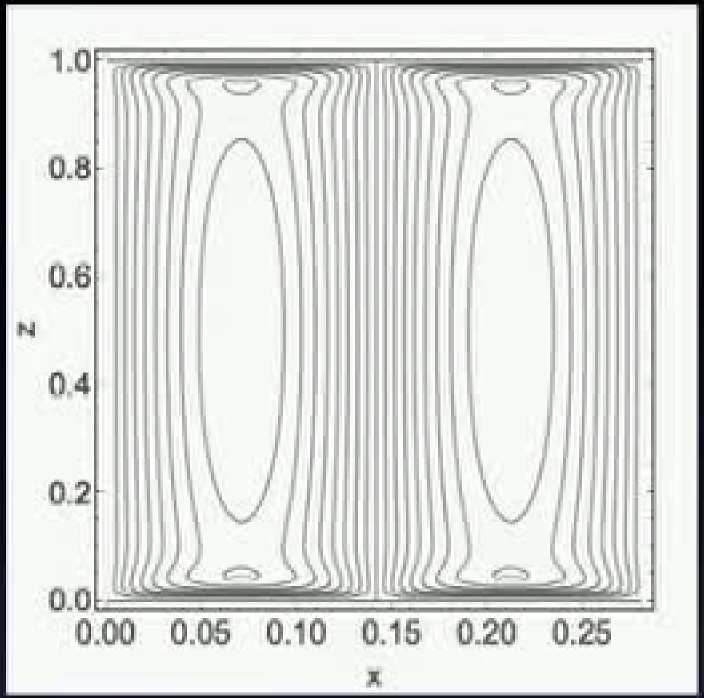

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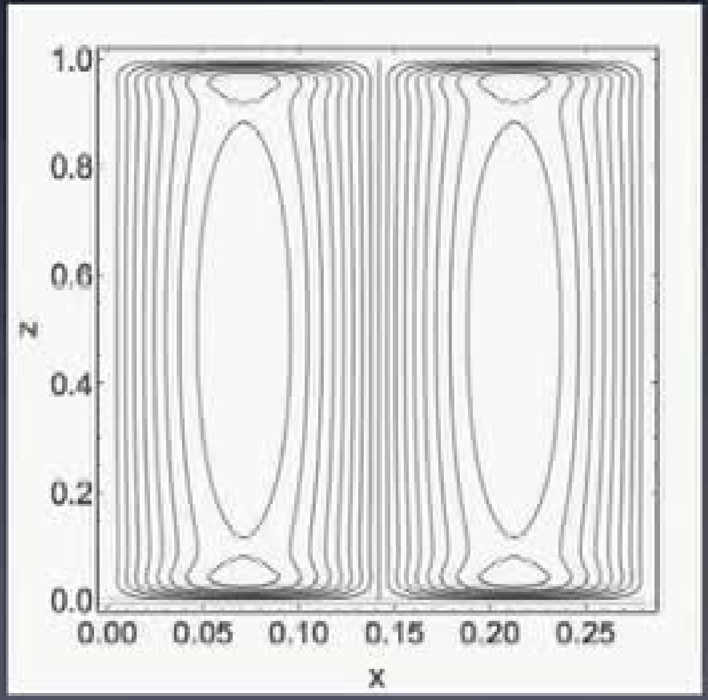

