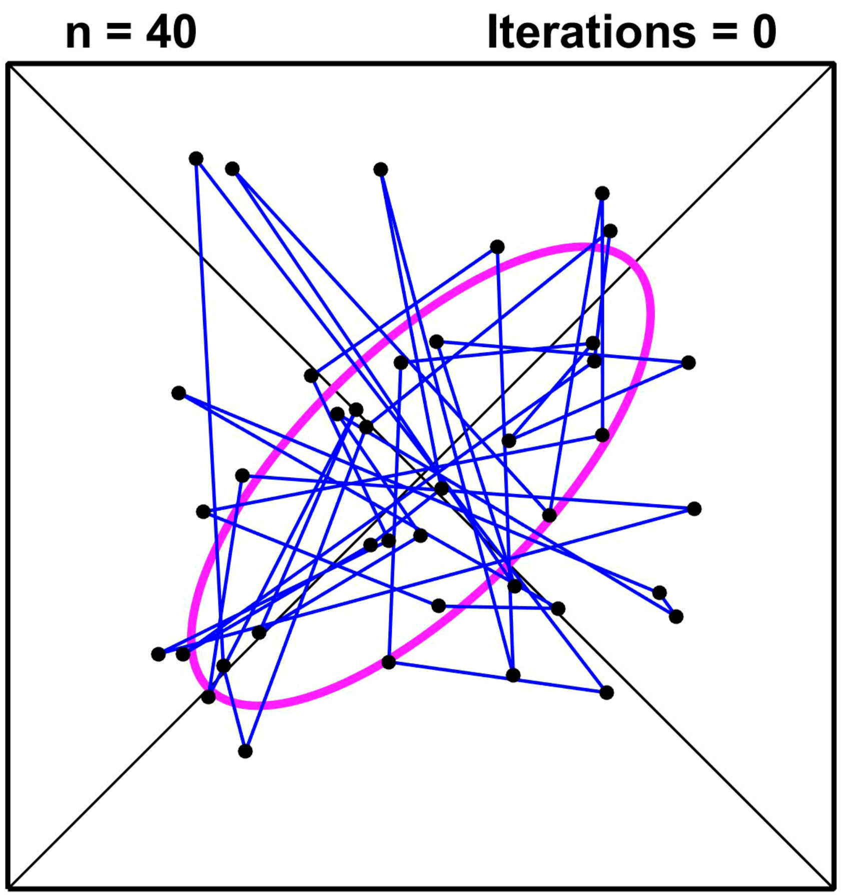
Untangling Random Polygons



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Outline

1. I will describe an elementary problem that I gave in an Introduction to Computing course that uses Matlab.

2. We will "play" with the problem and observe some interesting phenomena.

3. We will use various matrix decompositions and algorithms to explain those phenomena..

The Problem

Display a sequence of polygons where each polygon is obtained from its predecessor by connecting the midpoints of its sides.

Let the original polygon be random.

