EIGENVECTORS OF TENSORS

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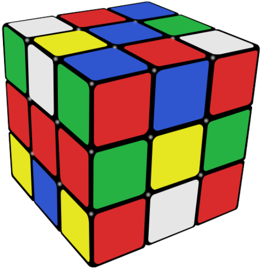
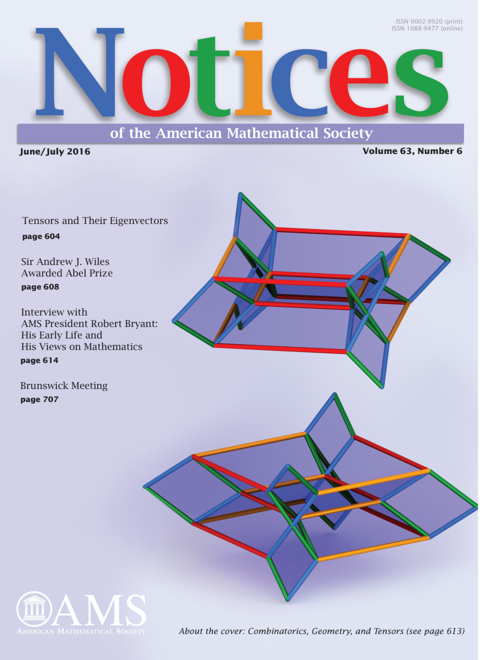


How many eigenvectors does a $3 \times 3 \times 3$*-tensor have?*
How many singular vector triples does a $3{\times}3{\times}3$*-tensor have?*

What is this?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About the cover: Combinatorics, Geometry, and Tensors (see page 613)

Tensors and their rank

A tensor is a d-dimensional array of numbers $T = (t_{i_1 i_2 \cdots i_d})$.
For $d = 1$ this is a vector, and for $d = 2$ this is a matrix.

A tensor T of format $n_1 \times n_2 \times \cdots \times n_d$ has $n_1 n_2 \cdots n_d$ entries.

T has rank 1 if it is the outer product of d vectors $\mathbf{u}, \mathbf{v}, \ldots, \mathbf{w}$:

$$t_{i_1 i_2 \cdots i_d} \;=\; u_{i_1} v_{i_2} \cdots w_{i_d}.$$

The set of tensors of rank 1 is the Segre variety.

A tensor has rank r if it is the sum of r tensors of rank 1. (not fewer).

Tensor decomposition:

- Express a given tensor as a sum of rank 1 tensors.
- Use as few summands as possible.

Textbook: JM Landsberg: *Tensors: Geometry and Applications*, 2012.

Symmetric tensors

An $n \times n \times \cdots \times n$-tensor $T = (t_{i_1 i_2 \cdots i_d})$ is symmetric if it is unchanged under permuting indices. Dimension is $\binom{n+d-1}{d}$.

T has rank 1 if it is the d-fold outer product of a vector $\mathbf{v}$:

$$t_{i_1 i_2 \cdots i_d} \;=\; v_{i_1} v_{i_2} \cdots v_{i_d}.$$

The set of symmetric tensors of rank 1 is the Veronese variety.

A symmetric tensor has rank r if it is the sum of r such tensors.

Symmetric tensors

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A symmetric tensor has rank r if it is the sum of r such tensors.

Open Problem [Comon's Conjecture] *Is the rank of every symmetric tensor equal to its rank as a general tensor?*

True for $d = 2$: every rank 1 decomposition of a symmetric matrix

$$T = \mathbf{u}_1^t \mathbf{v}_1 + \mathbf{u}_2^t \mathbf{v}_2 + \cdots + \mathbf{u}_r^t \mathbf{v}_r.$$

transforms into a decomposition with rank 1 symmetric matrices:

$$T = \mathbf{w}_1^t \mathbf{w}_1 + \mathbf{w}_2^t \mathbf{w}_2 + \cdots + \mathbf{w}_r^t \mathbf{w}_r$$

