Wrinkling of a twisted ribbon

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Outline

The experiment

Twist a ribbon and hold it with small tension. It should form wrinkles in the center.

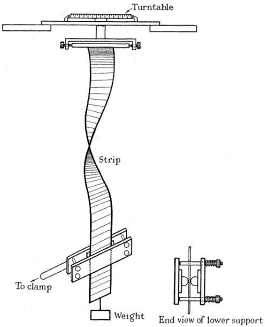


Figure: Left: A.E. Green, Proc. R. Soc. 1937[Gre37]. Right: Chopin and Kudrolli, PRL 2013[CK13] and Chopin et al, J. Elasticity 2015[CDD15]

Intuition

Why the ribbon wrinkles:

- Twisting makes the outside edges get longer.
- If you allow the ribbon to compress, but only a little, then the outside is under tension and the inside under compression.
- A one- or two-dimensional object can wrinkle to avoid compression.

The form of the energy

The form of the energy

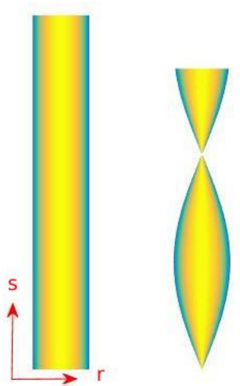
$$E^h = \int_\Omega \frac{1}{2}|M|^2 + h^2|B|^2$$

- The membrane term M measures the amount of stretching.
- B measures the amount of bending.
- The thickness h is small.

Optimistic goal: minimize E^h subject to boundary data.
Practical goal: find out how the minimum energy scales with h.

Variables for a twisted ribbon

- The **domain** $\Omega = (-1/2, 1/2) \times (0, l)$ is a rectangle. Points are parameterized by $(r, s) \in \Omega$.
- The **tangential displacement** $u : \Omega \to \mathbb{R}^2$ and **normal displacement** $v : \Omega \to \mathbb{R}$.
- **Twist** per unit length ω.
- **Displacement of the top:** $-\frac{1}{2}\omega^2 \tilde{r}^2$. Assume: $\tilde{r} < 1/2$.
- **Wrinkled zone:** for $|r| < \tilde{r}$, the ribbon is compressed in its reference state.



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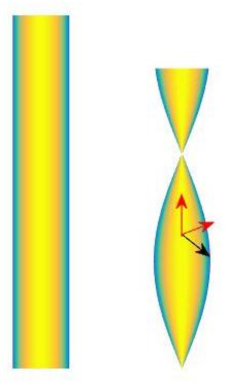

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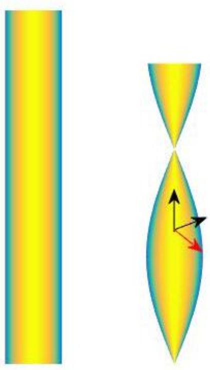

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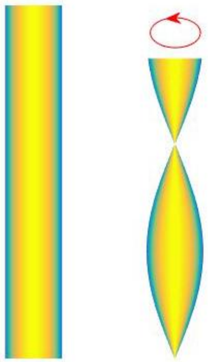

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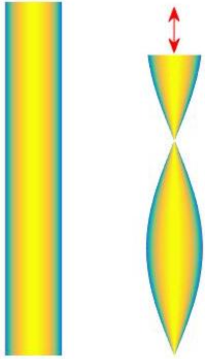

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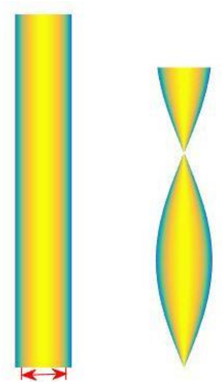


Small-slope energy of a twisted ribbon

We want to find out how the minimum of the energy scales with h:

$$E^h(u,v) = \int_\Omega \frac{1}{2}|M(u,v)|^2 + h^2|B(u,v)|^2$$

$$M(u,v) = e(u) + \frac{1}{2}\begin{pmatrix} v_r \\ v_s + \omega r \end{pmatrix} \otimes \begin{pmatrix} v_r \\ v_s + \omega r \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & \omega v \\ \omega v & \omega^2 \tilde{r}^2 \end{pmatrix}$$

$$B(u,v) = \nabla\nabla v + \begin{pmatrix} 0 & \omega \\ \omega & 0 \end{pmatrix}$$

with boundary data:

$$u(r,0) = u(r,1) = 0$$
$$v(r,0) = v(r,1) = 0$$

This energy is from Chopin et al [CDD15].