This is an Introduction-to-Vectors Problem

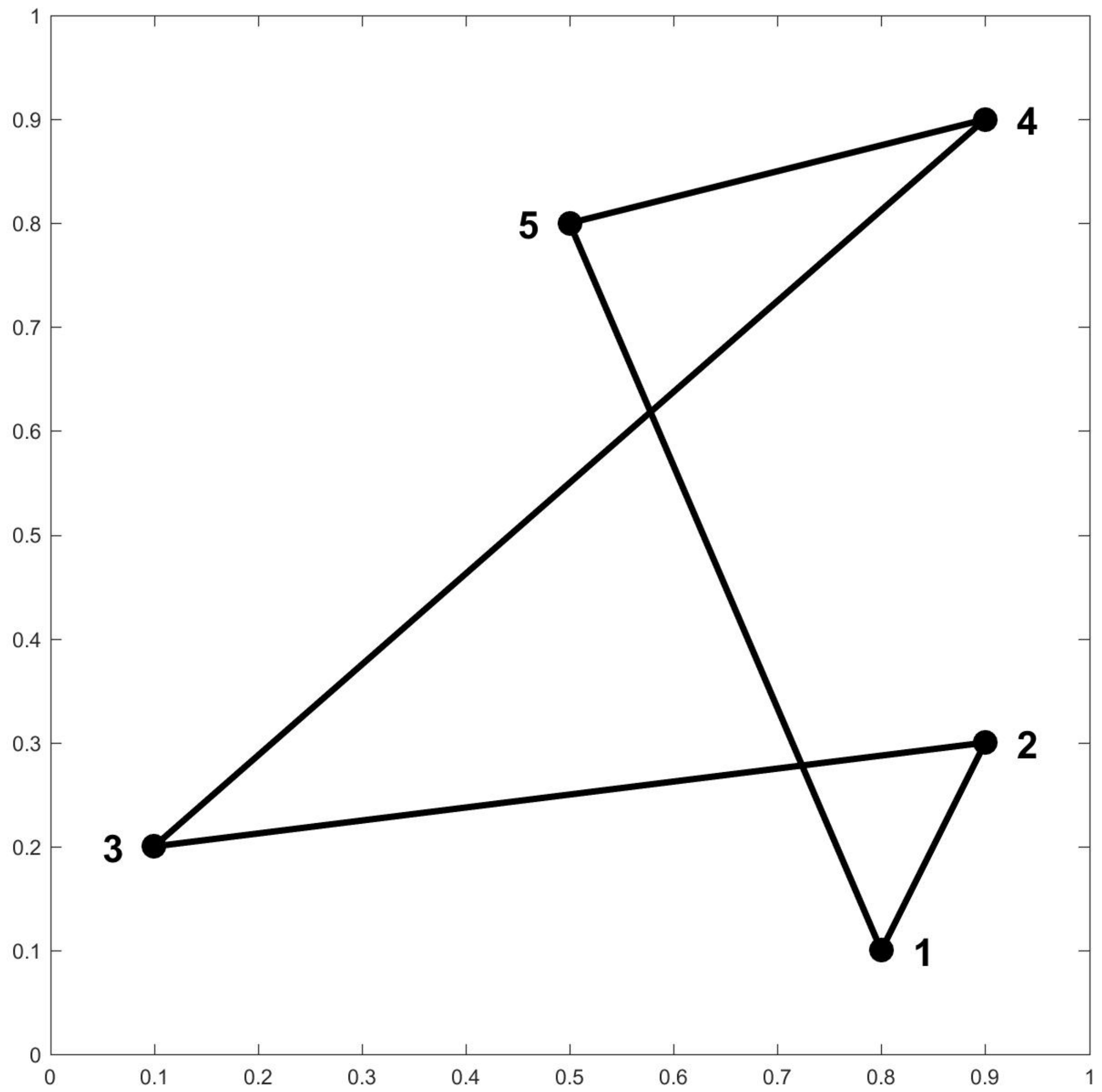
Current Polygon

x:	10	18	12	12	22
y:	16	20	14	16	10

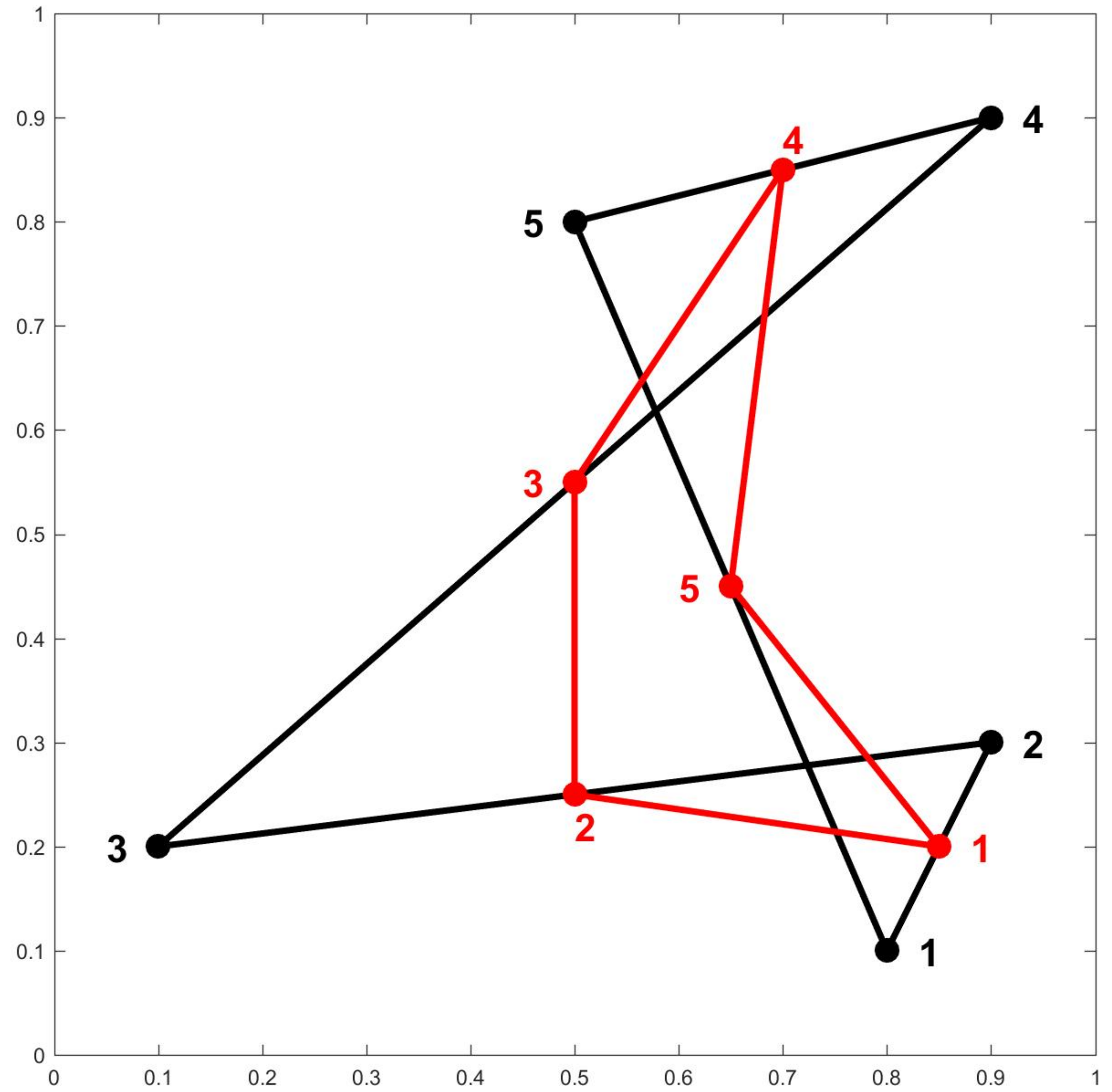
Next Polygon

x:	14	15	12	17	16
y:	18	17	15	13	13

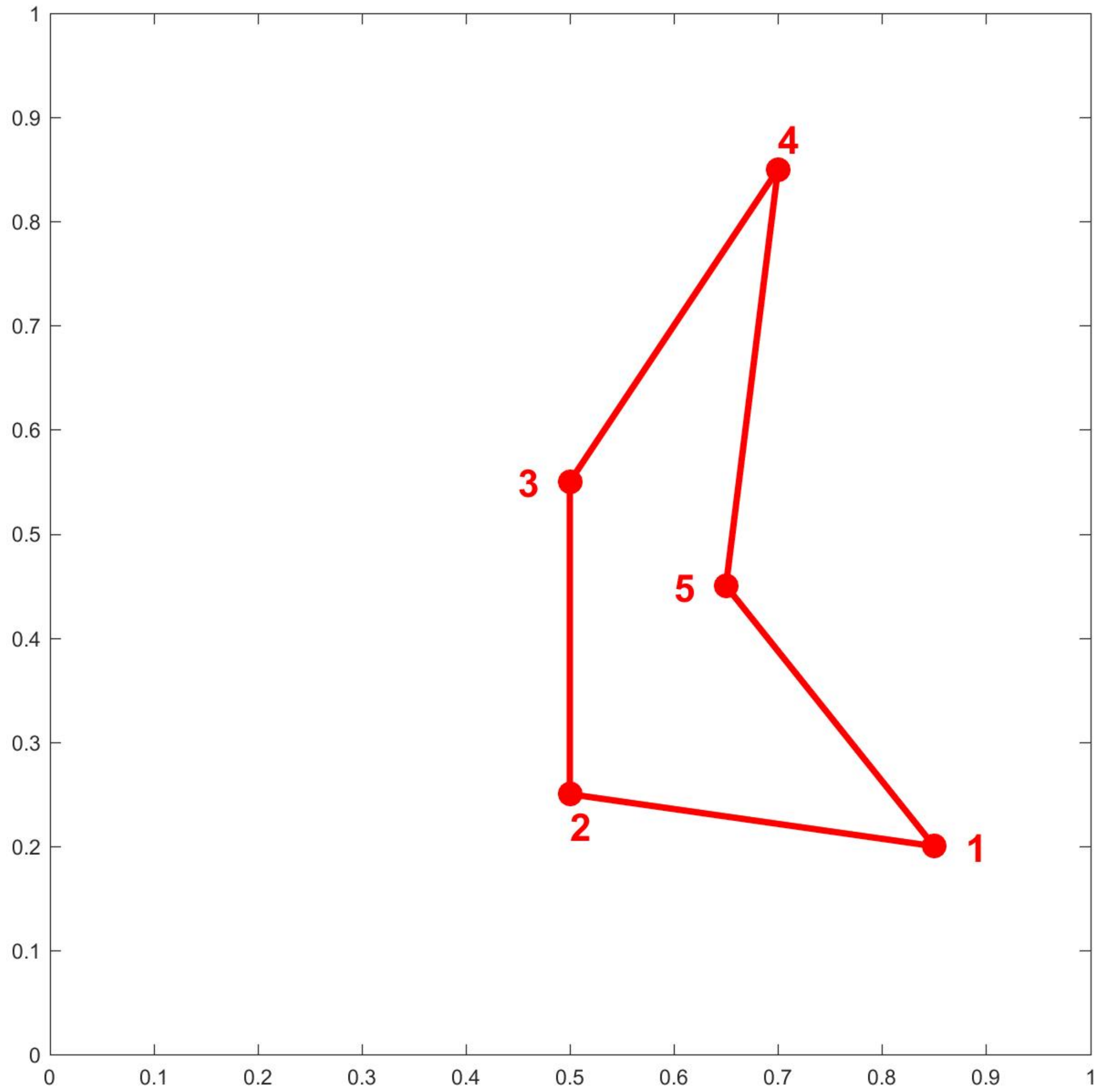
A Random Pentagon



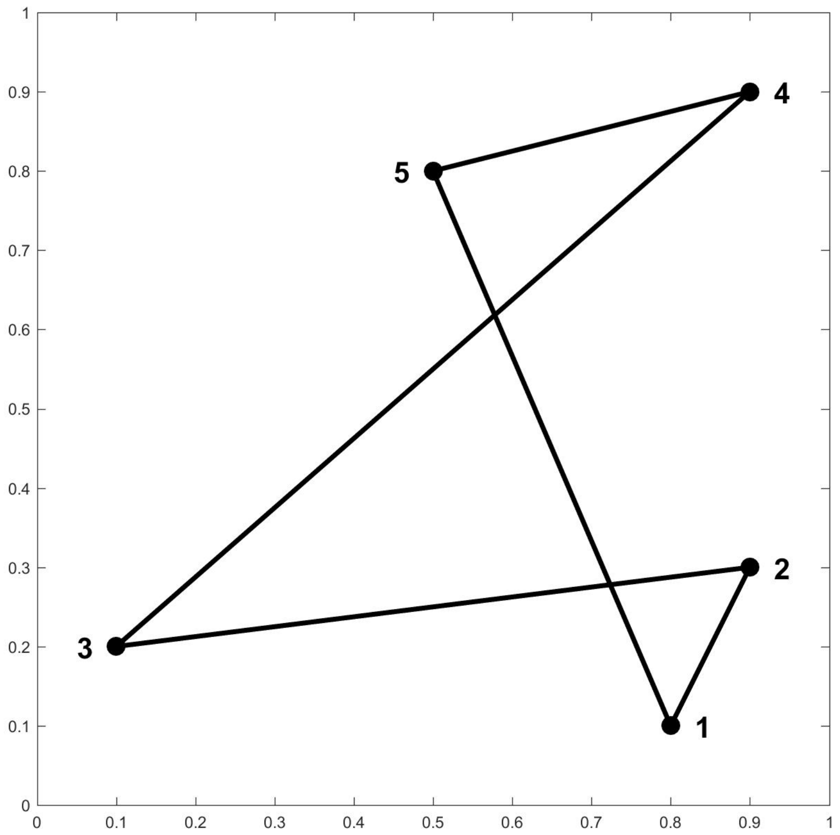
Connect the Side Midpoints

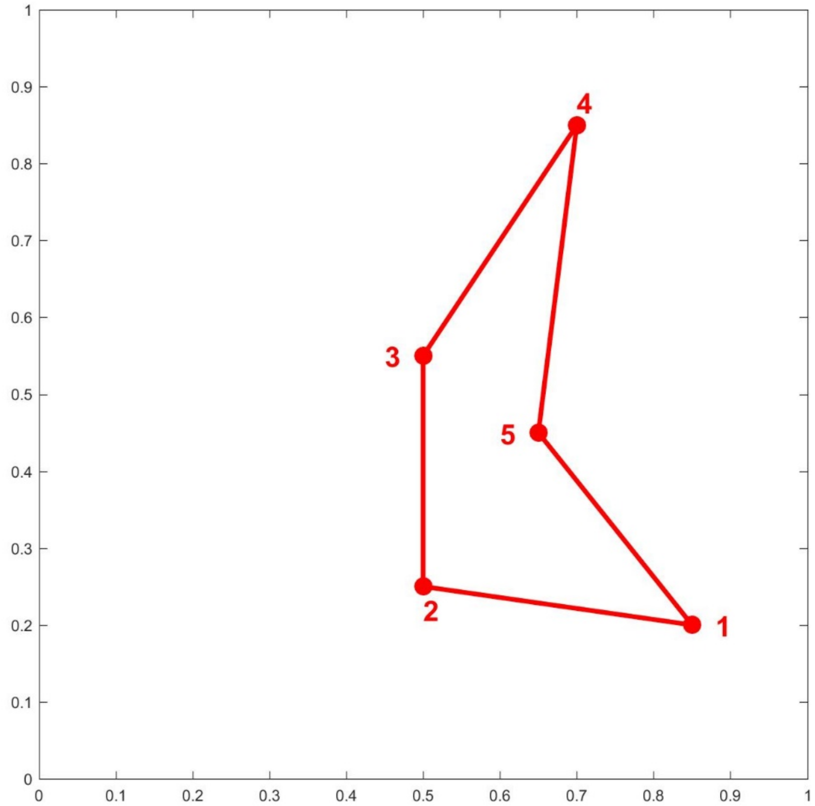


Obtain a New Pentagon



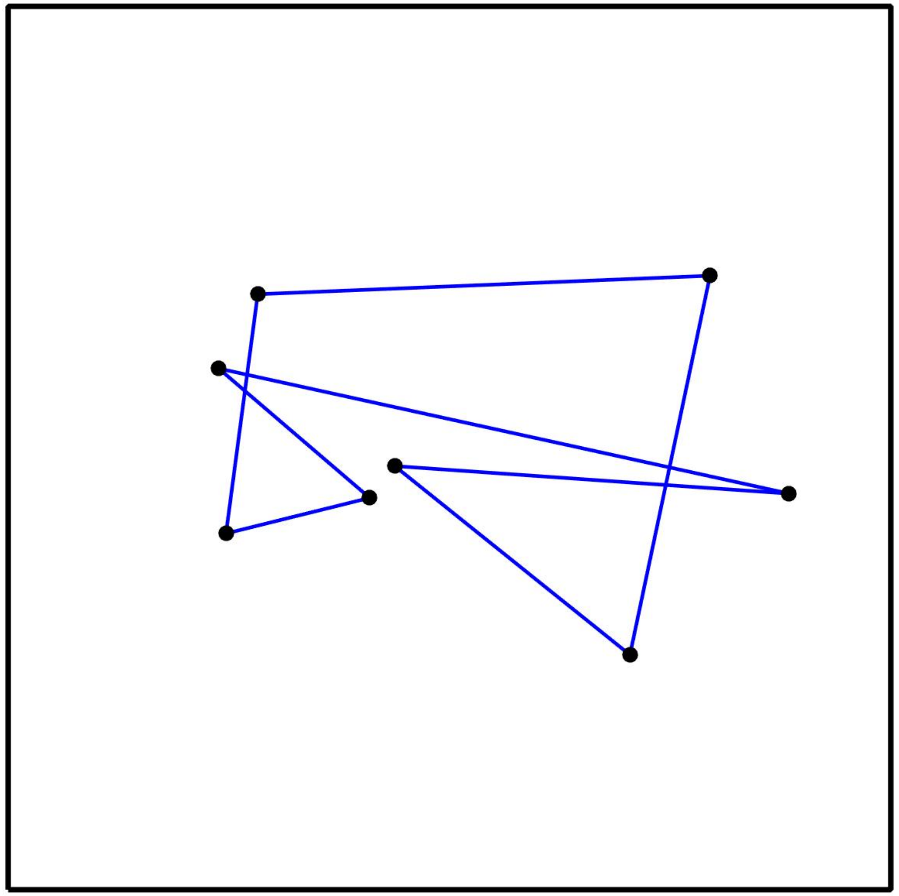
"Polygon Averaging"



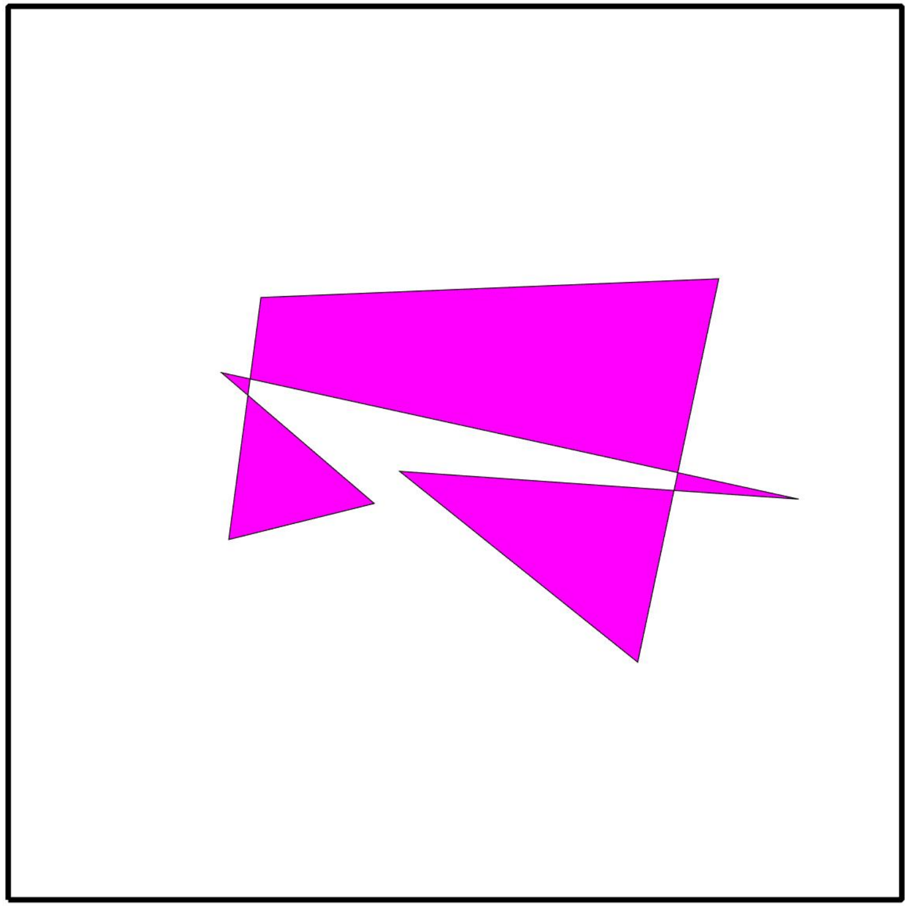


The process can obviously be repeated.

Graphics Note

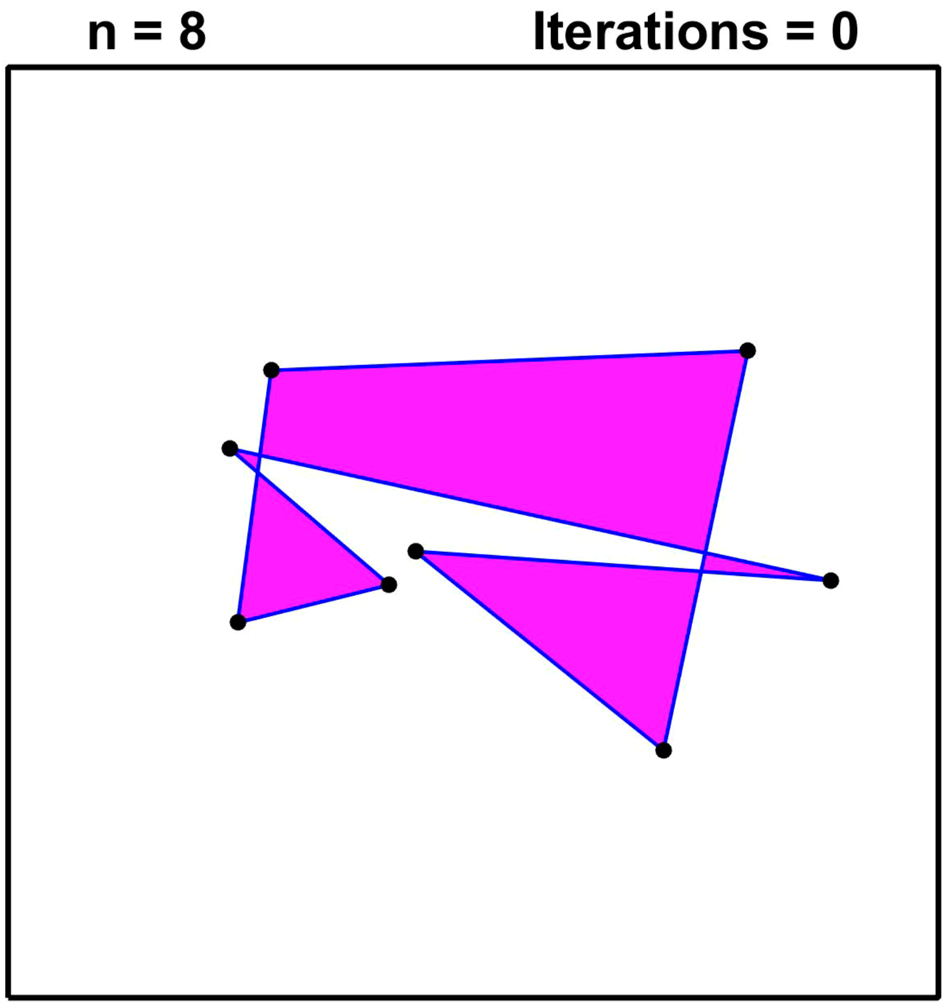


`plot([x;x(1)],[y;y(1)])`

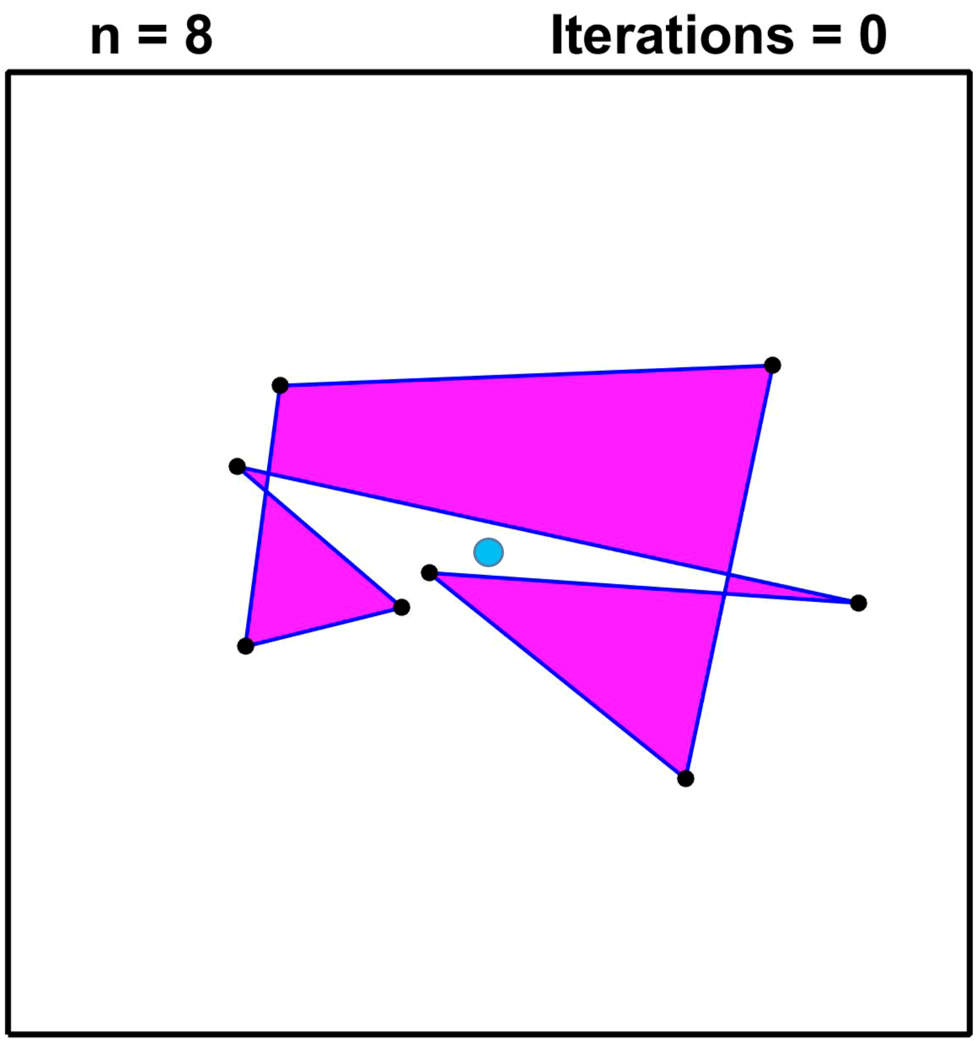


`fill(x,y,'m')`

Repeated Polygon Averaging on a Random Octagon



Repeated Polygon Averaging on a Random Octagon

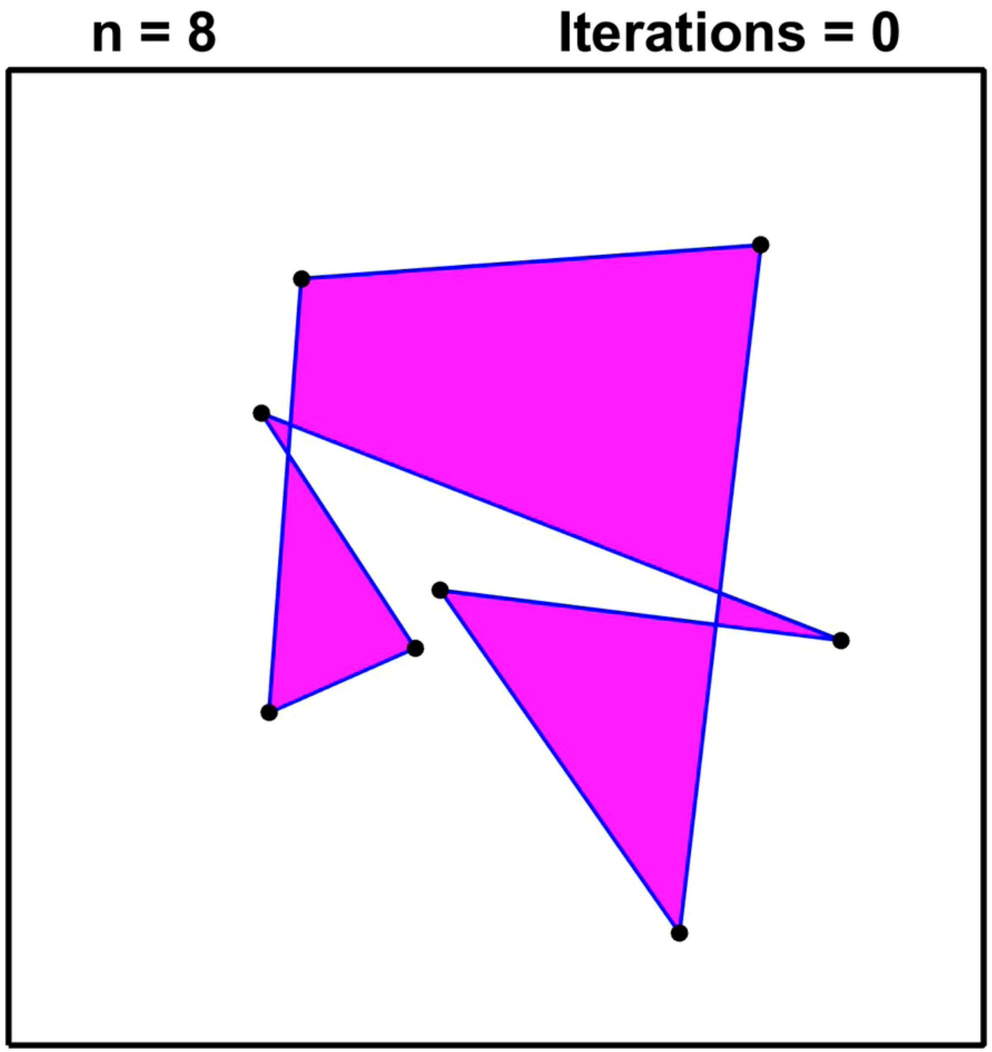


All vertices head towards the centroid.