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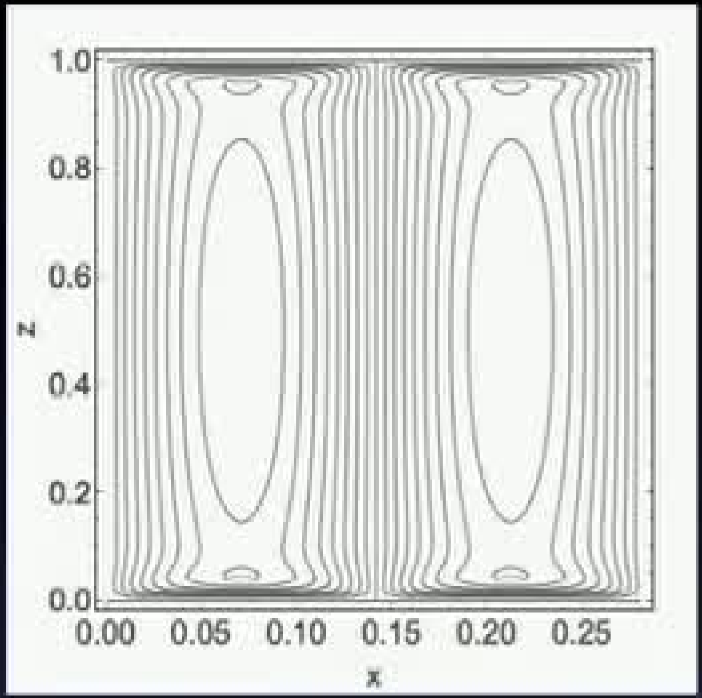

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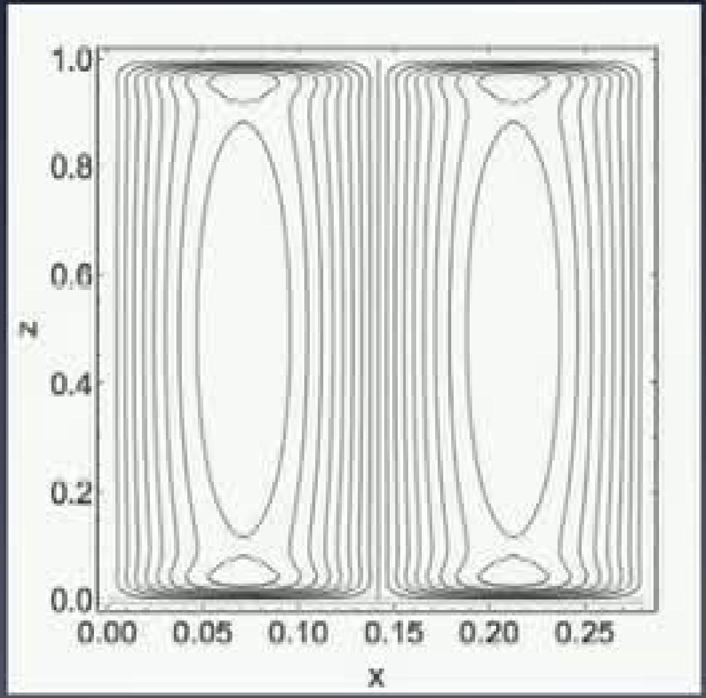

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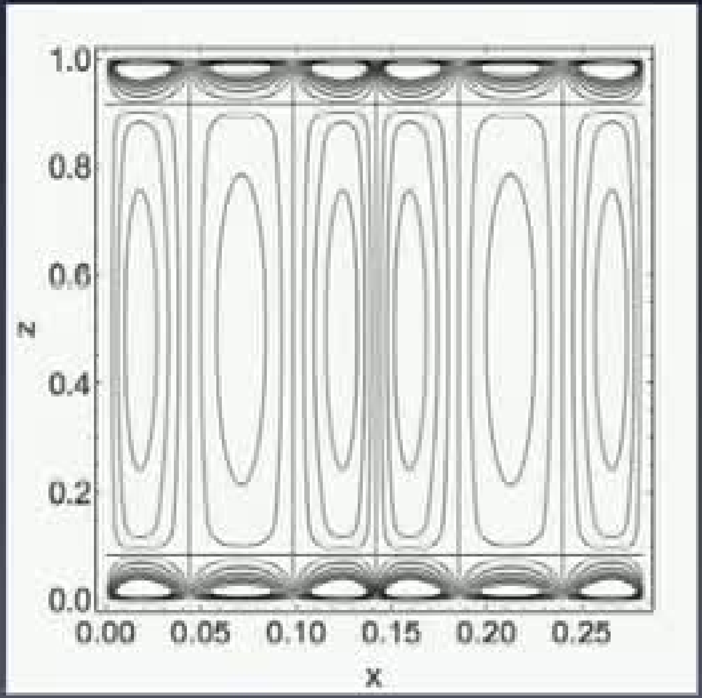

