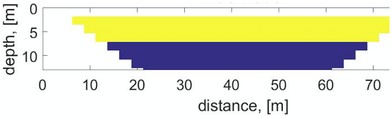
Discontinuous Boundaries Regularization Operators for Tikhonov Regularization

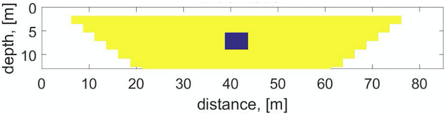
Jodi Mead
Department of Mathematics

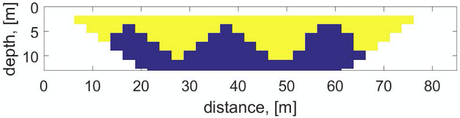
Thanks to: Hank Hetrick
NSF DMS-1418714

BOISE STATE UNIVERSITY

Imaging subsurface structure







Inverse Problem

$$\mathbf{d} = \mathbf{G}\mathbf{m} + \boldsymbol{\epsilon}$$

$\mathbf{d}$ - measurements

$\mathbf{G}$ - forward model, Jacobian for nonlinear model

$\mathbf{m}$ - unknown parameters

$\boldsymbol{\epsilon}$ - random noise

$$\min_{\mathbf{m}} \quad \|\mathbf{d} - \mathbf{G}\mathbf{m}\|_2^2$$

underdetermined, severely ill-conditioned

Tikhonov Regularization

$$\mathbf{m}_{\mathbf{L}_p} = argmin_{\mathbf{m}} \;\; \|\mathbf{W}_d(\mathbf{d} - \mathbf{G}\mathbf{m})\|_2^2 + \alpha^2 \|\mathbf{L}_p(\mathbf{m} - \mathbf{m}_{ref})\|_2^2$$

- $\mathbf{W}_d$ - data weight, typically data error covariance
- α - regularization parameter
  - α large $\rightarrow$ constraint: $\|\mathbf{L}_p(\mathbf{m} - \mathbf{m}_{ref})\|_2^2 \approx 0$
  - α small $\rightarrow$ problem may stay ill-conditioned
- $\mathbf{L}_p$ - pth derivative operator
- $\mathbf{m}_{ref}$ - initial parameter estimate

Outline

- Choice of $\mathbf{L}_p$
  - Adding information to the inverse problem
- Incorporating information about discontinuities
  - Regularization operators
- Electrical Resistance Tomography (ERT)
  - Two-layered, Anomaly and Sinusoidal models

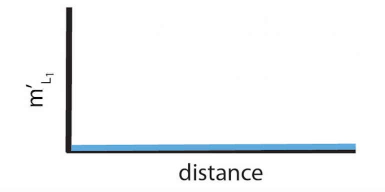
Choice of $\mathbf{L}_p$

$\mathbf{L}_0(\mathbf{m} - \mathbf{m}_{ref})$ - requires good initial estimate $\mathbf{m}_{ref}$, otherwise may not regularize the problem.

$\mathbf{L}_1(\mathbf{m} - \mathbf{m}_{ref})$ - typically $\mathbf{L}_1\mathbf{m}_{ref} = \mathbf{0}$, i.e. zero first derivative estimate, requires less prior information than $\mathbf{L}_0$.

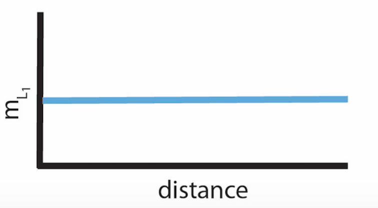
$\mathbf{L}_2(\mathbf{m} - \mathbf{m}_{ref})$ - typically $\mathbf{L}_2\mathbf{m}_{ref} = \mathbf{0}$, leaves more degrees of freedom than first derivative, so that data has more opportunity to inform parameter estimates.

Hypothetical results - $\min_{\mathbf{m}} \quad \|\mathbf{W}_d(\mathbf{d} - \mathbf{G}\mathbf{m})\|_2^2 + \alpha^2 \|\mathbf{L}_1 \mathbf{m}\|_2^2$



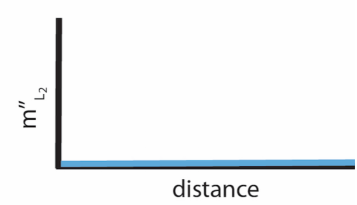
Large regularization parameter

$\Downarrow$



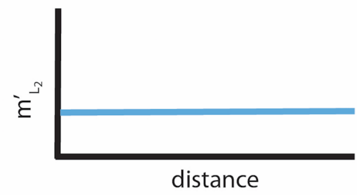
Data informs constant value

Hypothetical results - $\min_{\mathbf{m}} \quad \|\mathbf{W}_d(\mathbf{d} - \mathbf{G}\mathbf{m})\|_2^2 + \alpha^2\|\mathbf{L}_2\mathbf{m}\|_2^2$