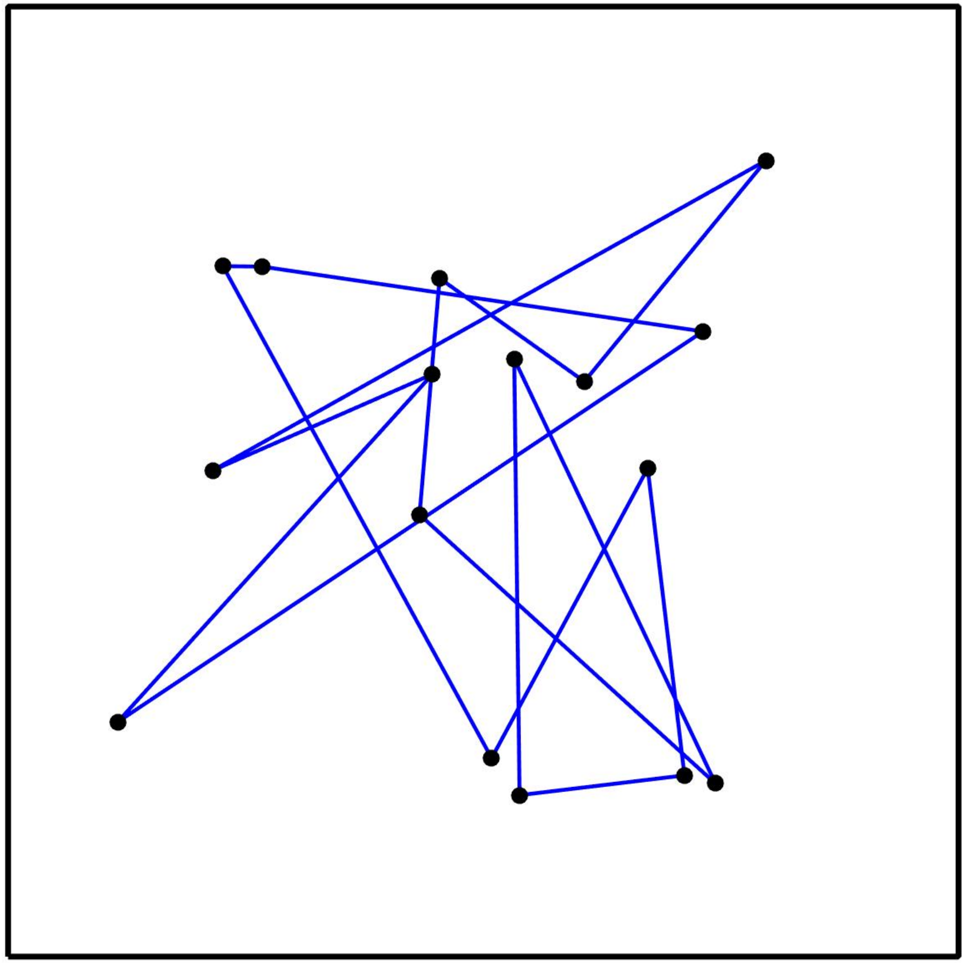
Normalized Repeated Polygon Averaging

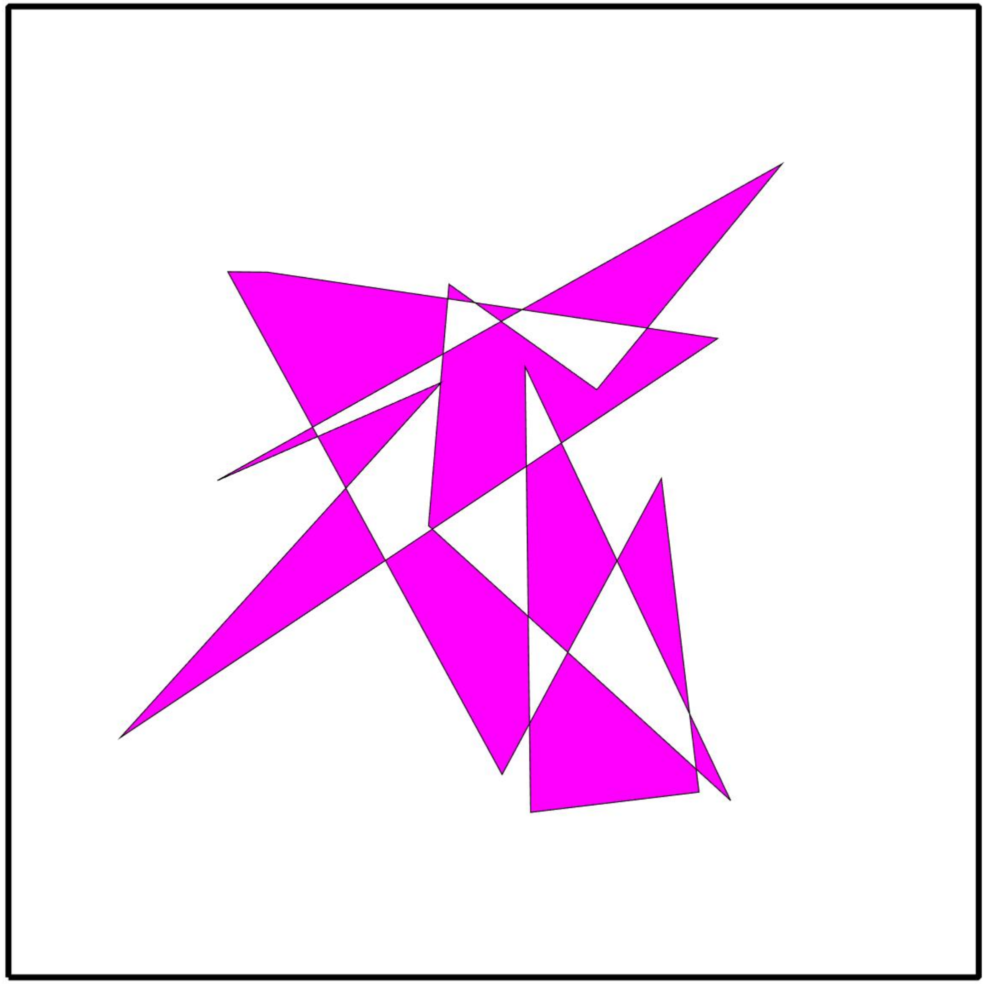
Maintain unit 2-norm vertex vectors:

```
x = x/norm(x)
y = y/norm(y)
```

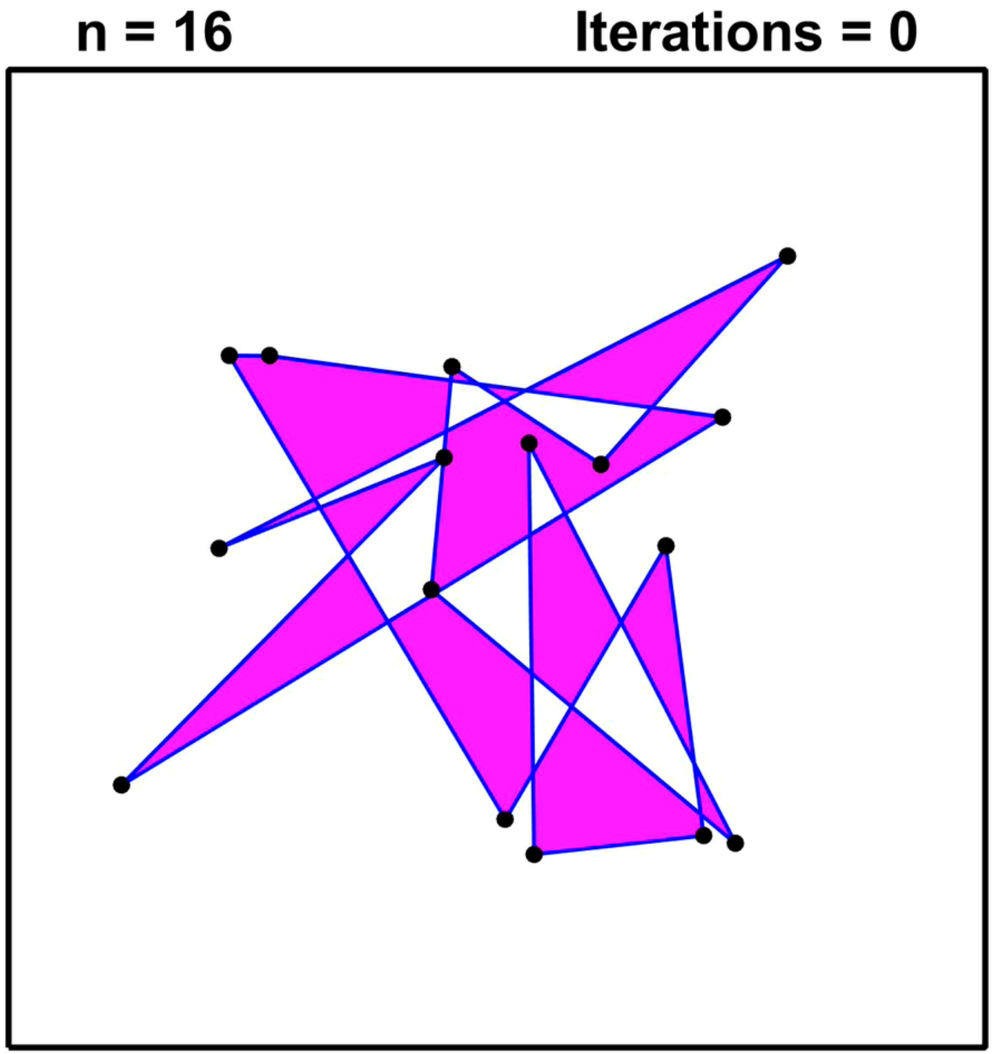


Example (n = 16)

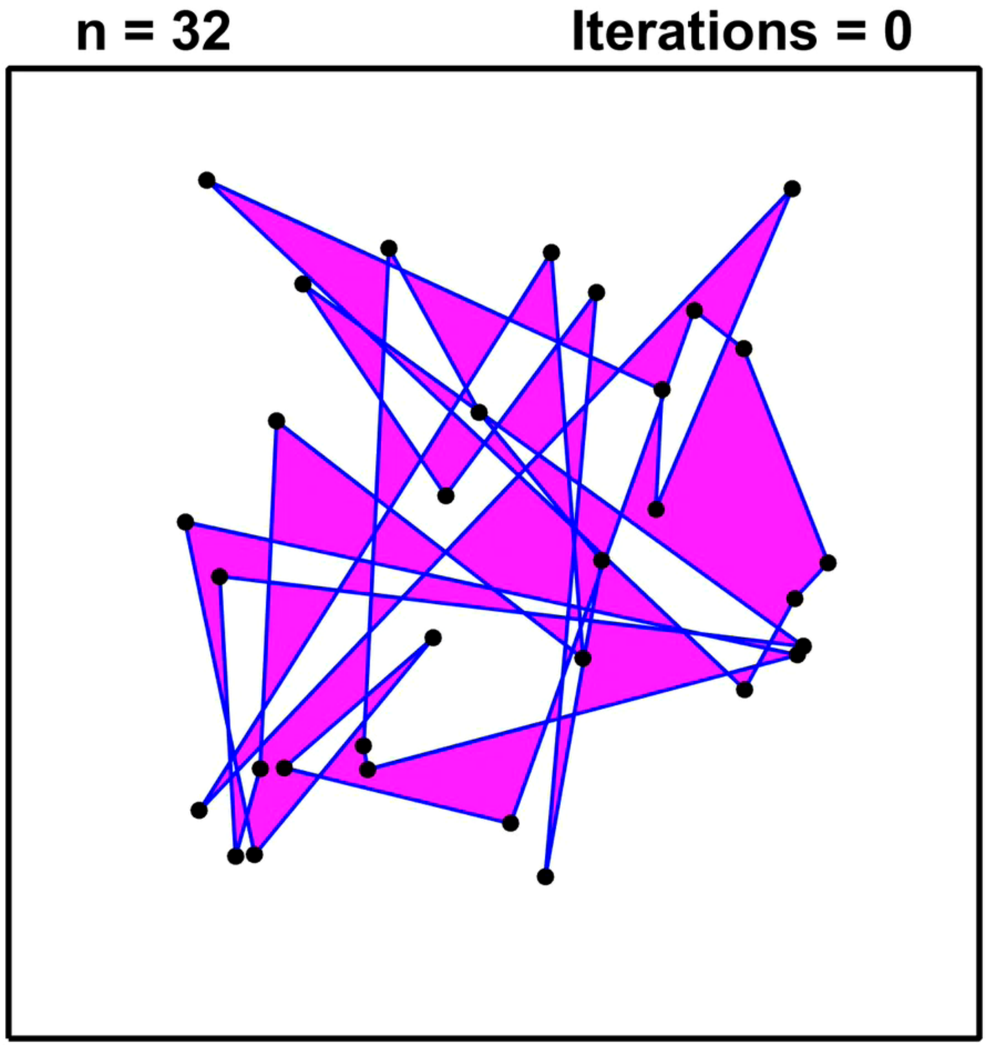




Example (n = 16)



Example (n = 32)



The points seem to converge to an ellipse with a 45-degree tilt.