Structure

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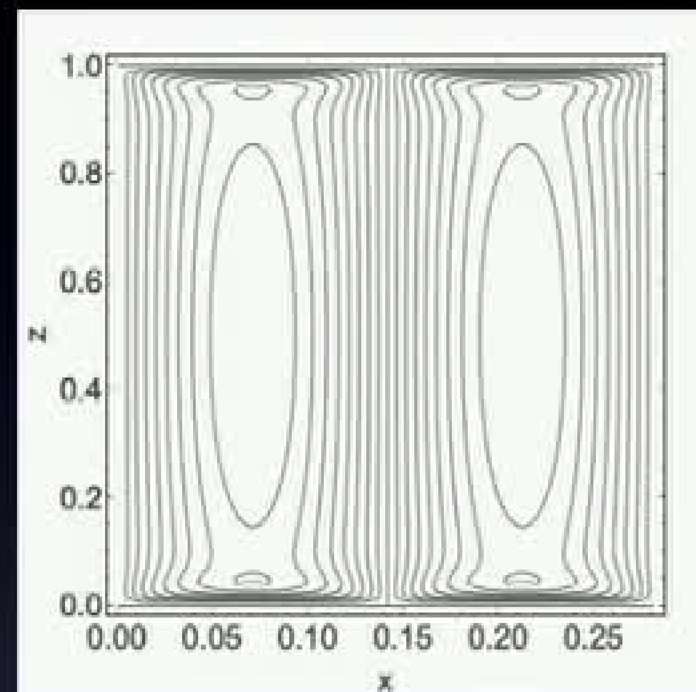

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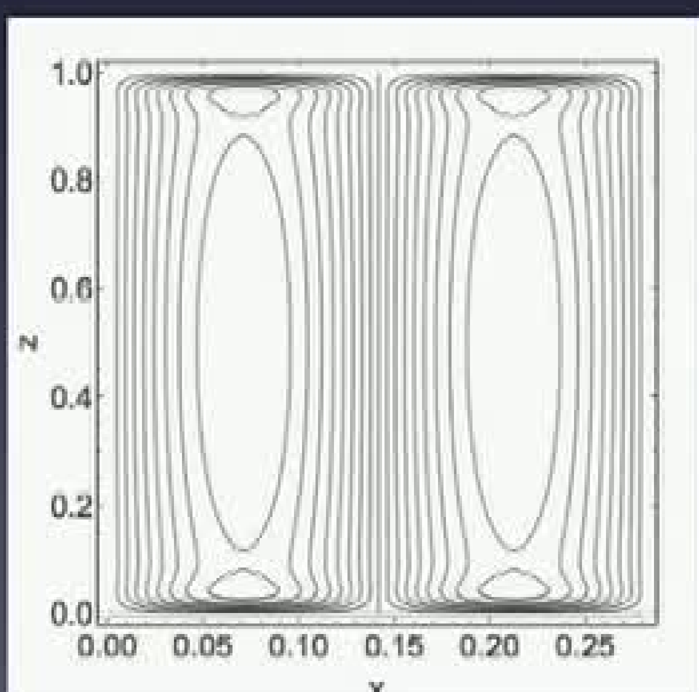