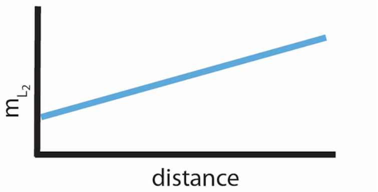
Large regularization parameter

$\Downarrow$



$\Rightarrow$

Data informs constant and slope

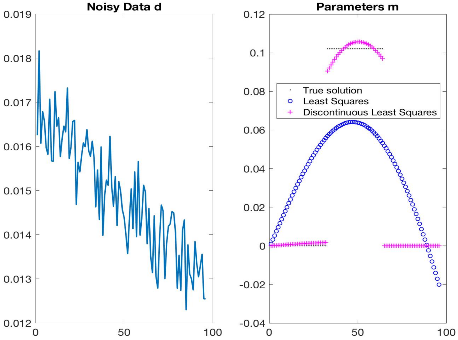


Incorporating discontinuities in Least Squares[1]

$$\min_{\mathbf{m}} \quad \|\mathbf{W}_d(\mathbf{d}-\mathbf{G}\mathbf{m})\|_2^2 + (\mathbf{m}-\mathbf{m}_{ref})^T \begin{bmatrix} \alpha_1\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \alpha_2\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \alpha_3\mathbf{I} \end{bmatrix} (\mathbf{m}-\mathbf{m}_{ref})$$

[1] Mead, 20013

Wing - 1D problem with discontinuous solution[2]



[2]P.C. Hansen, 2007

Regularization operator $\mathbf{R}$ for discontinuities

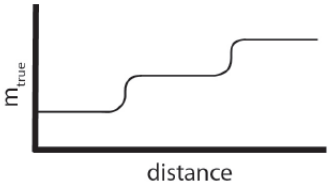
$$\min_{\mathbf{m}} \quad \|\mathbf{W}_d(\mathbf{d} - \mathbf{G}\mathbf{m})\|_2^2 + \alpha^2 \|\mathbf{R}\mathbf{L}_p\mathbf{m}\|_2^2$$

- $R = diag(r_1, \ldots, r_n)$, $r_i = 0$ or 1
- $r_i = 0 \rightarrow$ no regularization at discontinuity specified at i
  $\rightarrow$ no smoothness at i
- Only data informs parameter at discontinuity

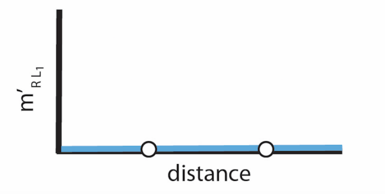
Toy Example - 3 layers, 2 discontinuities

$$\mathbf{R} = diag(1,0,1,0,1,1)$$

$$\mathbf{R}\mathbf{L}_1\mathbf{m} = \frac{1}{\Delta x}\begin{pmatrix} m_2 - m_1 \\ 0 \\ m_4 - m_3 \\ 0 \\ m_6 - m_5 \\ 0 \end{pmatrix}$$

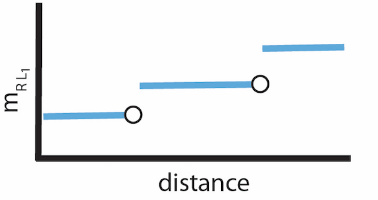


Toy Example - inferred results using $\alpha^2\|\mathbf{R}\mathbf{L}_1\mathbf{m}\|_2^2$



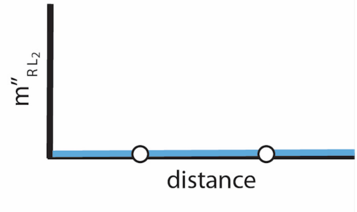
Large regularization parameter

$\Downarrow$



Data gives value at $\circ$
and informs constant values

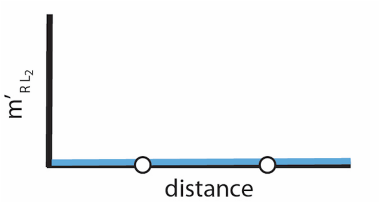
Toy Example - inferred results using $\alpha^2\|\mathbf{R}\mathbf{L}_2\mathbf{m}\|_2^2$