Interesting Questions

1. What is the limiting ellipse and why the 45-degree tilt?

2. Why do the vertices appear to “move” around the ellipse?

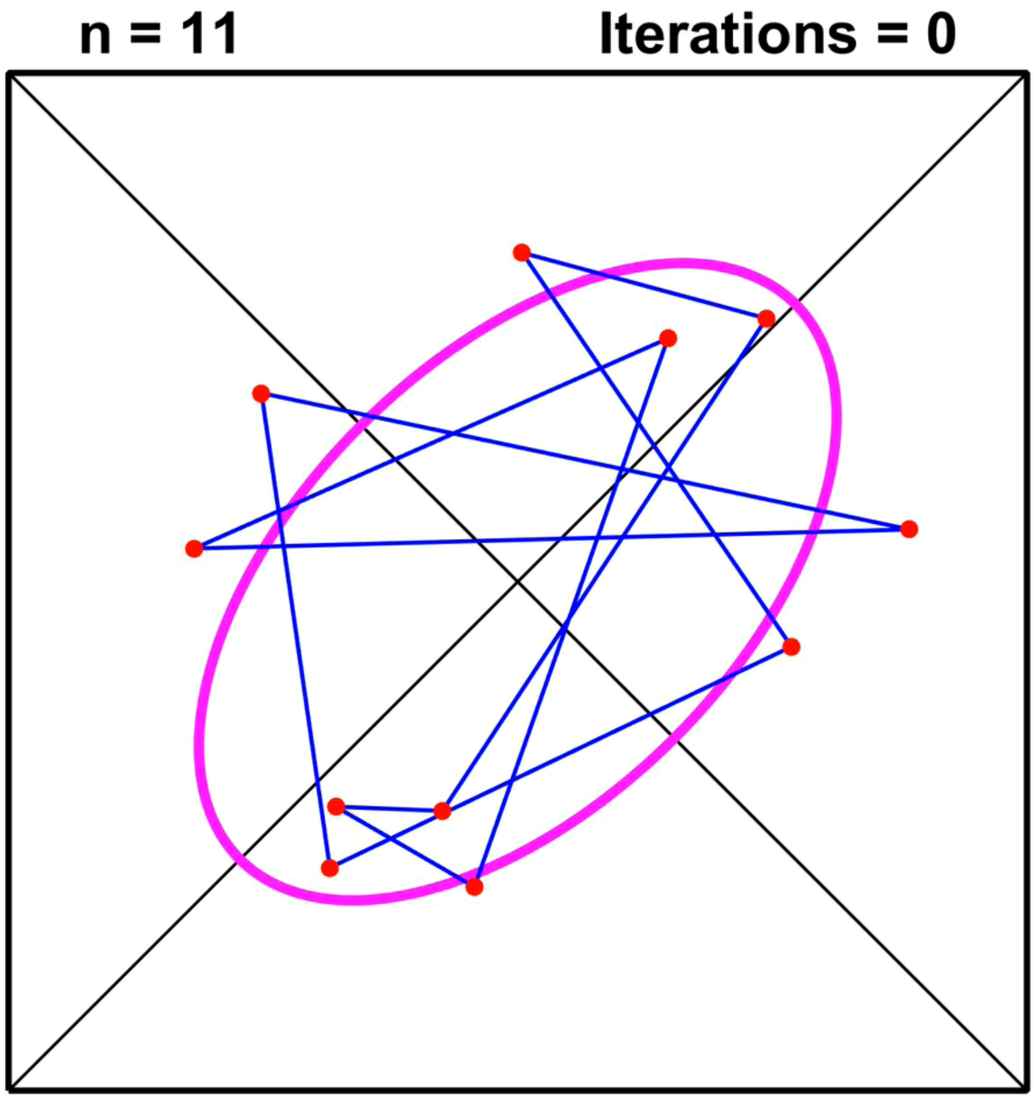
3. How long does it take to converge?

4. Does it always converge?

5. What is the inverse of the repeated polygon averaging process?

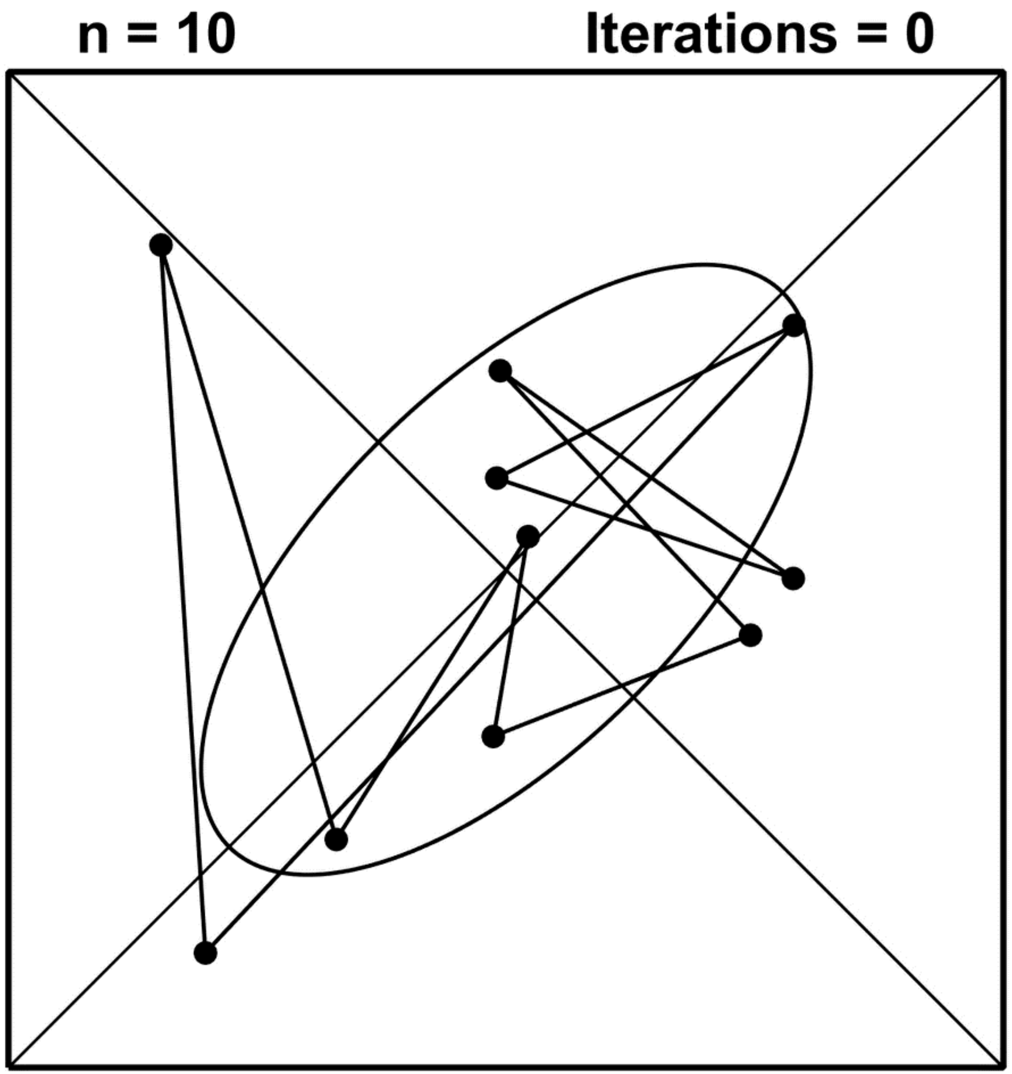
What is the limiting ellipse and why the 45-degree tilt?

The Ellipse Can Be Computed in Advance

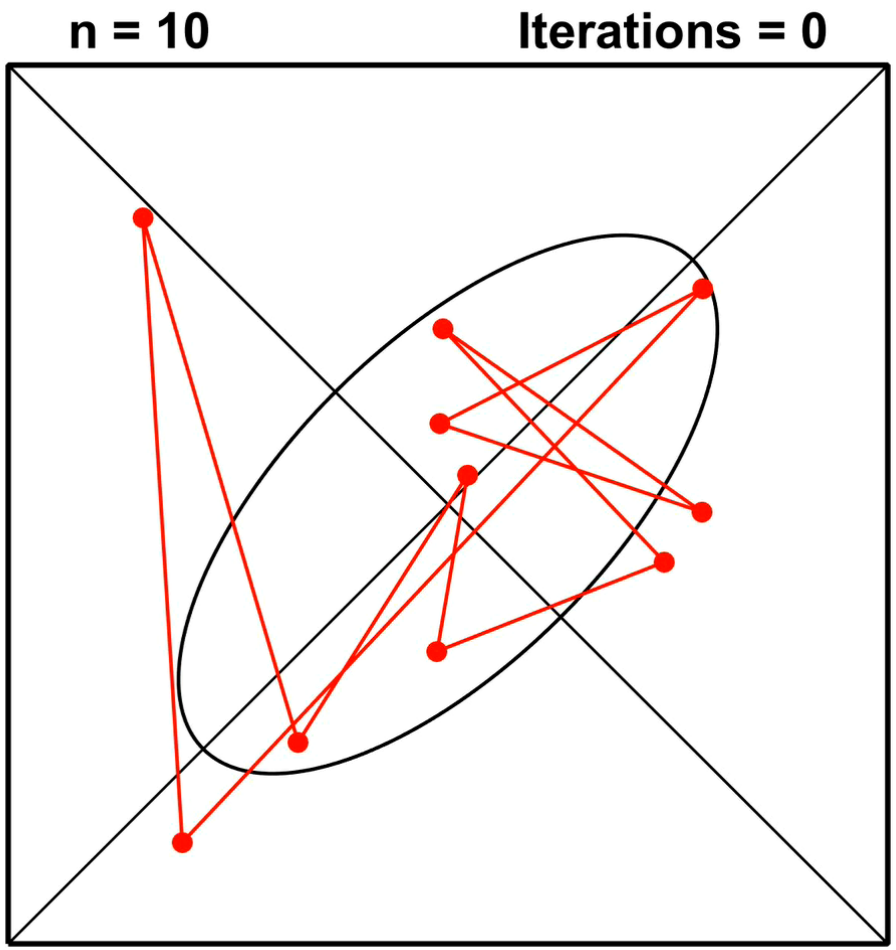


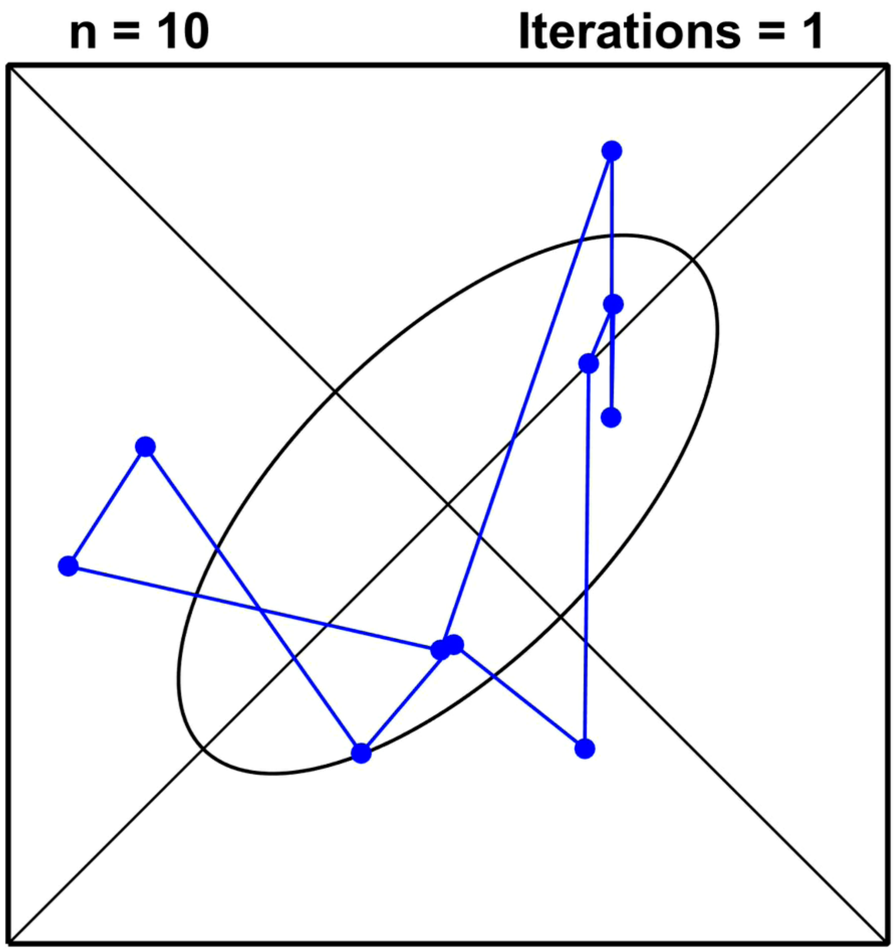
Why do the vertices appear to move around the ellipse?

The Vertices seem to Move Around the Ellipse



Look at Every Other Iteration

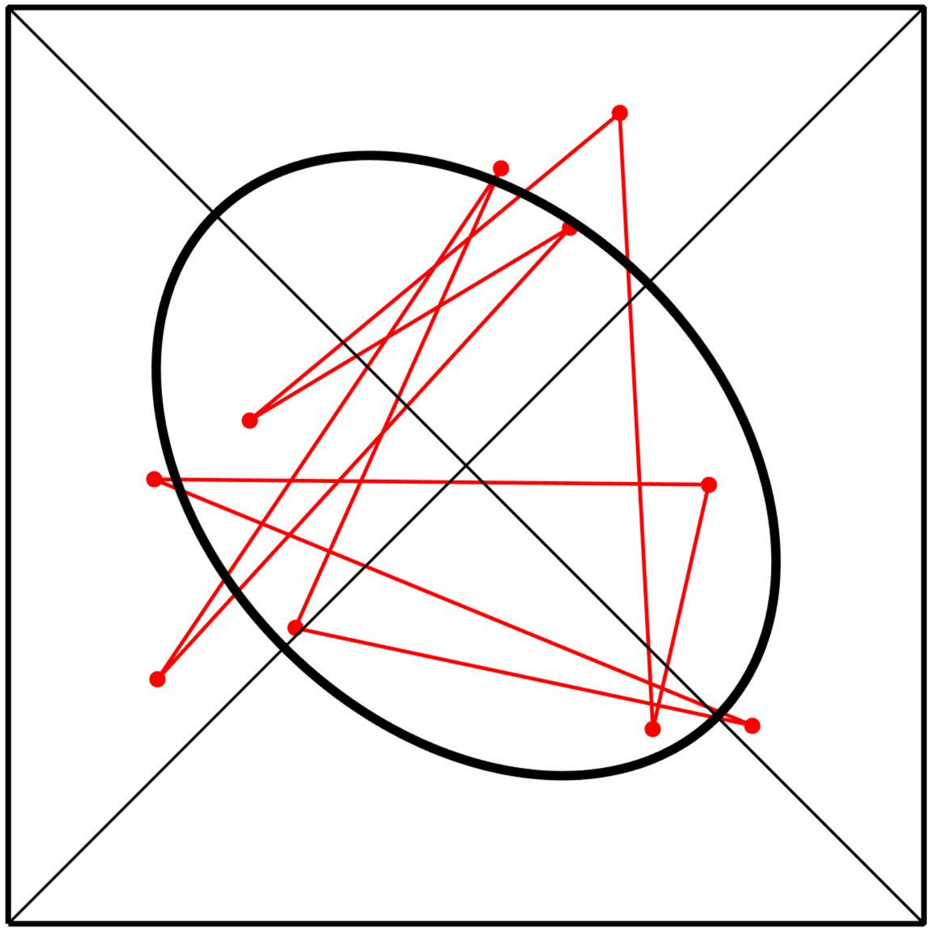




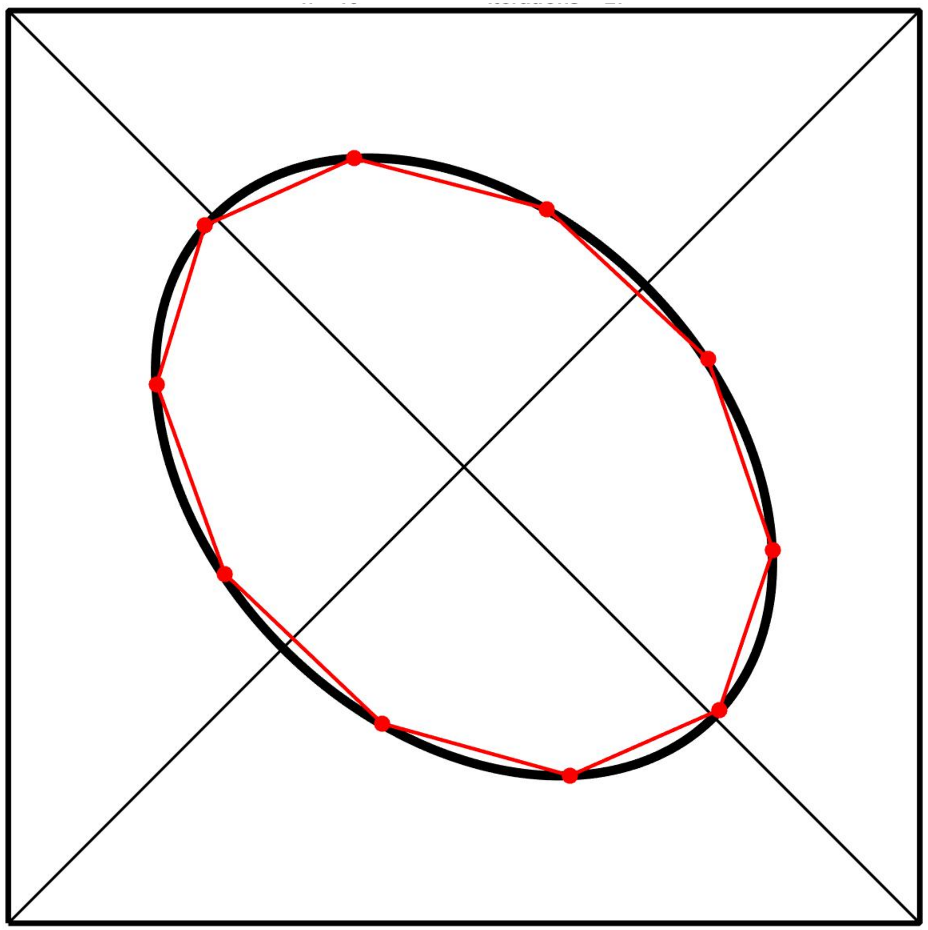
How long does it take for the vertices to converge to the limiting ellipse?