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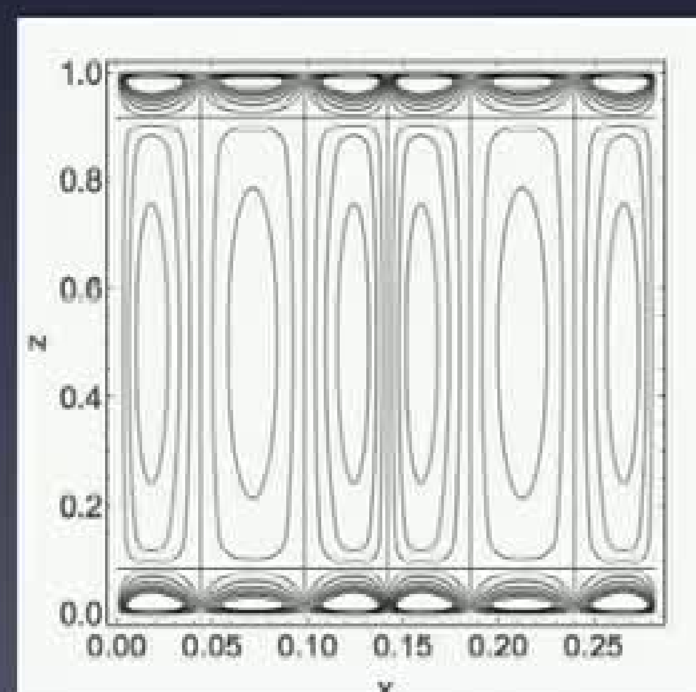

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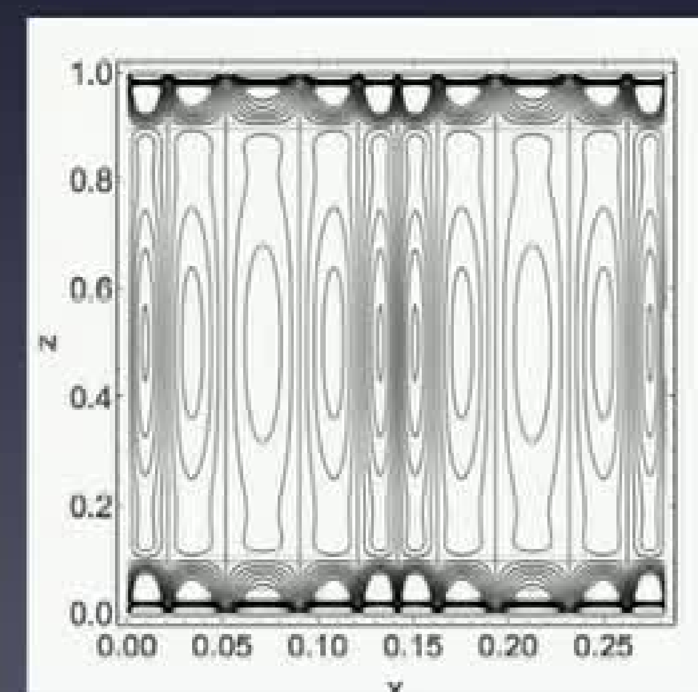

So What?

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- Utilizing a different formulation of the problem, it is possible to show that the scaling should be smaller