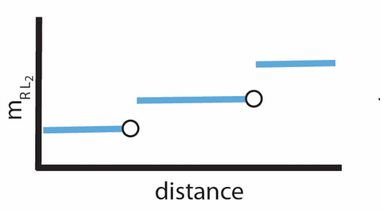
Large regularization parameter

Data gives ○ ⇓ *and informs zero*

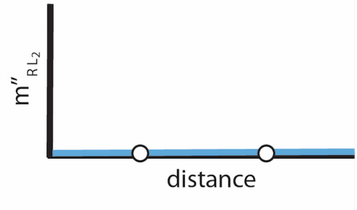
Data gives ○ *and informs constants*



⇒

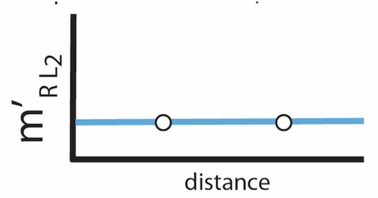


Toy Example - inferred results using $\alpha^2\|\mathbf{R}\mathbf{L}_2\mathbf{m}\|_2^2$



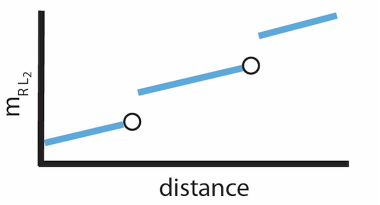
Large regularization parameter

Data gives ○ $\Downarrow$ *informs constant*



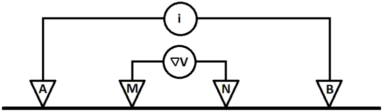
$\Rightarrow$

Data gives ○ *and informs constants*



Electrical Resistance Tomography (ERT)

$$\nabla \cdot [\sigma(\mathbf{r}) \nabla V(\mathbf{r})] = i[\delta(\mathbf{r} - \mathbf{r}_A) - \delta(\mathbf{r} - \mathbf{r}_B)]$$



Resistivity survey

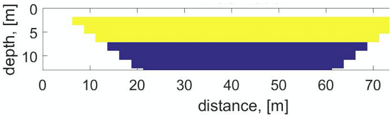
Numerical experiments

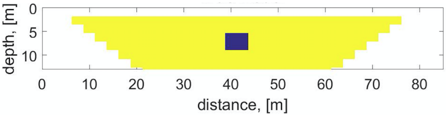
- 2.5D forward model - Fourier transform in y direction [3]
- 0.1% Gaussian noise

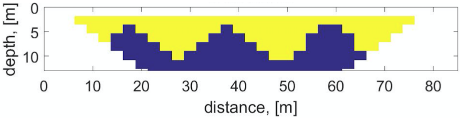
$$\|\mathbf{W}_d(\mathbf{d}-\mathbf{F}(\mathbf{m}))\|_2^2+\alpha^2\left\{\|\mathbf{R}_x\mathbf{L}_{px}\mathbf{m}\|_2^2+\|\mathbf{R}_z\mathbf{L}_{pz}\mathbf{m}\|_2^2\right\}$$

