Time Accurate Methods in CFD

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"Since time accurate solutions are expensive, steady solutions may be greeted as successes whether they are correct or not.
...on no account should the model be trusted to predict the transition points ... responsibility rests with the user ... through an educated guess." - P. Spalart

$$\text{Time Accuracy} \begin{cases} = & \text{Extending predictibility horizon} \\ \neq & \text{Estimates} \qquad ||error|| \leq C(u)e^{\alpha t_n} k^2 \end{cases}$$

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- Initialization (spinup), Model calibration & Legacy codes
- Newtonian indeterminancy, uncertainty & ensembles
- Time adaptivity
- Computational, Space & Cognitive complexity

Method from 1960's:

$$\frac{u^{n+1}-u^n}{k} + u^* \cdot \nabla u^{n+1} - \nu \triangle u^{n+1} + \nabla p^{n+1} = f^{n+1}$$

$$\varepsilon \frac{p^{n+1}-p^n}{k} + \nabla \cdot u^{n+1} = 0$$

Solve for u then Update p

$$\frac{u^{n+1}-u^n}{k} + u^* \cdot \nabla u^{n+1} - \frac{k}{\varepsilon}\nabla\nabla \cdot u^{n+1} - \nu \triangle u^{n+1} = f^{n+1} - \nabla p^n$$

$$\text{Then:} \quad p^{n+1} = p^n - \frac{k}{\varepsilon}\nabla \cdot u^{n+1}$$

Classic Theorem: Stable, Convergent, Error $= O(\varepsilon + k)$ so take $\varepsilon = k$

$$\vdots$$

1960's - 2018: little progress on

Higher Order, Variable Timestep, Time-Adaptive Methods

$\Rightarrow$

Time Accurate AC Methods elusive

The problem: Variable k & $\varepsilon \Rightarrow$ Energy Input

$$u_t - \nu \triangle u + \nabla p = 0 \quad \& \quad \boldsymbol{\varepsilon}(\mathbf{t}) p_t + \nabla \cdot u = 0$$

$$\Rightarrow$$

$$\frac{d}{dt} \frac{1}{2} \int |u|^2 + \varepsilon(t) p^2 dx + \int \nu |\nabla u|^2 dx = \dot{\boldsymbol{\varepsilon}}(\mathbf{t}) \int p^2 dx \ \ \textbf{(= Energy Input!)}$$

Resolution:

$$\frac{\varepsilon_{n+1} p^{n+1} - \widehat{\varepsilon} p^n}{k_{n+1}} + \nabla \cdot u^{n+1} = 0$$

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$$\text{where } \widehat{\varepsilon} = \begin{cases} \sqrt{\varepsilon_n \varepsilon_{n+1}} \\ or \\ \min\{\varepsilon_n, \varepsilon_{n+1}\} \end{cases}$$

Thm.[w McLaughlin & Ming Chen] Stable, Model Convergence:

$$u_\varepsilon \to u_{NSE} \ \& \ p_\varepsilon \to p_{NSE} \ as \ \varepsilon(t) \to 0 \ \& \ \dot{\boldsymbol{\varepsilon}}(\mathbf{t}) \to \mathbf{0}$$

IDEA: Double Adaptivity [w Michael McLaughlin]

$$\text{LTE}(p \text{ eqn}) = \mathcal{O}(\varepsilon) + \cdots$$
$$\&$$
$$\text{LTE}(u \text{ eqn}) = \mathcal{O}(k) + \cdots$$
.

$$\text{Adapt} \begin{cases} \varepsilon \text{ based on } \mathbf{EST}_p = \ ||\nabla \cdot u|| \\ . \\ k \ \text{ based on } \textbf{time filters} \end{cases}$$

$\mathbf{EST}_u$ forom Time Filters [BIT 2016]:

$$\frac{y^{n+1} - y^n}{k} = f(t_{n+1}, y^{n+1}),$$
.
$$y^{n+1} \Leftarrow y^{n+1} - \frac{\alpha}{2}\left\{y^{n+1} - 2y^n + y^{n-1}\right\} \equiv \mathcal{F}(y^{n+1})$$

Thm. [w Guzel] $\alpha = \frac{2}{3} \Rightarrow$ *A-stable,* $O(k^2)$ *and estimator*

$$EST = ||y^{n+1}_{post-\mathcal{F}} - y^{n+1}_{pre-\mathcal{F}}||$$

Double ε, k Adaptive Algorithm

Initialize:

$$\begin{aligned}
\widehat{\varepsilon} &= \sqrt{\varepsilon_n \varepsilon_{n+1}} \quad \text{or} \quad \min\{\varepsilon_n, \varepsilon_{n+1}\} \\
u^* &= (1 + \frac{k_{n+1}}{k_n})u^n - \frac{k_{n+1}}{k_n}u^{n-1}
\end{aligned}$$

I. **Solve for** u^{n+1}

$$\frac{u^{n+1} - u^n}{k_{n+1}} + u^* \cdot \nabla u^{n+1} - \frac{k_{n+1}}{\varepsilon_{n+1}}\nabla\nabla \cdot u^{n+1} - \nu \triangle u^{n+1} = f^{n+1} - \frac{\widehat{\varepsilon}}{\varepsilon_{n+1}}\nabla p^n$$

II. **Time Filter** $u^{n+1} \Leftarrow \mathcal{F}(u^{n+1})$ & Estimate:

$$\begin{aligned}
EST_m &= ||u^{n+1}_{post} - u^{n+1}_{pre}|| \\
&. \\
EST_c &= ||\nabla \cdot u^{n+1}||
\end{aligned}$$