How Long Does It Take?

n = 10

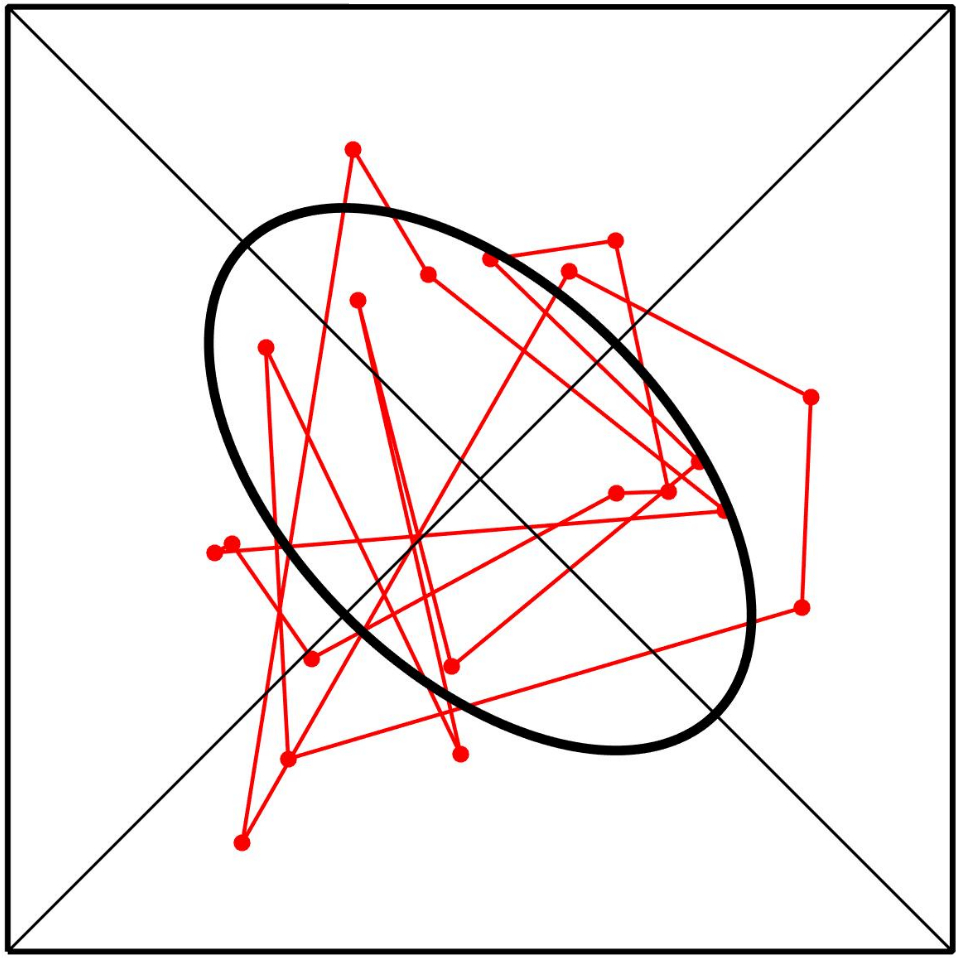


After 27 Iterations

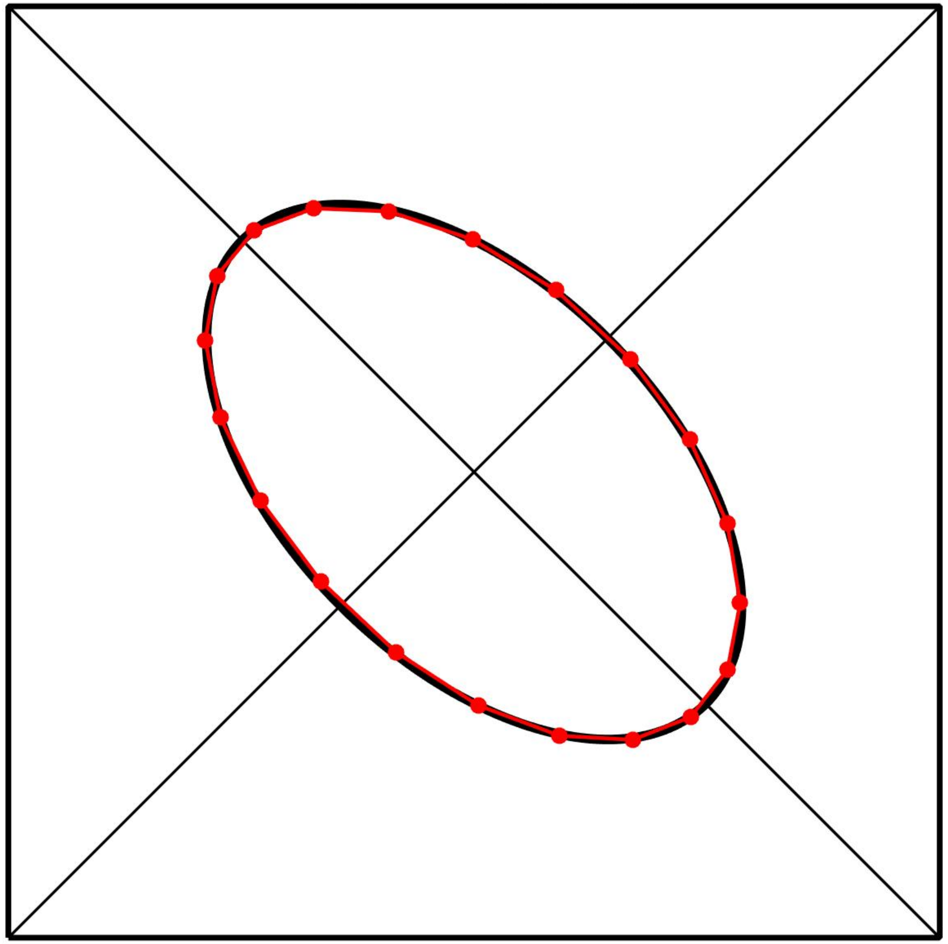


How Long Does It Take?

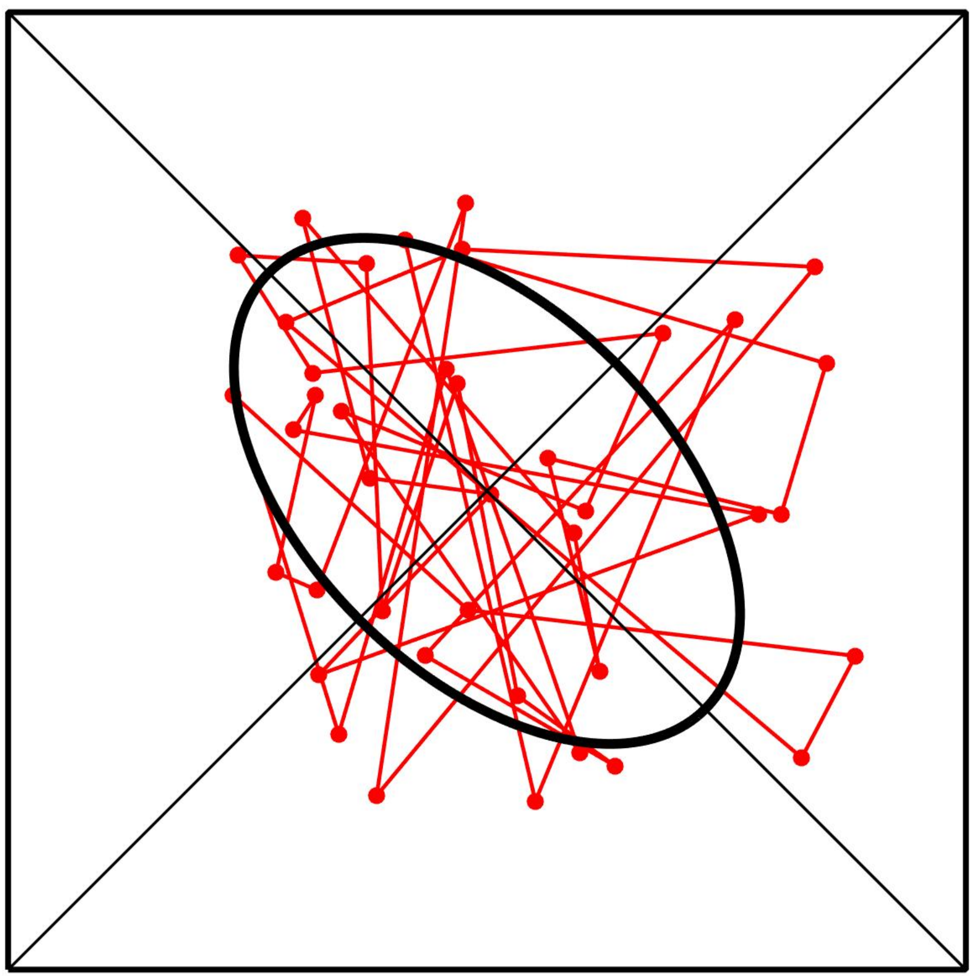
n = 20

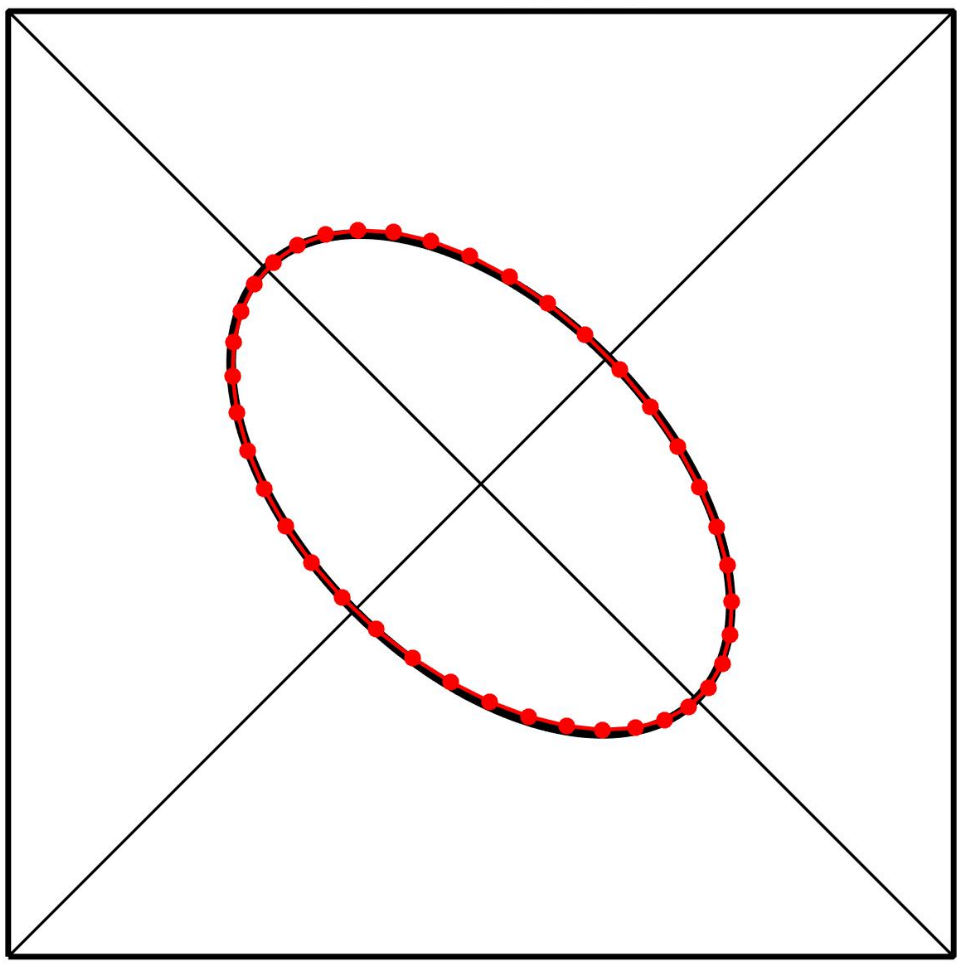


After 163 Iterations



How Long Does It Take?





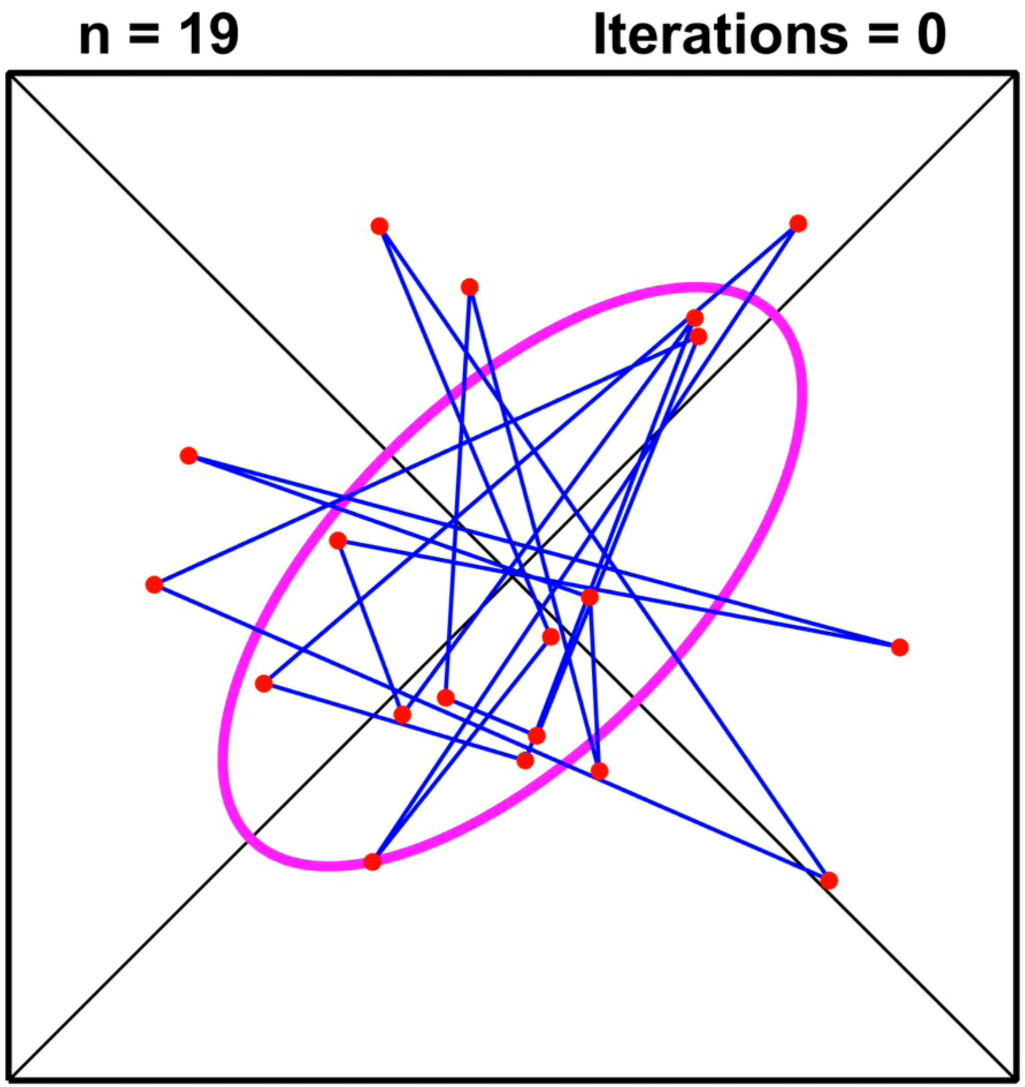
How Long Does It Take?