Polynomials and their eigenvectors

Symmetric tensors correspond to homogeneous polynomials

$$T \;=\; \sum_{i_1,\ldots,i_d=1}^{n} t_{i_1 i_2 \cdots i_d} \cdot x_{i_1} x_{i_2} \cdots x_{i_d}$$

The tensor has rank r if T is a sum of r powers of linear forms:

$$T \;=\; \sum_{j=1}^{r} \lambda_j \mathbf{v}_j^{\otimes d} \;=\; \sum_{j=1}^{r} \lambda_j (v_{1j}x_1 + v_{2j}x_2 + \cdots + v_{nj}x_n)^d.$$

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The *gradient of T* defines a polynomial map of degree $d-1$:

$$\nabla T : \mathbb{R}^n \to \mathbb{R}^n.$$

A vector $\mathbf{v} \in \mathbb{R}^n$ is an eigenvector of the tensor T if

$$(\nabla T)(\mathbf{v}) \;=\; \lambda \cdot \mathbf{v} \qquad \text{for some } \lambda \in \mathbb{R}.$$

What is this good for?

Consider the optimization problem of maximizing a homogeneous polynomial T over the unit sphere in $\mathbb{R}^n$.

Lagrange multipliers lead to the equations

$$(\nabla T)(\mathbf{v}) \;=\; \lambda \cdot \mathbf{v} \qquad \text{for some } \lambda \in \mathbb{R}.$$

Fact: *The critical points are the eigenvectors of T.*

It is convenient to replace $\mathbb{R}^n$ with **projective space** $\mathbb{P}^{n-1}$.

Eigenvectors of T are fixed points of $\nabla T \,:\, \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$.

Fact: *These are nonlinear dynamical systems on $\mathbb{P}^{n-1}$.*

[Lim, Ng, Qi: *The spectral theory of tensors and its applications*, 2013]

Linear maps

Real symmetric $n \times n$-matrices (t_{ij}) correspond to quadratic forms

$$T = \sum_{i=1}^{n} \sum_{j=1}^{n} t_{ij} x_i x_j$$

By the Spectral Theorem, there exists a real decomposition

$$T = \sum_{j=1}^{r} \lambda_j \, (v_{1j} x_1 + v_{2j} x_2 + \cdots + v_{nj} x_n)^2 .$$

Here r is the *rank* and the λ_j are the *eigenvalues* of T.
The *eigenvectors* $v_j = (v_{1j}, v_{2j}, \ldots, v_{nj})$ are orthonormal.

One can compute this decomposition by the *Power Method*:

Iterate the linear map $\nabla T : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$

Fixed points of this dynamical system are eigenvectors of T.

Quadratic maps

Symmetric $n \times n \times n$-tensors (t_{ijk}) correspond to cubic forms

$$T \;=\; \sum_{i=1}^{n} \sum_{j=1}^{n} \sum_{k=1}^{n} t_{ijk} x_i x_j x_k$$

We are interested in low rank decompositions

$$T \;=\; \sum_{j=1}^{r} \lambda_j \left(v_{1j} x_1 + v_{2j} x_2 + \cdots + v_{nj} x_n\right)^3 .$$

One idea to find this decomposition is the *Tensor Power Method*:

Iterate the quadratic map $\nabla T : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$

Fixed points of this dynamical system are eigenvectors of T.

Bad News: The eigenvectors are usually not the vectors $\mathbf{v}_i$ in the low rank decomposition ... unless the tensor is *odeco*.

Odeco tensors

A symmetric tensor T is *odeco* (= orthogonally decomposable) if

$$T \;=\; \sum_{j=1}^{n} \lambda_j \mathbf{v}_j^{\otimes d} \;=\; \sum_{j=1}^{n} \lambda_j (v_{1j}x_1 + \cdots + v_{nj}x_n)^d,$$

where $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of $\mathbb{R}^n$.

The tensor power method works well for odeco tensors:

Theorem

If $\lambda_j > 0$ then the $\mathbf{v}_i$ are precisely the robust eigenvectors of T.