The membrane term: heuristics

Vertical stretching:

$$\begin{aligned} m_{22} &= u_{2,s} + \frac{1}{2}(v_s + \omega r)^2 - \frac{1}{2}\omega^2 \tilde{r}^2 \\ &= u_{2,s} + \omega r v_s + \frac{1}{2}\left(v_s^2 - \omega^2\left(\tilde{r}^2 - r^2\right)\right) \end{aligned}$$

Red: Mean-0 in s. Blue: Positive. Green: Sign depends on r.

- Vertical lines are stretched if $|r| > \tilde{r}$ and (in the reference state) compressed if $|r| < \tilde{r}$.
- Compression can be relieved by wrinkling (choosing u_2 and v oscillatory).

Outline

The main result

Theorem

There exists constants E_0, C, C' such that

$$E_0 + Ch^{4/3} \leq \min_{u,v} E^h(u,v) \leq E_0 + C'h^{4/3}.$$

Two parts of the proof:

- The lower bound requires an argument for any u and v.
- The upper bound is an ansatz (a choice of u and v).

The leading-order energy E_0

Main point: the zones under vertical tension always contribute energy E_0, and making u, v nonzero can only increase the energy.

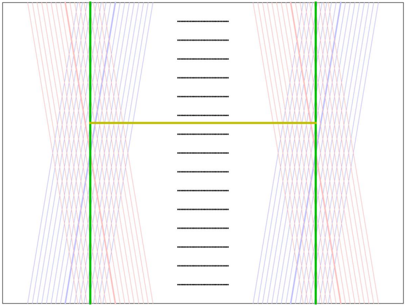
$$\begin{aligned} E^h(u,v) &= \int_{-1/2}^{1/2}\int_0^1 \frac{1}{2}|M|^2 + h^2|B|^2 ds\, dr \\ &\geq \frac{1}{2}\int_{-1/2}^{1/2}\left(\int_0^1 m_{22} ds\right)_+^2 dr \\ &\geq \frac{1}{4}\int_{\tilde{r}}^{1/2} \omega^4 (r^2 - \tilde{r}^2)^2 dr = E_0 \end{aligned}$$

Remark: We are minimizing the relaxed problem.

An outline of the lower bound

We assume that $E^h(u, v) < E_0 + \varepsilon$ and find a contradiction if ε is too small. This proof has two main steps:

1. The outer edges contain rigid lines: displacements are small.
2. Horizontal lines are stretched if the wrinkles have large amplitude, but bending resistance keeps the amplitude from being too small.



Sources: Strauss [Str73]; Bella, Kohn [BK14a].

Lower bound sketch (1.1)

$$E^h - E_0 = \int_\Omega \frac{1}{2}\left|\tilde{M}\right|^2 + h^2|B|^2 + \frac{1}{4}\omega^2\left(r^2 - \tilde{r}^2\right)_+ (v_s)^2 < \varepsilon$$

where $\tilde{M}$ is the excess strain:

$$\tilde{M} = M - \frac{1}{2}\omega^2\left(r^2 - \tilde{r}^2\right)_+ \hat{s} \otimes \hat{s}$$

1. Tension in the vertical direction: for any $R > \tilde{r}$, $\|v_s\|_{L^2(|r|>R)} \lesssim \varepsilon^{1/2}$.
2. Small displacement: $\|v\|_{L^2(|r|>R)L^\infty(s)} \lesssim \varepsilon^{1/2}$.
3. There is some $r_0 > R$ such that $\|v(\pm r_0, s)\|_{L^\infty(s\in[0,1])} \lesssim \varepsilon^{1/2}$

Next: Control $u(\pm r_0, s)$.

Lower bound sketch (1.2)

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- **An observation:** Tension in direction a (unit vector) gives control on $\langle a, \tilde{M}a\rangle$, which gives control on $\langle a, e(u)a\rangle$.