n = # Vertices	# Iterations Until "Converged"
10	27
20	163
40	688

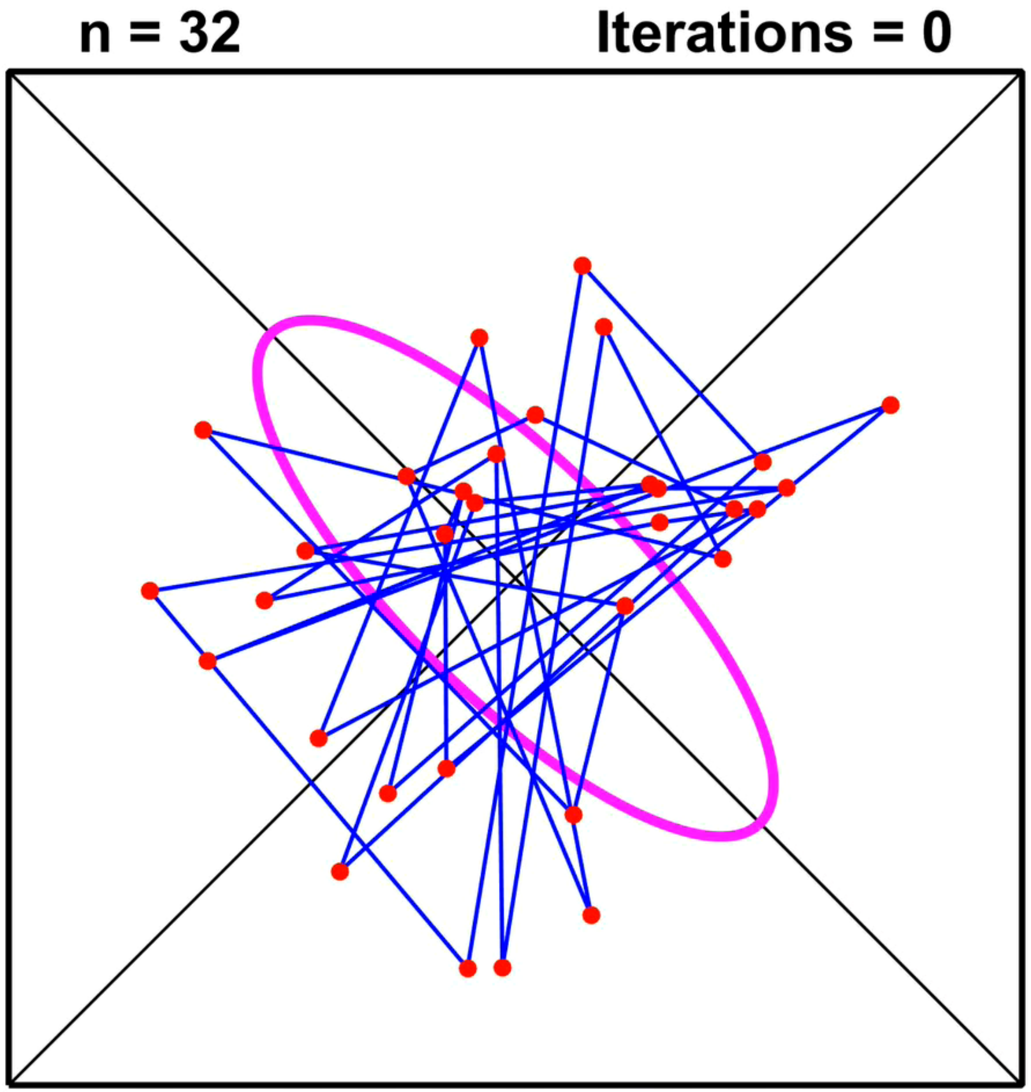
Looks like $O(n^2)$

Does the process always converge?

Do the Vertices Always Move to the Ellipse?



Do the Vertices Always Move to the Ellipse?



What is the inverse of the repeated averaging process?

What is the Inverse of the Polygon Averaging Process?

Notation:

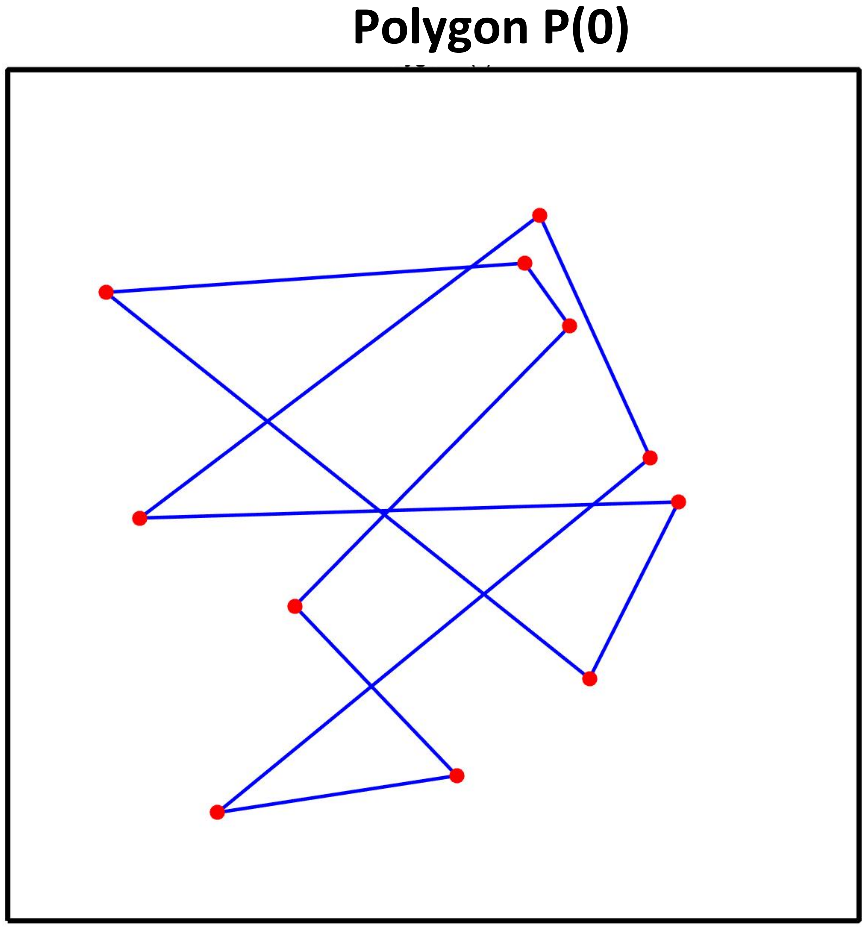
P(0) = the given polygon

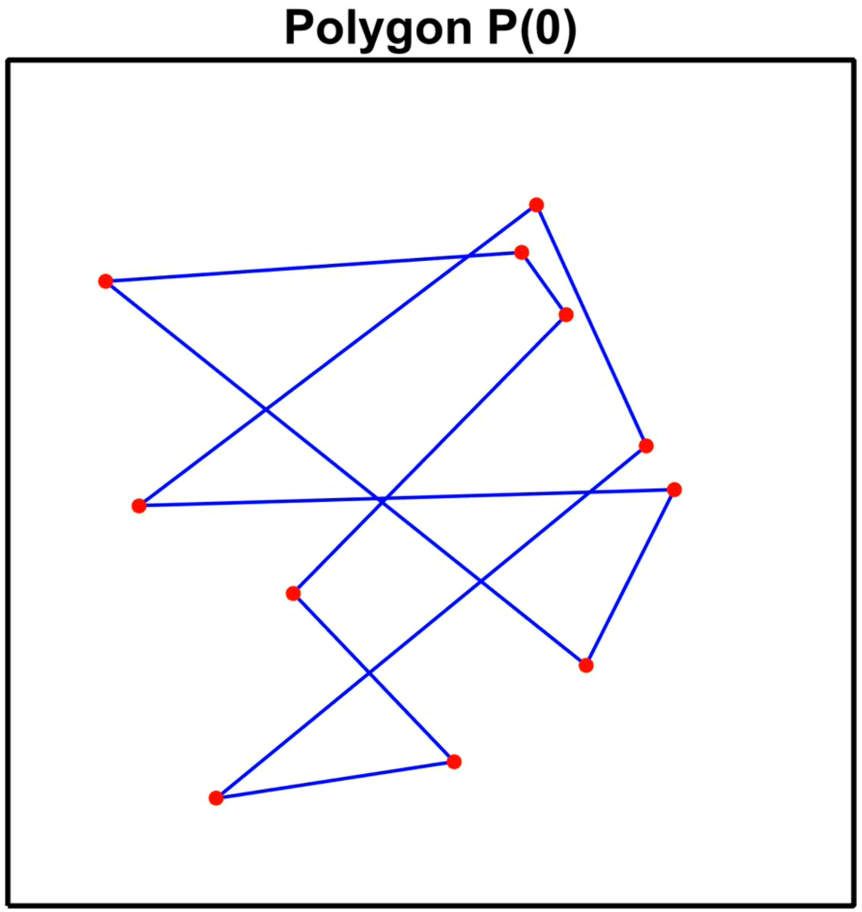
P(k) = the kth polygon obtained from P(k-1) via averaging.

… → P(-3) → P(-2) → P(-1) → **P(0)** → P(1) → P(2) → …

? ? ?

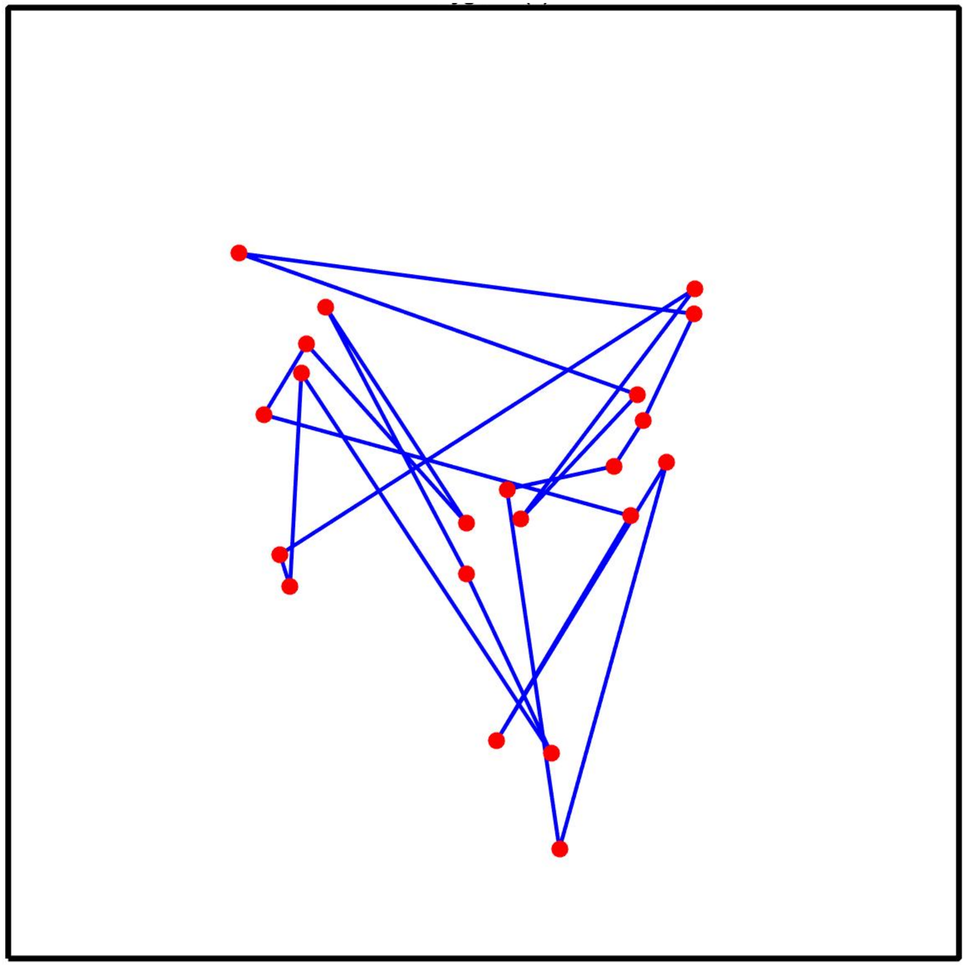
Run the Process Backwards (n=11)



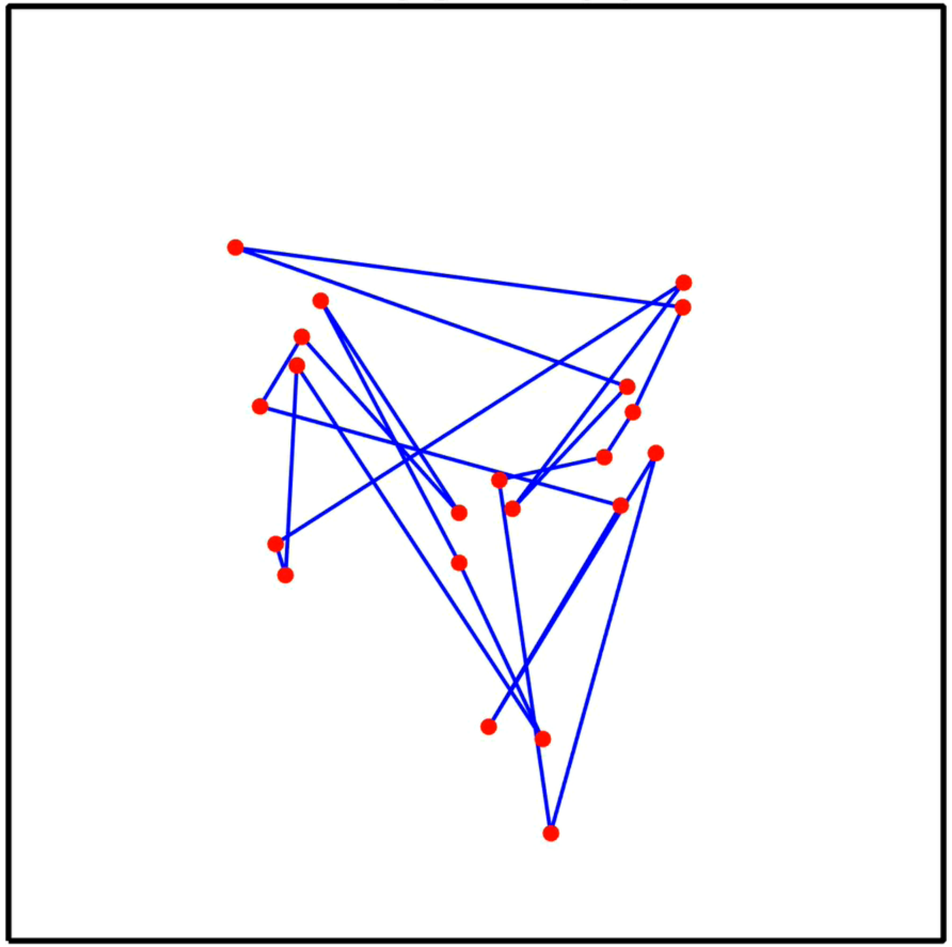


Run the Process Backwards (n=21)

Polygon P(0)



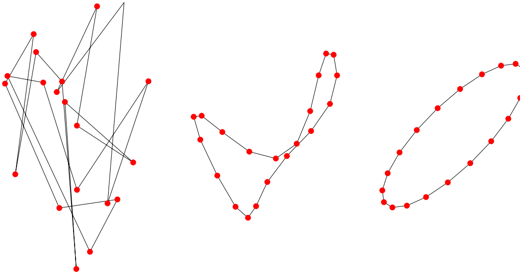
Polygon P(0)



Now let’s look at the math!