[3] Pidlisecky and Knight, 2008

Test Problems

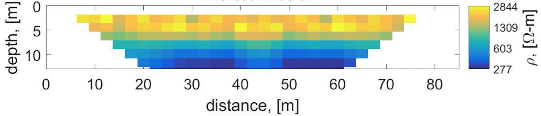




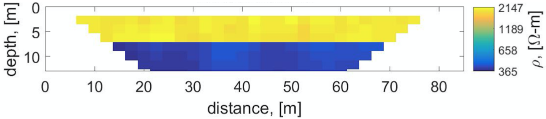


Inverted layered model with constant variability in subregions

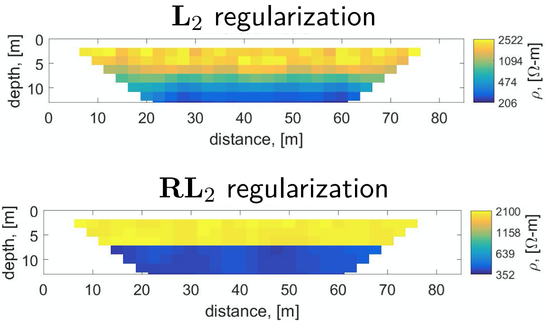
$\mathbf{L}_1$ regularization



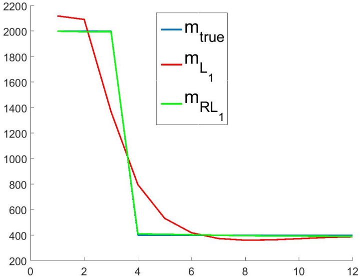
$\mathbf{RL}_1$ regularization

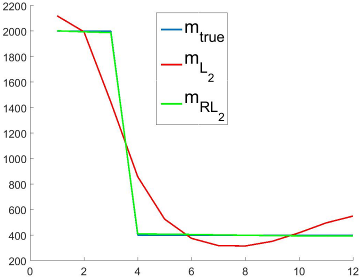


Inverted layered model with constant variability in subregions



Averaged vertical slices of resistivity

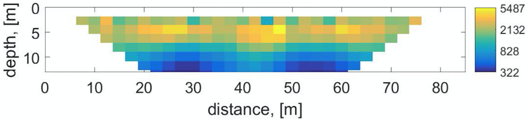




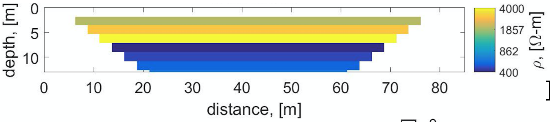
Layered model with constant variability

Inverted layered model with moderate linear variability in subregions

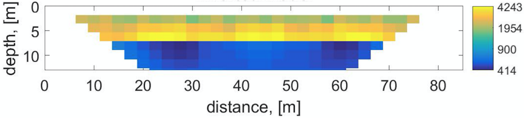
$\mathbf{L}_1$ regularization



True resistivity

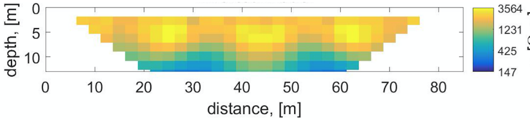


$\mathbf{RL}_1$ regularization

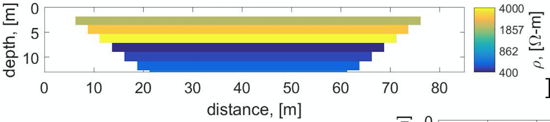


Inverted layered model with moderate linear variability in subregions

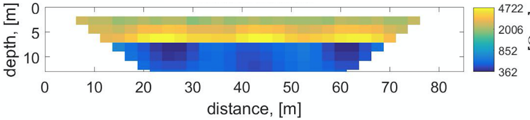
$\mathbf{L}_2$ regularization



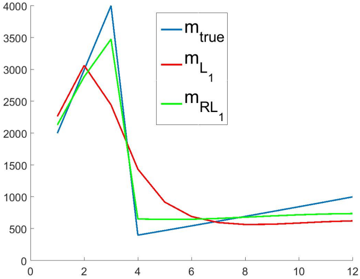
True resistivity

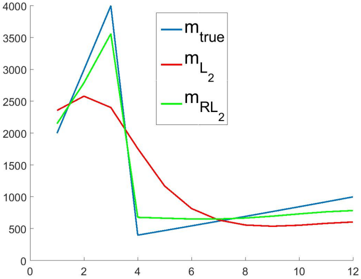


$\mathbf{RL}_2$ regularization



Averaged vertical slices of resistivity

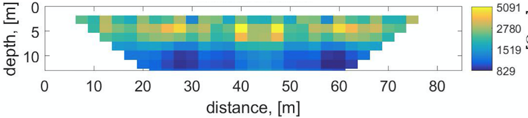




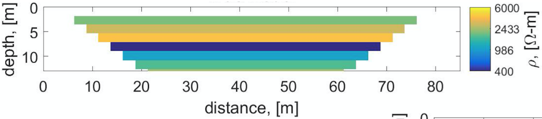
Layered model with moderate linear variability

Inverted layered model with strong linear variability in subregions

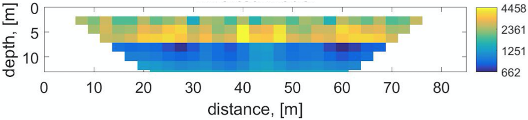
$\mathbf{L}_1$ regularization



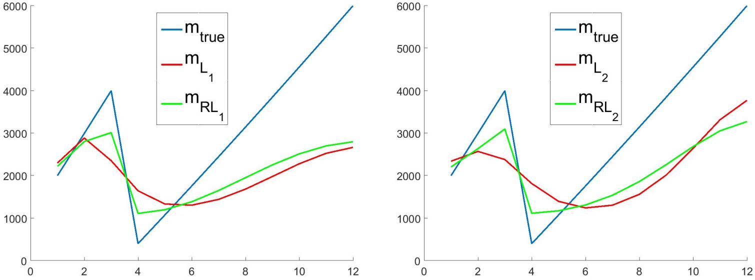
True resistivity



$\mathbf{RL}_1$ regularization



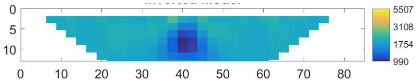
Averaged vertical slices of resistivity



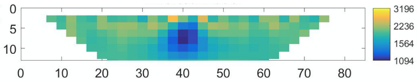
Layered model with strong linear variability