[Anandkumar, Ge, Hsu, Kakade, Telgarsky: *Tensor decompositions for learning latent variable models*, J. Machine Learning Research, 2014]

[Kolda: *Symmetric orthogonal tensor decomposition is trivial*, 2015]

The set of odeco tensors is a *very nice* **variety** of dimension $\binom{n+1}{2}$.

[Robeva: *Orthogonal decomposition of symmetric tensors*, 2015]

Associativity

Fact: *Every $n \times n \times n$-tensor T defines an algebra structure on $\mathbb{R}^n$.*

Example: Fix $\mathbb{R}^2$ with basis $\{a, b\}$. A $2 \times 2 \times 2$-tensor $T = (t_{ijk})$ defines

$$a \star a = t_{000} a + t_{001} b \qquad a \star b = t_{010} a + t_{011} b$$
$$b \star a = t_{100} a + t_{101} b \qquad b \star b = t_{110} a + t_{111} b$$

This algebra is generally not associative:

$$\begin{aligned} b \star (a \star a) &= (t_{000} t_{100} + t_{001} t_{110}) a + (t_{000} t_{101} + t_{001} t_{111}) b \\ (b \star a) \star a &= (t_{000} t_{100} + t_{101} t_{100}) a + (t_{100} t_{001} + t_{101}^2) b \end{aligned}$$

Suppose that T is a symmetric tensor, corresponding to a binary cubic

$$\begin{aligned} & t_{000} x^3 + (t_{001} + t_{010} + t_{100}) x^2 y + (t_{011} + t_{101} + t_{110}) x y^2 + t_{111} y^3 \\ = \; & t_{000} x^3 + 3 t_{001} x^2 y + 3 t_{011} x y^2 + t_{111} y^3 \end{aligned}$$

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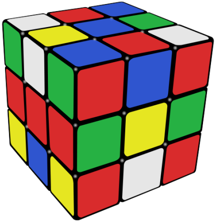
$b \star (a \star a) = (b \star a) \star a$ iff $t_{000}t_{011} + t_{001}t_{111} = t_{001}^2 + t_{011}^2$ iff T odeco

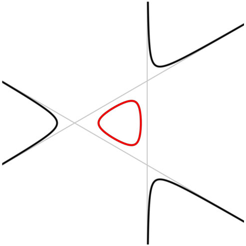
Theorem (Boralevi-Draisma-Horobeţ-Robeva 2015)

The odeco equations say that T defines an **associative** *algebra.*

Our question

How many eigenvectors does a symmetric $3 \times 3 \times 3$-tensor have ?





How many critical points does a cubic have on the unit 2-sphere?

Our question

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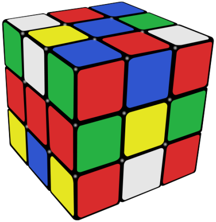
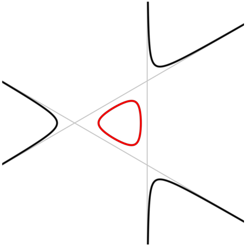




How many critical points does a cubic have on the unit 2-sphere?

Fermat: Odeco tensor : $T = x^3 + y^3 + z^3$

$$\nabla T : \mathbb{P}^2 \dashrightarrow \mathbb{P}^2, \ (x : y : z) \mapsto (x^2 : y^2 : z^2)$$

$$(1:0:0), (0:1:0), (0:0:1),$$
$$(1:1:0), (1:0:1), (0:1:1), (1:1:1)$$

Answer: **Seven.**

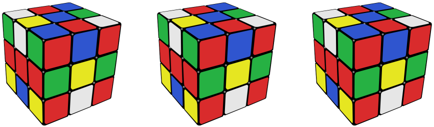
Let's count

Theorem

Consider a general symmetric tensor T of format $n \times n \times \cdots \times n$. The number of complex eigenvectors in $\mathbb{P}^{n-1}$ equals

$$\frac{(d-1)^n - 1}{d-2} \;=\; \sum_{i=0}^{n-1} (d-1)^i .$$

[Cartwright, St: The number of eigenvalues of a tensor, 2013]
[Fornaess, Sibony: Complex dynamics in higher dimensions, 1994]



Q: How many eigenvectors does a $3 \times 3 \times 3 \times 3$-tensor have?
A: Plug $n = 3$ and $d = 4$ into the formula. The answer is **13**.