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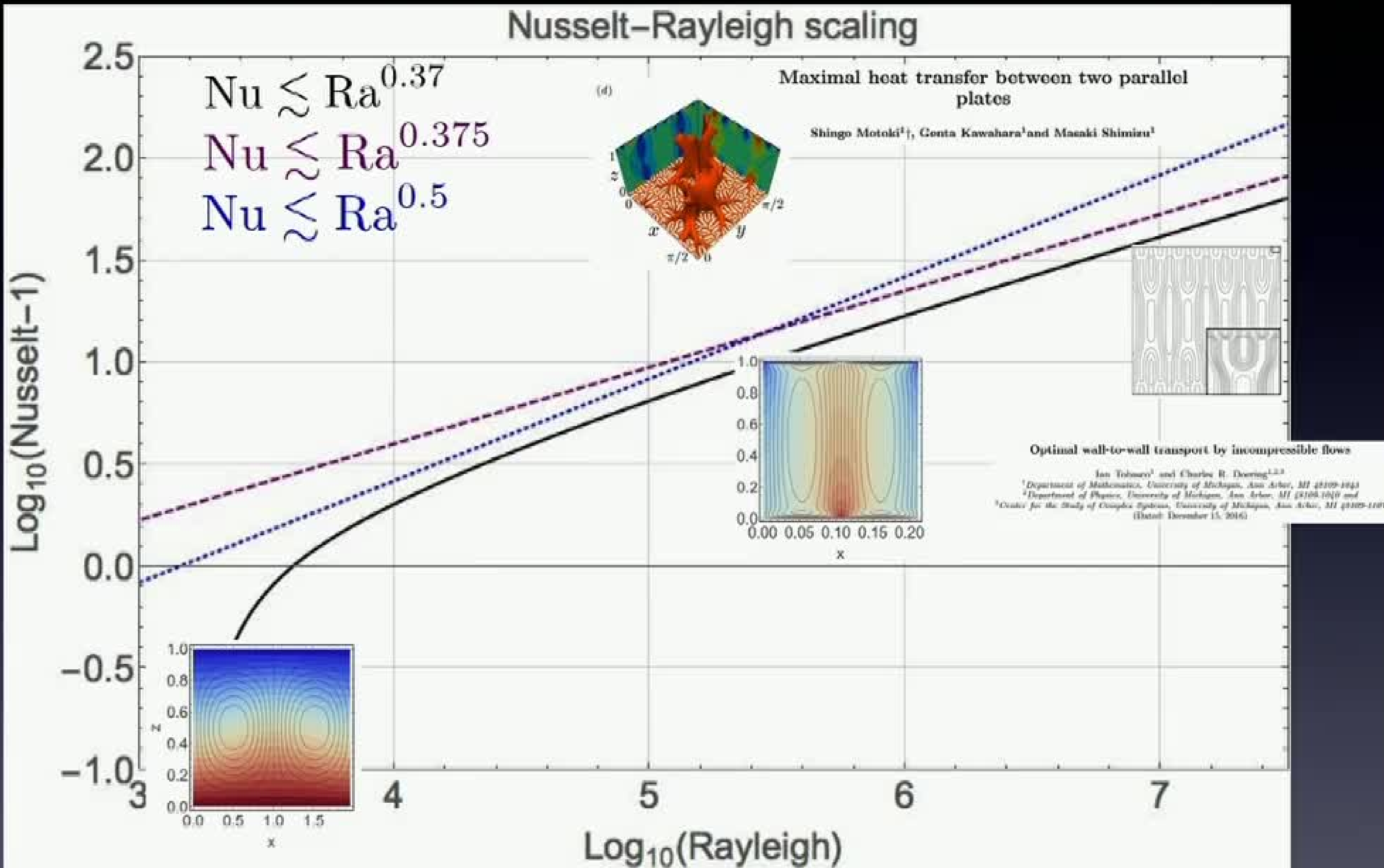
- Utilizing a different formulation of the problem, it is possible to show that the scaling should be smaller (with more assumptions)

Basic Idea

$$\inf_{\eta} \mathcal{S} \equiv \left\langle 2u_3\xi - |\nabla\Delta^{-1}(\mathbf{u}\cdot\nabla\xi)|^2 - |\nabla\xi|^2 + \frac{\mu}{2}\left(\mathrm{Pe}^2 - |\nabla\mathbf{u}|^2\right) + \nabla p\cdot\mathbf{u}\right\rangle$$