Untangling Random Polygons

Let's Do the Math!



Polygon Averaging

Generating Polygons $P_1, P_2, \ldots$

P_0 a random n-gon.

for $k = 1, 2, \ldots$

  Connect the edge midpoints of P_{k-1} to get P_k.

end

P_k is an average of P_{k-1} and $\text{Shift}(P_{k-1})$, e.g.,

$$\frac{\{(x_1, y_1), (x_2 . y_2), (x_3, y_3), (x_4, y_4)\} \; + \; \{(x_2, y_2), (x_3, y_3), (x_4, y_4), (x_1, y_1)\}}{2}$$

Reference and Acknowledgement

A.N. Elmachtoub, C.F. Van Loan (2010), *From random polygon to ellipse: an eigenanalysis,* SIAM Rev. 52, 151–170.

Adam Elmachtoub was a Cornell Operations Research Undergraduate (2005-2009) and an MIT Phd Student (2009-2015).

He is now on the faculty at Columbia.

A Progression

Intro Programming with Matlab (2008)

$\downarrow$

Intro Matrix Computations (2009)

$\downarrow$

SIAM Review (2010)

$\downarrow$

SIAM News (2018)

Related Work

G. Darboux (1878). *Sur un probleme de geometrie elementaire*, Bulletin des Sciences Mathematique et Astronomiques, Deuxieme Serie, 298-304.

I.J. Schoenberg, *The Finite Fourier series and elementary geometry*, Amer. Math. Monthly 57 (1950), 390–404.

E.R. Berlekamp, E.N. Gilbert, F.W. Sinden (1965), *A polygon problem*, Amer. Math. Monthly 72, 233–241.

G. Sapiro and A.M. Bruckstein (1995). *The Ubiquitous Ellipse*, ACTA Applicandae Mathematicae, 38, 149–161.

A.M. Bruckstein, G. Sapiro, and D. Shaked (1995), *Evolution of Planar Polygons*, J. Pattern Recognition and Artifical Intelligence, 9, 991–1014.

I.A. Wagner and A.M. Bruckstein (1997), *Row Straightening via Local Interactions*, Circuit Systems and Signal Processing, 15(3),287–305.

F. Oggier and A.M. Bruckstein (2012), *On Cyclic and Nearly Cyclic Multiagent Interactions in the Plane*, Operator Theory Advances and Applications, 218, 513–539.

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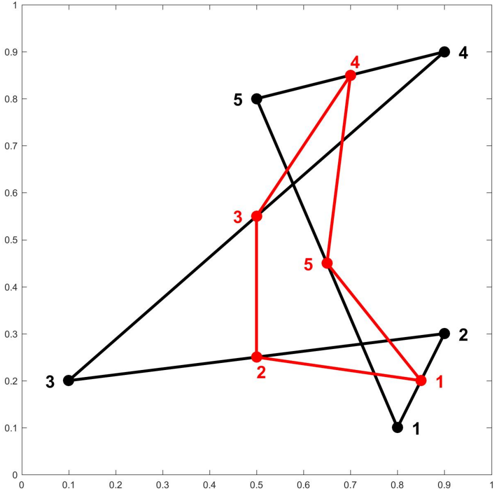
Connnect the Edge Midpoints



$$(x_1, y_1) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(x_2, y_2) = \left(\frac{x_2 + x_3}{2}, \frac{y_2 + y_3}{2}\right)$$

$$(x_3, y_3) = \left(\frac{x_3 + x_4}{2}, \frac{y_3 + y_4}{2}\right)$$

$$(x_4, y_4) = \left(\frac{x_4 + x_5}{2}, \frac{y_4 + y_5}{2}\right)$$

$$(x_5, y_5) = \left(\frac{x_5 + x_1}{2}, \frac{y_5 + y_1}{2}\right)$$

Centroid Preservation: $(\bar{x}, \bar{y}) = (\bar{x}, \bar{y})$

Polygon Averaging (Shifted to Origin)

Generating Polygons $P_1, P_2, \ldots$

```
x = rand(n,1); x = x - mean(x);
y = rand(n,1); y = y - mean(y);
for k = 1,2,...
    x = ( x +[x(2:end);x(1)] )/2;
    y = ( y +[y(2:end);y(1)] )/2;
end
```

The Vertex Vector Update in Matrix-Vector Terms

$$\begin{bmatrix} (x_1+x_2)/2 \\ (x_2+x_3)/2 \\ (x_3+x_4)/2 \\ (x_4+x_5)/2 \\ (x_5+x_1)/2 \end{bmatrix} = \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix}}_{M_5} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

Polygon Averaging (in Matrix Terms)

Generating Polygons $P_1, P_2, \ldots$

```
x = rand(n,1); x = x - mean(x);
y = rand(n,1); y = y - mean(y);
for k = 1,2,...
    x = M*x;
    y = M*y;
end
```

We have two copies of the **power method**:

The k-th **x**-vector is $\mathbf{M^k}\cdot$**(initial x-vector)**.
The k-th **y**-vector is $\mathbf{M^k}\cdot$**(initial y-vector)**.

Analysis requires an understanding of M's eigensystem.

M_n has a Highly Structured Real Schur Decomposition

$Q^T M Q = T$ where Q is orthogonal and T upper quasi-triangular.

If $M = M_5$ then

```
Q =
      0.4472    0.6325         0    0.6325         0
      0.4472    0.1954    0.6015   -0.5117    0.3717
      0.4472   -0.5117    0.3717    0.1954   -0.6015
      0.4472   -0.5117   -0.3717    0.1954    0.6015
      0.4472    0.1954   -0.6015   -0.5117   -0.3717


T =
       1.0000         0         0         0         0
            0    0.6545    0.4755         0         0
            0   -0.4755    0.6545         0         0
            0         0         0    0.0955    0.2939
            0         0         0   -0.2939    0.0955
```

Eigenvalues: 1.0000, $.6545 \pm .4755i$, $.0955 \pm .2939i$

The Matrix M_n

The "update" matrix M_n is given by

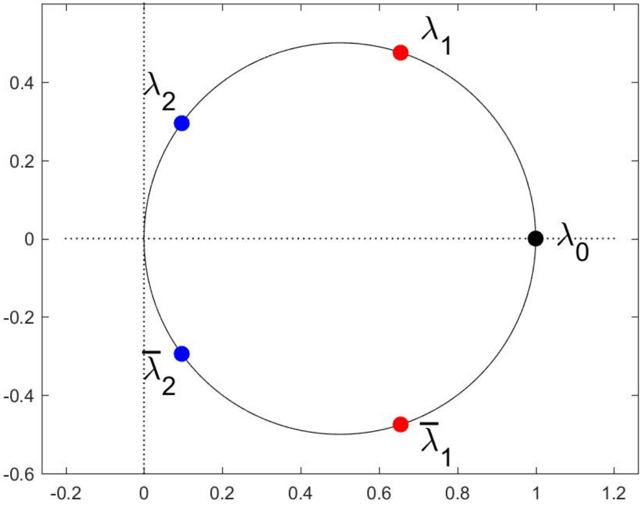
$$M_n = (I_n + S_n)/2$$

where S_n is the n-by-n upshift matrix

$$S_n \;=\; \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad (n = 5)$$

The eigenvalues and eigenvectors of S_n are <u>completely</u> known.

The Eigenvalues of M_5.



The Real Schur Decomposition: $Q^T M_5 Q = T$

$$Q = \begin{bmatrix} x & \color{red}{x} & \color{red}{x} & \color{blue}{x} & \color{blue}{x} \\ x & \color{red}{x} & \color{red}{x} & \color{blue}{x} & \color{blue}{x} \\ x & \color{red}{x} & \color{red}{x} & \color{blue}{x} & \color{blue}{x} \\ x & \color{red}{x} & \color{red}{x} & \color{blue}{x} & \color{blue}{x} \\ x & \color{red}{x} & \color{red}{x} & \color{blue}{x} & \color{blue}{x} \end{bmatrix} \qquad T = \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & \color{red}{x} & \color{red}{x} & 0 & 0 \\ 0 & \color{red}{x} & \color{red}{x} & 0 & 0 \\ 0 & 0 & 0 & \color{blue}{x} & \color{blue}{x} \\ 0 & 0 & 0 & \color{blue}{x} & \color{blue}{x} \end{bmatrix}$$

M_5 has three invariant subspaces of interest:

Invariant Subspace	Associated Eigenvalue(s)
Black	λ_0
Red	$\lambda_1 , \bar{\lambda}_1$
Blue	$\lambda_2 , \bar{\lambda}_2$

M_5: The Black Invariant Subspace

$$Q = \begin{bmatrix} \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} & \mathbf{x} \end{bmatrix} \qquad T = \begin{bmatrix} \mathbf{x} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{x} & \mathbf{x} & 0 & 0 \\ 0 & \mathbf{x} & \mathbf{x} & 0 & 0 \\ 0 & 0 & 0 & \mathbf{x} & \mathbf{x} \\ 0 & 0 & 0 & \mathbf{x} & \mathbf{x} \end{bmatrix}$$

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad \begin{bmatrix} \mathbf{x} \end{bmatrix} = \begin{bmatrix} 1 \end{bmatrix}$$

$\lambda_0 = 1$ is the largest eigenvalue and `ones(n,1)` is the eigenvector.

M_5: The Red Invariant Subspace

$$Q = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \qquad T = \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

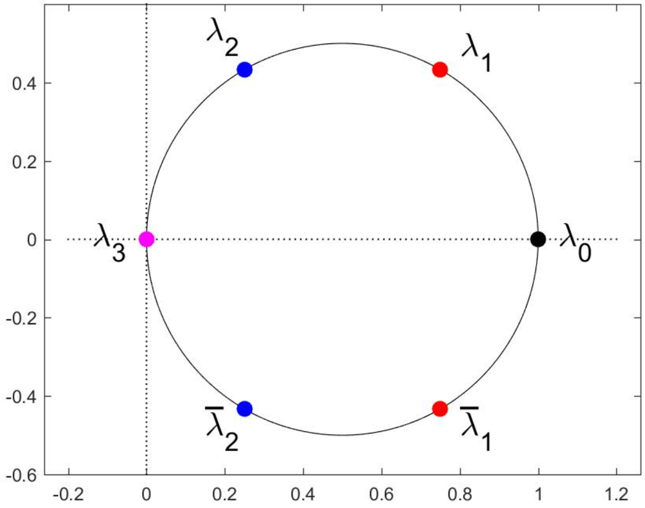
$$\sqrt{\frac{2}{5}} \begin{bmatrix} \cos(0\pi/5) & \sin(0\pi/5) \\ \cos(2\pi/5) & \sin(2\pi/5) \\ \cos(4\pi/5) & \sin(4\pi/5) \\ \cos(6\pi/5) & \sin(6\pi/5) \\ \cos(8\pi/5) & \sin(8\pi/5) \end{bmatrix} \qquad \frac{1}{2} \begin{bmatrix} 1+\cos(2\pi/5) & \sin(2\pi/5) \\ -\sin(2\pi/5) & 1+\cos(2\pi/5) \end{bmatrix}$$

M_5: The Blue Invariant Subspace

$$Q = \begin{bmatrix} x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \\ x & x & x & x & x \end{bmatrix} \qquad T = \begin{bmatrix} x & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & x & x & 0 & 0 \\ 0 & 0 & 0 & x & x \\ 0 & 0 & 0 & x & x \end{bmatrix}$$

$$\sqrt{\frac{2}{5}} \begin{bmatrix} \cos(0\pi/5) & \sin(0\pi/5) \\ \cos(4\pi/5) & \sin(4\pi/5) \\ \cos(8\pi/5) & \sin(8\pi/5) \\ \cos(12\pi/5) & \sin(12\pi/5) \\ \cos(16\pi/5) & \sin(16\pi/5) \end{bmatrix} \qquad \frac{1}{2} \begin{bmatrix} 1 + \cos(4\pi/5) & \sin(4\pi/5) \\ -\sin(4\pi/5) & 1 + \cos(4\pi/5) \end{bmatrix}$$

The Eigenvalues of M_6



M_n is singular if n is even.

No Surprise that M_n is Singular if n is Even

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$$

The Real Schur Decomposition: $Q^T M_6 Q = T$

$$Q = \begin{bmatrix} x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \\ x & x & x & x & x & x \end{bmatrix} \qquad T = \begin{bmatrix} x & 0 & 0 & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 & 0 \\ 0 & x & x & 0 & 0 & 0 \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & x & x & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

M_6 has four invariant subspaces of interest:

Invariant Subspace	Associated Eigenvalue(s)
Black	λ_0
Red	λ_1 , $\bar{\lambda}_1$
Blue	λ_2 , $\bar{\lambda}_2$
Purple	0

M_6: The Purple Invariant Subspace

$$Q = \begin{bmatrix} \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \\ \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} & \mathsf{x} \end{bmatrix} \qquad T = \begin{bmatrix} \mathsf{x} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 \\ 0 & \mathsf{x} & \mathsf{x} & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathsf{x} & \mathsf{x} & 0 \\ 0 & 0 & 0 & \mathsf{x} & \mathsf{x} & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{0} \end{bmatrix}$$

$$\begin{bmatrix} \mathsf{x} \\ \mathsf{x} \\ \mathsf{x} \\ \mathsf{x} \\ \mathsf{x} \\ \mathsf{x} \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} \mathbf{1} \\ -\mathbf{1} \\ \mathbf{1} \\ -\mathbf{1} \\ \mathbf{1} \\ -\mathbf{1} \end{bmatrix}$$

Let's Use the Real Schur to Track the Vertex Vectors

$$M^k x = \left(QTQ^T\right)^k x = QT^k(Q^T x)$$

$$M^k y = \left(QTQ^T\right)^k y = QT^k(Q^T y)$$

```
Q =
    0.4472    0.6325         0    0.6325         0
    0.4472    0.1954    0.6015   -0.5117    0.3717
    0.4472   -0.5117    0.3717    0.1954   -0.6015
    0.4472   -0.5117   -0.3717    0.1954    0.6015
    0.4472    0.1954   -0.6015   -0.5117   -0.3717


T =
     1.0000         0         0         0         0
          0    0.6545    0.4755         0         0
          0   -0.4755    0.6545         0         0
          0         0         0    0.0955    0.2939
          0         0         0   -0.2939    0.0955
```

Expand the Vertex Vectors in the Real Schur Basis

$$x = \alpha_0 \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \end{bmatrix} + \color{red}{\begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix}} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + \color{blue}{\begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix}} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

$$y = \gamma_0 \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \end{bmatrix} + \color{red}{\begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix}} \begin{bmatrix} \gamma_1 \\ \delta_1 \end{bmatrix} + \color{blue}{\begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix}} \begin{bmatrix} \gamma_2 \\ \delta_2 \end{bmatrix}$$

We're showing $n = 5$ but the expansion starts out like this for any n.

Implication of `mean(x) = 0`

This is an orthonormal basis expansion:

$$x = \alpha_0 \begin{bmatrix} \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \\ \mathbf{x} \end{bmatrix} + \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

Since x has zero mean we have...

$$\alpha_0 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}^T \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = (x_1 + x_2 + x_3 + x_4 + x_5)/\sqrt{5} = 0$$

The Dominant Eigenvector is Not Around

Thus,

$$x = \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

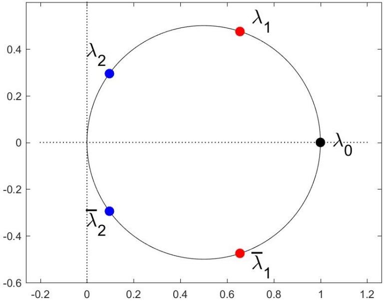
and so after the k-th iterate this vertex vector is given by

$$x = M^k \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + M^k \begin{bmatrix} \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \\ \mathbf{x} & \mathbf{x} \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$

As $k \to \infty$ this vector goes to zero <u>and</u> as this happens the red component increasingly dominates the blue component.