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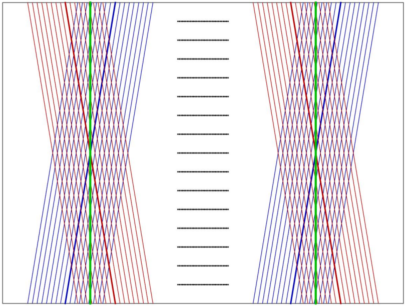
- **A problem:** We have vertical, but not horizontal, tension. We want control on $u(\pm r_0, s)$.
- **The resolution:** Use tension in two diagonal directions $a_\pm$.

Lower bound sketch (1.3)

Take vectors $a_\pm$, sets $\Omega^\pm$ as shown. $\Omega^0 = \Omega^+ \cap \Omega^-$.

Goal: Show that u is small on the green lines.



- Blue: Lines parallel to a_+ shading region Ω^+.
- Red: Lines parallel to a_- shading region Ω^-.
- Green: Lines $r = \pm r_0$.

Lower bound sketch (1.4)

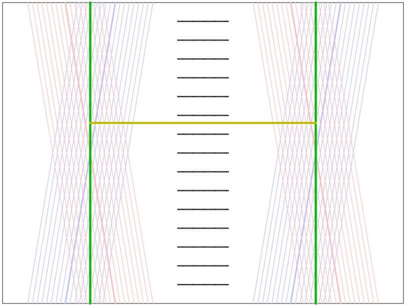
1. Integrate along diagonal lines:

$$\begin{aligned} \varepsilon \quad &\gtrsim \quad \int_{\Omega^\pm} \left|\tilde{M}\right|^2 \gtrsim \int_{\Omega^\pm} \langle a_\pm, \tilde{M} a_\pm \rangle^2 \\ &\gtrsim \quad \left(\int_{\Omega^\pm} \left\langle a_\pm, \left[\frac{1}{2} \nabla v \otimes \nabla v - \begin{pmatrix} 0 & \omega v \\ \omega v & 0 \end{pmatrix} \right] a_\pm \right\rangle \right)^2 \end{aligned}$$

2. Conclude that $\|\nabla v\|_{L^2(\Omega^0)} \lesssim \varepsilon^{1/4}$.
3. Triangle Inequality: $\|e(u)\|_{L^1(\Omega^0)} \lesssim \varepsilon^{1/2}$.
4. Another diagonal line argument: $\left| \int_0^1 u(\pm r_0, s) ds \right| \lesssim \varepsilon^{1/2}$.

Lower bound part 2: picture

Goal: Show that v is small along the gold line (the wrinkles have small amplitude).



- Green: Lines $r = \pm r_0$. Displacements are small.
- Gold: Line across wrinkles. Cannot be stretched much.

Lower bound sketch (2.1)

We control the horizontal stretching across the wrinkles to show that v cannot be too large. First: Jensen's Inequality on the 11 membrane term.

$$\begin{aligned} \varepsilon &\geq \int_\Omega \frac{1}{2}(\tilde{m}_{11})^2 \geq \frac{1}{2}\left(\int_{\{|r|<r_0\}} u_{1,r} + \frac{1}{2}v_r^2\right)^2 \\ &\gtrsim \left(\int_{\{|r|<r_0\}} v_r^2\right)^2 - \left|\int_{\{|r|<r_0\}} u_{1,r}\right|^2 \end{aligned}$$

so $\|v_r\|_{L^2(\{|r|<r_0\})} \lesssim \varepsilon^{1/4}$, and therefore $\|v\|_{L^2(\{|r|<r_0\})} \lesssim \varepsilon^{1/4}$.

Lower bound sketch (2.2)

The membrane term prefers that v be small. The bending term prefers to have v_{ss} small:

$$\int_{\Omega} h^2 v_{ss}^2 \leq \varepsilon$$

By Gagliardo-Nirenberg Interpolation, the slopes must be small:

$$\|v_s\|_{L^2(\{|r|<r_0\})} \leq \|v\|_{L^2(\{|r|<r_0\})}^{1/2} \|v_{ss}\|_{L^2(\{|r|<r_0\})}^{1/2} \lesssim \left(\varepsilon^{3/4} h^{-1}\right)^{1/2}$$

Lower bound sketch (2.3)

We now have a contradiction: the wrinkles must waste an $O(1)$ amount of arclength.