Inverted anomaly model with constant variability

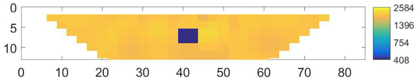
$\mathbf{L}_2$ regularization



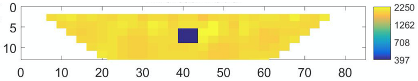
$\mathbf{L}_1$ regularization



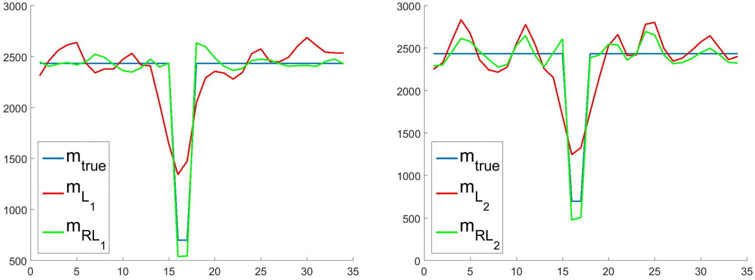
$\mathbf{RL}_2$ regularization



$\mathbf{RL}_1$ regularization

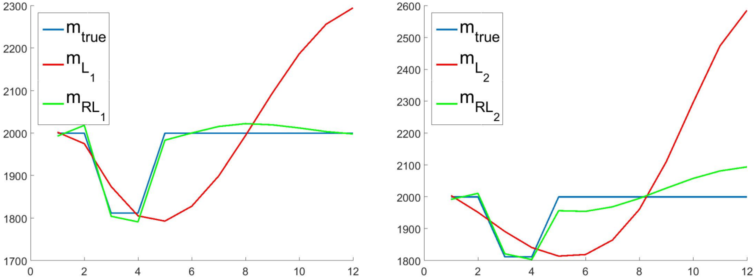


Averaged horizontal slices of resistivity



Anomaly model with constant variability

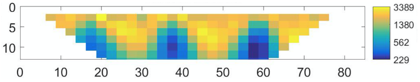
Averaged vertical slices of resistivity



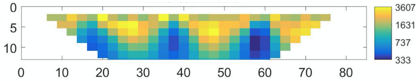
Anomaly model with constant variability

Inverted sinusoidal model with linear variability

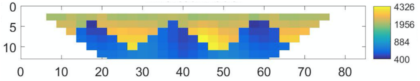
$\mathbf{L}_2$ regularization



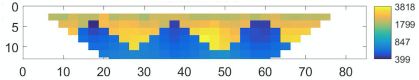
$\mathbf{L}_1$ regularization



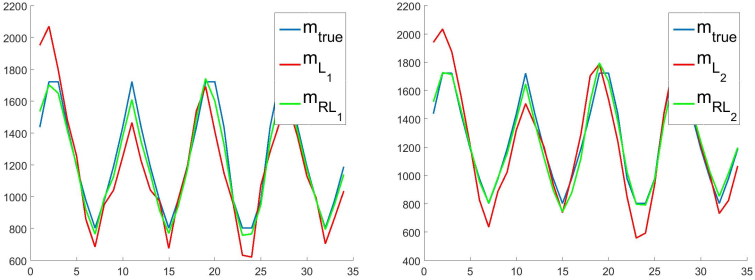
$\mathbf{RL}_2$ regularization



$\mathbf{RL}_1$ regularization



Averaged horizontal slices of resistivity



Sinusoidal model with linear variability

Summary

- Sharp discontinuities can be recovered with Tikhonov regularization through regularization operators $\mathbf{R}$
  - requires knowledge of the boundaries.
- Smoothing constraints can be viewed as prior information
  - derivatives don't require good initial estimates.
  - second derivative offers more degrees of freedom.
- Concepts applied to ERT synthetic inversion
  - analysis verified on distant disconitnuities, small anomaly, and complex boundary geometry.

Questions?