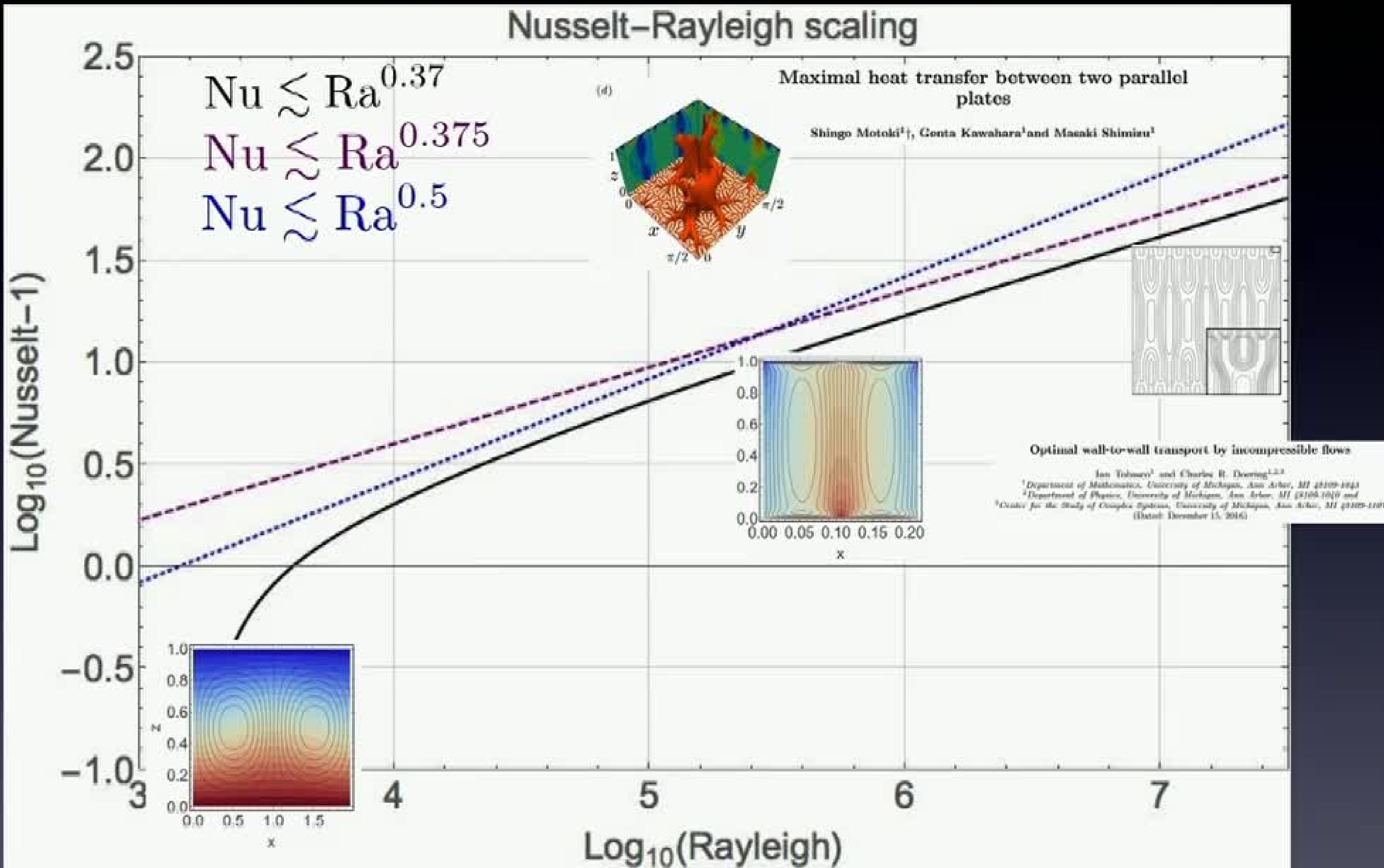
$$u_3(x,z) \approx X(x)Z(z)$$
$$\xi(x,z) \approx X(x)\zeta(z)$$



Nusselt−Rayleigh scaling
Nu ≲ Ra^0.37
Nu ≲ Ra^0.375
Nu ≲ Ra^0.5
Maximal heat transfer between two parallel plates
Shingo Motoki, Genta Kawahara and Masaki Shimizu
Optimal wall-to-wall transport by incompressible flows
Log10(Nusselt−1)
Log10(Rayleigh)
2.5
2.0
1.5
1.0
0.5
0.0
−0.5
−1.0
3
4
5
6
7
x
z

Questions

- Can time-dependent flow fields do better?
- How far can the tensor product approach be extended?
- Are there better upper/lower bounds to maximal transport in 2D?
- In 3D 1/2 possible asymptotically?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