M^5x: Intuition

$$M^k x = M^k \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} + M^k \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix}$$



Why Red Dominates Blue:

$$\frac{|\lambda_2|}{|\lambda_1|} = .3820$$

Why the whole things goes to zero:

$$\| Mx \| \leq |\lambda_1| \| x \| = .6545 \cdot \| x \|$$

Why the Polygons Collapse to (0,0)

If x has zero mean then

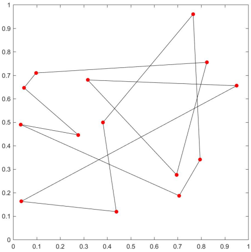
$$\| M_n^k x \| \le |\lambda_1|^k \| x \| = \cos(2\pi/n)^k \| x \|$$

The exact reduction in norm each step:

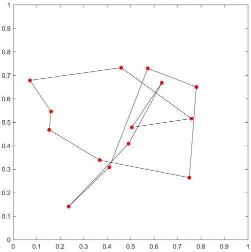
$$\begin{aligned} \| Mx \|_2^2 &= \| x \|_2^2 - \frac{1}{4} \| (I - S)x \|_2^2 \\ &= \| x \|_2^2 - \frac{1}{4} \sum_{i=1}^{n} (x_i - x_{i+1})^2 \end{aligned}$$

Let's Prevent This

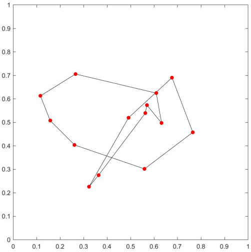
P_0



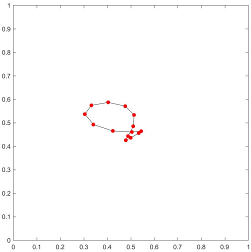
P_1



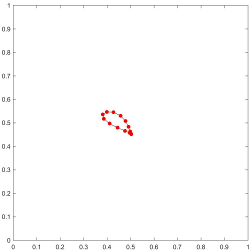
P_2



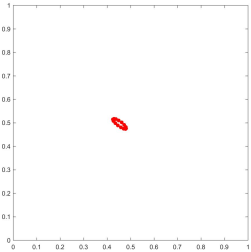
P_{15}



P_{30}



P_{60}



Polygon Averaging (with 2-norm normalization)

Generating Polygons $P_1, P_2, \ldots$

```
x = rand(n,1); x = x - mean(x); x = x/norm(x);
y = rand(n,1); y = y - mean(y); y = y/norm(y);
```

for $k = 1, 2, \ldots$

```
    x = M*x; x = x/norm(x);
    y = M*y; y = y/norm(y);
```

end

Two copies of the **power method** with normalization:

The k-th x is a unit vector in the direction of $\mathbf{M^k}\cdot$**(initial x)**.

The k-th y is a unit vector in the direction of $\mathbf{M^k}\cdot$**(initial y)**.

Let's look at these vectors!

A Fact About M^k Acting on an Invariant Subspace

To understand the red and blue components of the k-th vertex vectors we need to work with

$$M^k \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \begin{bmatrix} \text{Re}(\lambda_1) & \text{Im}(\lambda_1) \\ \text{-Im}(\lambda_1) & \text{Re}(\lambda_1) \end{bmatrix}^k$$

$$M^k \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} = \begin{bmatrix} x & x \\ x & x \\ x & x \\ x & x \\ x & x \end{bmatrix} \begin{bmatrix} \text{Re}(\lambda_2) & \text{Im}(\lambda_2) \\ \text{-Im}(\lambda_2) & \text{Re}(\lambda_2) \end{bmatrix}^k$$

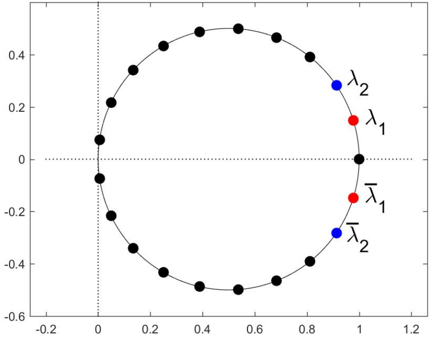
$M^k x \;\longrightarrow\;$ Red Subspace

$$M^k x = \begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix} \left(\begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix}^k \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \right) + \begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix} \left(\begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix}^k \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right)$$

$$\frac{\left\| \begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix}^k \begin{bmatrix} \alpha_2 \\ \beta_2 \end{bmatrix} \right\|}{\left\| \begin{bmatrix} \text{x} & \text{x} \\ \text{x} & \text{x} \end{bmatrix}^k \begin{bmatrix} \alpha_1 \\ \beta_1 \end{bmatrix} \right\|} = \left| \frac{\cos(2\pi/5)}{\cos(\pi/5)} \right|^k \cdot \sqrt{\frac{\alpha_2^2 + \beta_2^2}{\alpha_1^2 + \beta_1^2}}$$

$$\text{Damping Factor} = \left| \frac{\cos(2\pi/5)}{\cos(\pi/5)} \right| = .3820.$$

The Damping Factor for General n



n	ρ_n
5	.3820
10	.8507
20	.9629
30	.9835
40	.9907
50	.9941
100	.9985

$$\rho_n = \frac{|\lambda_2|}{|\lambda_1|} = \left|\frac{\cos(2\pi/n)}{\cos(\pi/n)}\right| = 1 - \frac{3}{2}\left(\frac{\pi}{n}\right)^2 + O\left(\frac{1}{n^4}\right)$$

Now Let's Figure Out the Limiting Ellipse

Because of damping we may assume that the initial unit 2-norm vertex vectors x and y are given by

$$x = \begin{bmatrix} \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} \cos(\theta_x) \\ \sin(\theta_x) \end{bmatrix} \qquad \begin{bmatrix} \cos(\theta_x) \\ \sin(\theta_x) \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$$

$$y = \begin{bmatrix} \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \\ \times & \times \end{bmatrix} \begin{bmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{bmatrix} \qquad \begin{bmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{bmatrix} = \begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$$

The scalars $\cos(\theta_x)$, $\sin(\theta_x)$, $\cos(\theta_y)$, and $\sin(\theta_y)$ are **computable** since we know the red matrix and initial vertex vectors x and y.

The $\{(x_i, y_i)\}$ Sit on an Ellipse

Recall that the Red Matrix is made of cosines and sines and so

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \sqrt{\frac{2}{5}} \begin{bmatrix} \cos(0\tau) & \sin(0\tau) \\ \cos(1\tau) & \sin(1\tau) \\ \cos(2\tau) & \sin(2\tau) \\ \cos(3\tau) & \sin(3\tau) \\ \cos(4\tau) & \sin(4\tau) \end{bmatrix} \begin{bmatrix} \cos(\theta_x) \\ \sin(\theta_x) \end{bmatrix} \qquad \tau = \sqrt{\frac{2\pi}{5}}$$

$$\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} = \sqrt{\frac{2}{5}} \begin{bmatrix} \cos(0\tau) & \sin(0\tau) \\ \cos(1\tau) & \sin(1\tau) \\ \cos(2\tau) & \sin(2\tau) \\ \cos(3\tau) & \sin(3\tau) \\ \cos(4\tau) & \sin(4\tau) \end{bmatrix} \begin{bmatrix} \cos(\theta_y) \\ \sin(\theta_y) \end{bmatrix} \qquad \tau = \sqrt{\frac{2\pi}{5}}$$

i.e.,

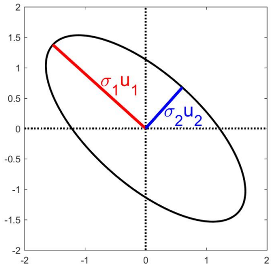
$$\begin{bmatrix} x_i \\ y_i \end{bmatrix} = \left(\sqrt{\frac{2}{5}} \begin{bmatrix} \cos(\theta_x) & \sin(\theta_x) \\ \cos(\theta_y) & \sin(\theta_y) \end{bmatrix} \right) \begin{bmatrix} \cos(t_i) \\ \sin(t_i) \end{bmatrix} \qquad t_i = (i-1)\tau,\ i = 1{:}5$$

The SVD Tells Us All About the Ellipse

The ellipse

$$\mathcal{E} = \left\{ \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} : 0 \leq t \leq 2\pi \right\}$$

looks like this



where $A = U\Sigma V^T$ and $U = [u_1 \; u_2]$ and $\Sigma = \text{diag}(\sigma_1, \sigma_2)$.

And the Ellipse has a 45-Degree Tilt

2x2 SVD Theorem

If

$$A = \mu \begin{bmatrix} \cos(\theta_x) & \sin(\theta_x) \\ \cos(\theta_y) & \sin(\theta_y) \end{bmatrix} \qquad \leftarrow \text{Rows of equal length.}$$

then its SVD $A = U\Sigma V^T$ is given by

$$U = \begin{bmatrix} \cos(\pi/4) & -\sin(\pi/4) \\ \sin(\pi/4) & \cos(\pi/4) \end{bmatrix} \qquad V = \begin{bmatrix} \cos(a) & -\sin(a) \\ \sin(a) & \cos(a) \end{bmatrix}$$

$$\Sigma = \mu \begin{bmatrix} \sqrt{2}\cos(b) & 0 \\ 0 & \sqrt{2}\sin(b) \end{bmatrix}$$

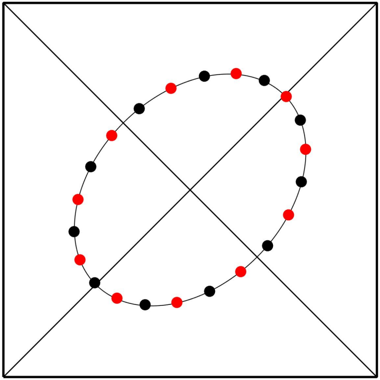
where

$$a = \frac{\theta_x + \theta_y}{2} \qquad \text{and} \qquad b = \frac{\theta_x - \theta_y}{2}.$$

Three Parting Shots

1. The vertices converge to the limiting ellipse but continue to move.
2. The inverse polygon averaging problem can be explained.
3. Kepler's "Centered" Second Law

The Vertices Appear to Move Around the Limiting Ellipse



$n = 11$

Red vertices depict the polygon after an even number of averagings.

Black vertices depict the polygon after an odd number of averagings.

Reason: Structured Sines and Cosines in the Red Matrix

$$
\begin{array}{ccccccccccc}
\text{even} & & \text{odd} & & \text{even} & & \text{odd} & & \text{even} & & \\
\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} & \rightarrow & \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \\ x_5' \end{bmatrix} & \rightarrow & \begin{bmatrix} x_5 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} & \rightarrow & \begin{bmatrix} x_5' \\ x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix} & \rightarrow & \begin{bmatrix} x_4 \\ x_5 \\ x_1 \\ x_2 \\ x_3 \end{bmatrix} & \rightarrow \\
\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix} & \rightarrow & \begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ y_4' \\ y_5' \end{bmatrix} & \rightarrow & \begin{bmatrix} y_5 \\ y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} & \rightarrow & \begin{bmatrix} y_5' \\ y_1' \\ y_2' \\ y_3' \\ y_4' \end{bmatrix} & \rightarrow & \begin{bmatrix} y_4 \\ y_5 \\ y_1 \\ y_2 \\ y_3 \end{bmatrix} & \rightarrow
\end{array}
$$

Downshifted versions of the grandparent.

The Inverse Polygon Averaging Problem

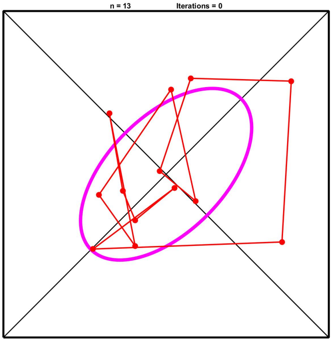
Generating Polygons $P_{-1}, P_{-2}, \ldots$

```
x = rand(n,1); x = x - mean(x); x = x/norm(x)
y = rand(n,1); y = y - mean(y); y = y/norm(y)
for k = 1, 2, ...
    x = inv(M)*x; x = x/norm(x)
    y = inv(M)*y; y = y/norm(y)
end
```

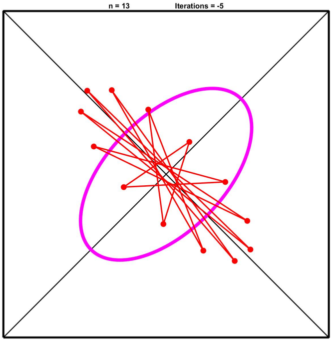
$$M = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \qquad M^{-1} = \begin{bmatrix} 1 & -1 & 1 & -1 & 1 \\ 1 & 1 & -1 & 1 & -1 \\ -1 & 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 & -1 \\ -1 & 1 & -1 & 1 & 1 \end{bmatrix}$$

The Inverse Polygon Averaging Problem

$P(0)$



$P(-5)$



The invariant subspace associated with M's smallest complex eigenvalue is relevant.

The Inverse Polygon Averaging Problem

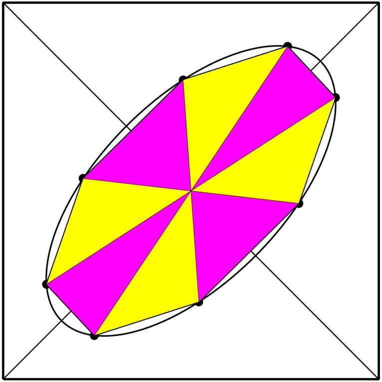
For $n = 13$, the columns of this matrix span that space:

$$\begin{bmatrix} 0.3922 & 0 \\ -0.3808 & 0.0939 \\ 0.3473 & -0.1823 \\ -0.2936 & 0.2601 \\ 0.2228 & -0.3228 \\ -0.1391 & 0.3667 \\ 0.0473 & -0.3894 \\ 0.0473 & 0.3894 \\ -0.1391 & -0.3667 \\ 0.2228 & 0.3228 \\ -0.2936 & -0.2601 \\ 0.3473 & 0.1823 \\ -0.3808 & -0.0939 \end{bmatrix}$$

Having the maximum number of sign changes explains why the limiting inverse polygon has a maximal number of "edge crossings."

Kepler's "Centered" Second Law

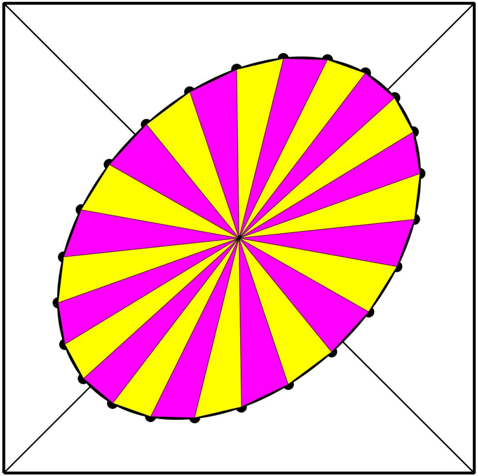
Conjecture: The vertices on the limiting ellipse define triangular "pizza slices" with equal area:



Triangle Areas
7.065091311397755e-02
7.065091311397753e-02
7.065091311397756e-02
7.065091311397756e-02
7.065091311397759e-02
7.065091311397755e-02
7.065091311397756e-02
7.065091311397755e-02

An "equal-area/equal-time" planet travels along the perimeter of limiting polygon as it orbits the Sun. The time it takes to travel from vertex to vertex is uniform.

Proof By Matlab



.010121592325505**50**
.010121592325505**51**
.010121592325505**51**
.010121592325505**51**
.010121592325505**52**
.010121592325505**53**
.010121592325505**51**
.010121592325505**49**
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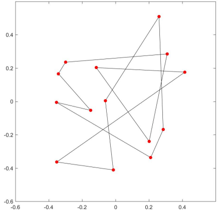
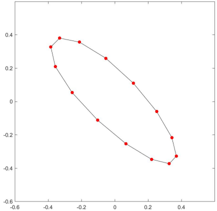
.010121592325505**52**
.010121592325505**51**
.010121592325505**50**
.010121592325505**50**
.010121592325505**54**

The triangle areas agree through 15 digits.

Summary





$$A = U\Sigma V^T$$

$$A = QTQ^T$$