Discriminant

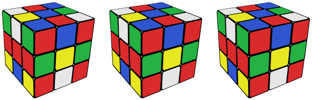
The *eigendiscriminant* is the irreducible polynomial in the entries $t_{i_1 i_2 \cdots i_d}$ which vanishes when two eigenvectors come together.

Theorem

The degree of eigendiscriminant is $n(n-1)(d-1)^{n-1}$.

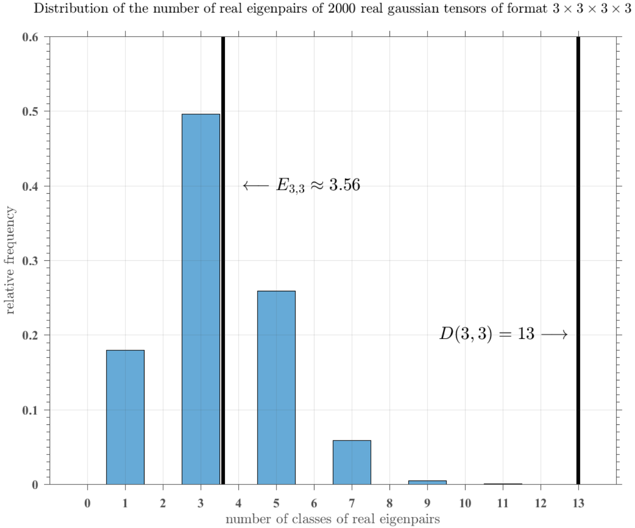
[Abo, Seigal, St: Eigenconfigurations of tensors, 2015]

Example 1 $(d = 2)$ The discriminant of the characteristic polynomial of an $n \times n$-matrix is an equation of degree $n(n-1)$.

Example 2 $(n = 3, d = 4)$ The eigendiscriminant for $3{\times}3{\times}3{\times}3$ tensors  is an equation of degree **54**.

Note: The eigendiscriminant divides tensor space into regions where the number of **real** solutions is constant. Average number?

Get Real



[Breiding: The expected number of eigenvalues of a real Gaussian tensor, 2016] gives an exact formula in terms of hypergeometric integrals.

Line Arrangements

Open Problem: Can all eigenvectors be real?

Yes, if $n = 3$: *All* $1 + (d-1) + (d-1)^2$ *fixed points can be real.*

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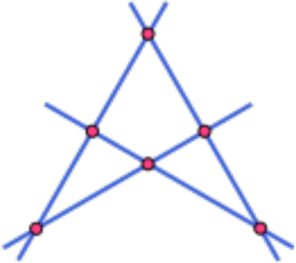

$\mathbf{6} + \mathbf{7} = \mathbf{13}$

Proof: Let T be a product of d linear forms.

The $\binom{d}{2}$ vertices of the line arrangement are the base points.
The analytic centers of the $\binom{d}{2} + 1$ regions are the fixed points.

Singular vectors

Given a rectangular matrix T, one seeks to solve the equations

$$T\mathbf{u} = \sigma\mathbf{v} \quad \text{and} \quad T^t\mathbf{v} = \sigma\mathbf{u}.$$

The scalar σ is a singular value and $(\mathbf{u}, \mathbf{v})$ is a singular vector pair.

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Gradient Dynamics: Matrices correspond to bilinear forms

$$T \;=\; \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} t_{ij}x_i y_j$$

This defines a rational map

$$\begin{array}{rccc} (\nabla_{\mathbf{x}} T, \nabla_{\mathbf{y}} T): & \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} & \dashrightarrow & \mathbb{P}^{n_1-1} \times \mathbb{P}^{n_2-1} \\ & (\mathbf{u}\,,\, \mathbf{v}) & \mapsto & (\,T^t\mathbf{v}\,,\; T\mathbf{u}\,) \end{array}$$

The fixed points of this map are the singular vector pairs of T.