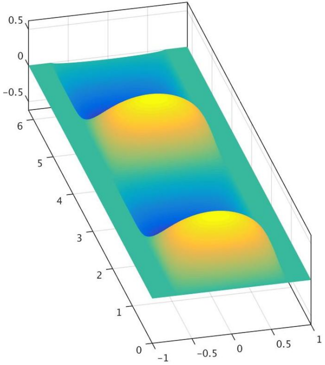
$$\begin{aligned} \varepsilon^{1/2} &\gtrsim \int_{\{|r|<r_0\}} \left| v_s^2 + \frac{1}{2}\omega^2(\tilde{r}^2 - r^2)_+ \right| \\ &\geq \int_{\{|r|<r_0\}} \left| \frac{1}{2}\omega^2(\tilde{r}^2 - r^2)_+ \right| - \int_{\{|r|<r_0\}} \left| v_s^2 \right| \end{aligned}$$

This gives a contradiction if $\varepsilon < Ch^{h/4}$ for some C.

The main point this proves a lower bound for the energy. Along the way we showed inequalities about any low energy state.

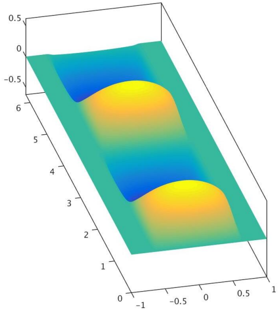
Ansatz sketch: first attempt

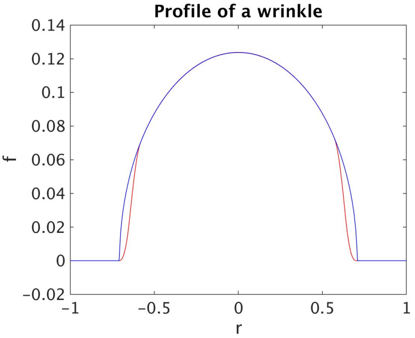
- **The basic idea:** Wrinkling can waste arclength to avoid compression. The lower bound suggests the wavelength.
- **A natural first attempt:** $v(r, s) = f(r) \sin\left(\frac{s}{\lambda}\right)$ where is λ the wavelength and $f(r)$ controls the amplitude.
- **Choosing u:** pick u to cancel out the two highest-order membrane terms m_{22} and $m_{12} = m_{21}$.
- **The problem:** The optimal $f(r) = \lambda\omega\sqrt{2(\tilde{r}^2 - r^2)_+}$ is not $W^{2,2}$, which gives infinite energy.



Ansatz sketch: second attempt

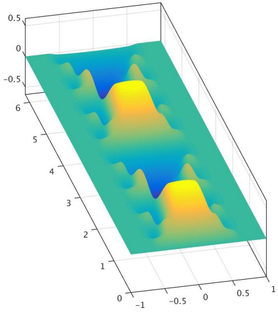
- **The basic idea:** Smooth out f in a boundary layer near $|r| = \tilde{r}$.
- **The problem:** This gives $O(h)$ scaling, not $O(h^{4/3})$.
- **The reason:** The leading-order membrane term m_{11} is singular, so smoothing over a small width gives a large contribution.

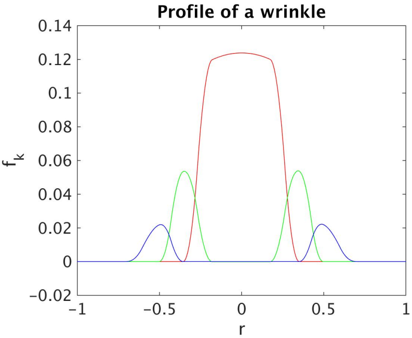




Ansatz sketch: refinement

- **Idea:** We have two parameters to play with: the wavelength and the amplitude. Varying both with r allows us to make f less singular.
- **The old ansatz (reminder):** $v(r,s) = f(r)\sin\left(\frac{s}{\lambda}\right)$
- **The new ansatz:** $v(r,s) = \sum_{k=0}^{N} f_k(r)\sin\left(\frac{s}{\lambda_k}\right)$





Outline

Results

- We proved a lower bound for our energy and found a matching ansatz.
- The energy scales as $E_0 + Ch^{4/3}$, which indicates that some zone is stretched (E_0) and that there is microstructure ($h^{4/3}$).
- The lower bound does not identify the shape of the wrinkles, or tell us if there are multiple length scales.
- In proving the lower bound, we showed that low energy states are rigid near the edges and wrinkle in the center.
- The ansatz uses a cascade of wrinkles.

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