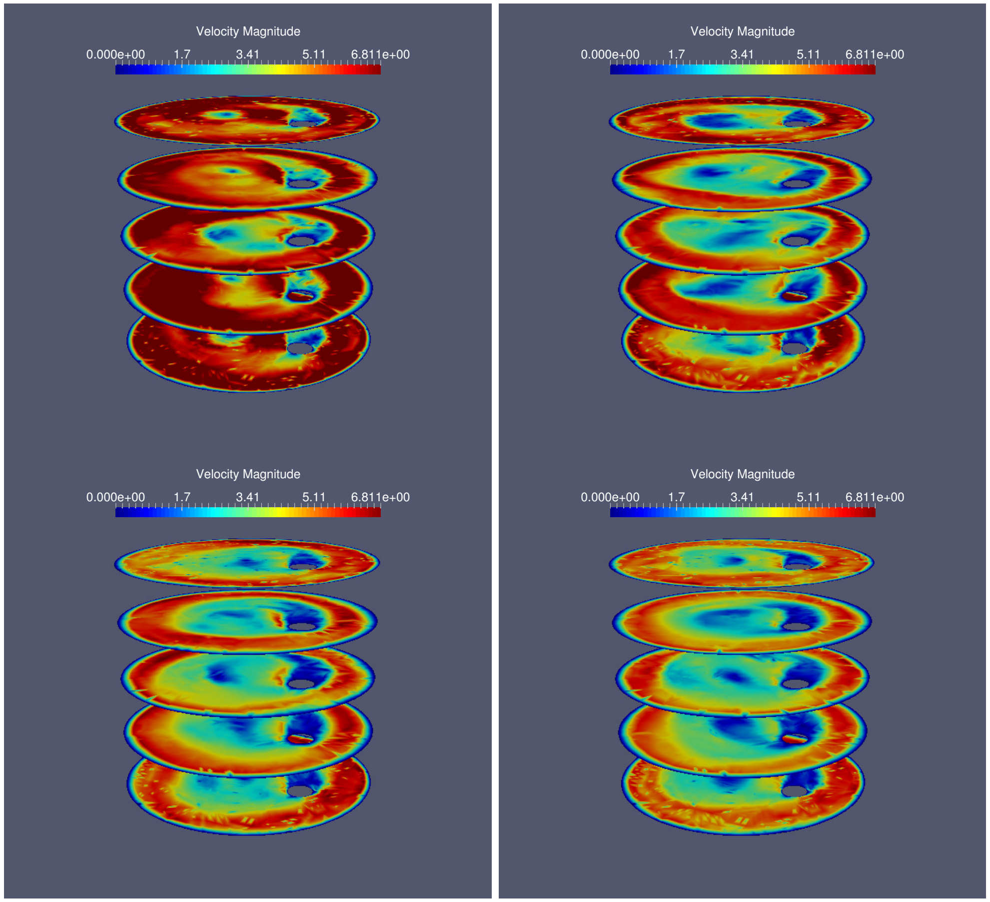
III. **Decision tree:** Adapt ε, k

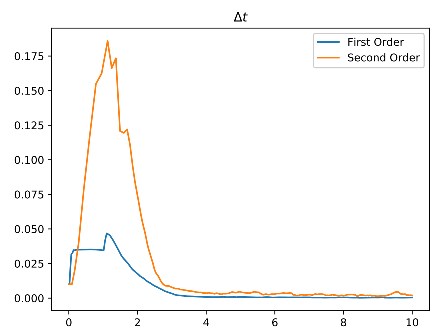
EST too Big? EST too Small? EST just Right?

IV. **Update:**

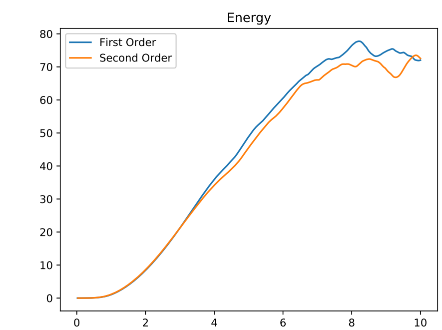
$$p^{n+1} = \frac{\widehat{\varepsilon}}{\varepsilon_{n+1}}p^n - \frac{k_{n+1}}{\varepsilon_{n+1}}\nabla \cdot u^{n+1}$$

Offset cylinder problem

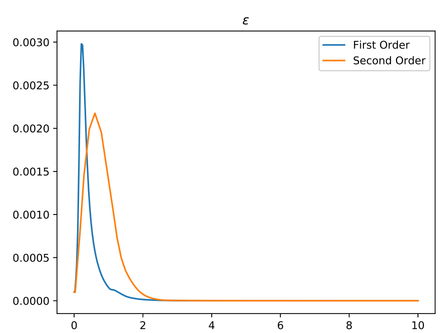




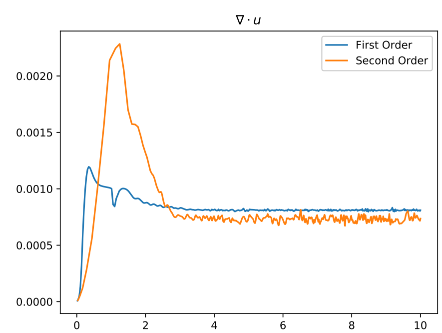
timestep evolution



energy evolution



ε evolution



||div u|| evolution

Onwards!

Higher order + Adaptive + AC + Ensembles

$\Downarrow$

Extend the predictability horizon of turbulent flows