Multilinear forms

Tensors T in $\mathbb{R}^{n_1 \times n_2 \times \cdots \times n_d}$ correspond to multilinear forms. The singular vector tuples of T are fixed points of the gradient map

$$\nabla T \; : \; \mathbb{P}^{n_1 - 1} \times \cdots \times \mathbb{P}^{n_d - 1} \dashrightarrow \mathbb{P}^{n_1 - 1} \times \cdots \times \mathbb{P}^{n_d - 1}.$$

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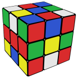
Theorem

For a general $n_1 \times n_2 \times \cdots \times n_d$-tensor T, the number of singular vector tuples is the coefficient of $z_1^{n_1-1} \cdots z_d^{n_d-1}$ in the polynomial

$$\prod_{i=1}^{d} \frac{(\widehat{z_i})^{n_i} - z_i^{n_i}}{\widehat{z_i} - z_i} \quad \textit{where} \quad \widehat{z_i} = z_1 + \cdots + z_{i-1} + z_{i+1} + \cdots + z_d.$$

[Friedland, Ottaviani: The number of singular vector tuples...., 2014]



Example: $d = 3, n_1 = n_2 = n_3 = 3$:

$$(\widehat{z_1}^2 + \widehat{z_1} z_1 + z_1^2)(\widehat{z_2}^2 + \widehat{z_2} z_2 + z_2^2)(\widehat{z_3}^2 + \widehat{z_3} z_3 + z_3^2) = \cdots + \mathbf{37} z_1^2 z_2^2 z_3^2 + \cdots$$

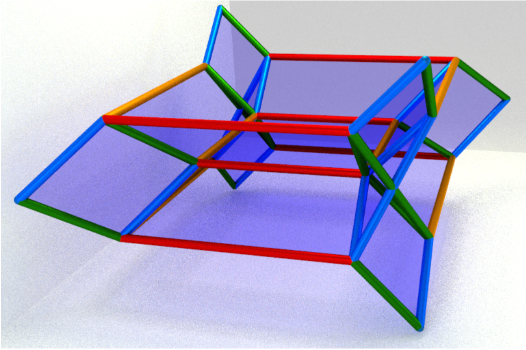
Odeco Tensors

A general tensor of format $3{\times}3{\times}2{\times}2$ has **98** singular vector tuples. What happens for orthogonally decomposable tensors

$$T \quad = \quad x_0 y_0 z_0 w_0 + x_1 y_1 z_1 w_1 \quad ?$$

[Robeva, Seigal: Singular vectors of odeco tensors, 2016]

The gradient map $\nabla T : \mathbb{P}^2{\times}\mathbb{P}^2{\times}\mathbb{P}^1{\times}\mathbb{P}^1 \dashrightarrow \mathbb{P}^2{\times}\mathbb{P}^2{\times}\mathbb{P}^1{\times}\mathbb{P}^1$ has only **18** fixed points. In addition, there is a *surface of base points*:



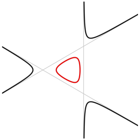
Conclusion

Eigenvectors of square matrices are central to linear algebra.

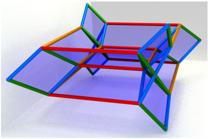
Eigenvectors of tensors are a natural generalization. Pioneered in numerical multilinear algebra, these now have many applications.

[Lek-Heng Lim: Singular values and eigenvalues of tensors...., 2005]
[Liqun Qi: Eigenvalues of a real supersymmetric tensor, 2005]

Fact: *This lecture serves as an invitation to applied algebraic geometry.*



The word **variety** is <u>not</u> scary.



The terms **Segre variety** and **Veronese variety** refer to tensors of rank 1. Given some data, getting close to these is highly desirable.

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