The Generalized Kuramoto Model:

Odd Dimensions are Different; Even Dimensions are Deceptively Similar

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Introduction

- The Kuramoto model is quite well known with several applications
- Our generalization is primarily motivated by:
  - Description of alignment of directions in two dimensions, hence used for alignment of herds of animals on a plane
  - XY model of interacting spins with frozen-in noise
- How do you describe higher-dimensional analogues of these problems?

2D Kuramoto Model: Equations

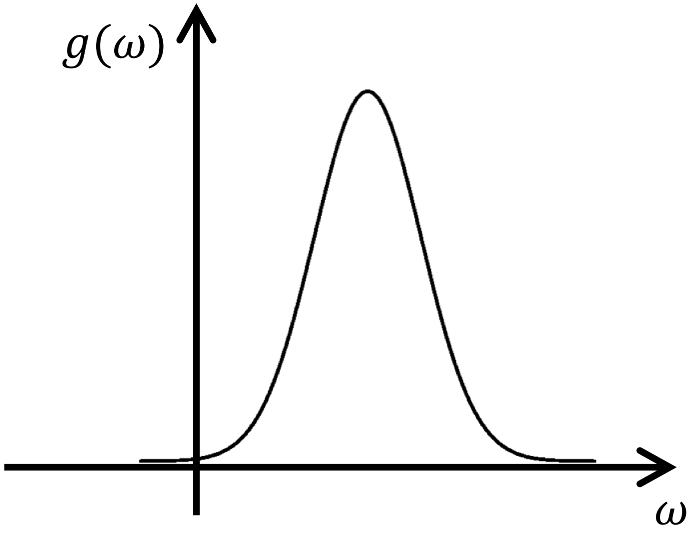
$$\partial_t \theta_i = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \omega_i$$

- θ_i represents the phase of each oscillator, representing a two-dimensional unit vector as a point on a circle
- ω_i represents the natural frequency of each oscillator
- K is the coupling strength

2D Kuramoto Model: Equations

$$\partial_t \theta_i = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \omega_i$$

ω_is chosen according to a distribution $g(\omega)$



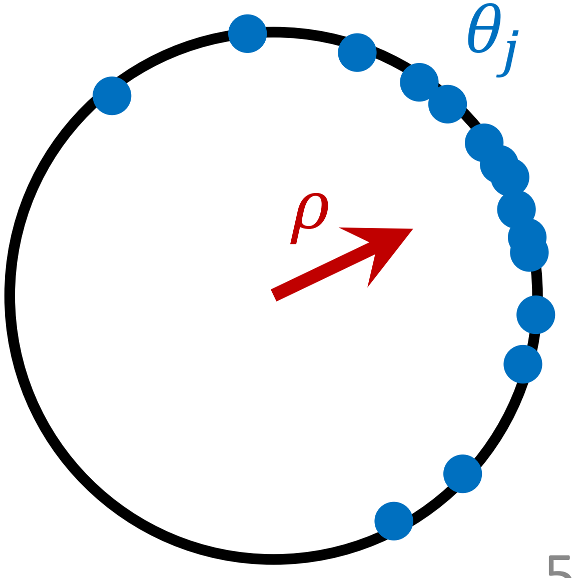
2D Kuramoto Model: Equations

$$\partial_t \theta_i = \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \omega_i$$

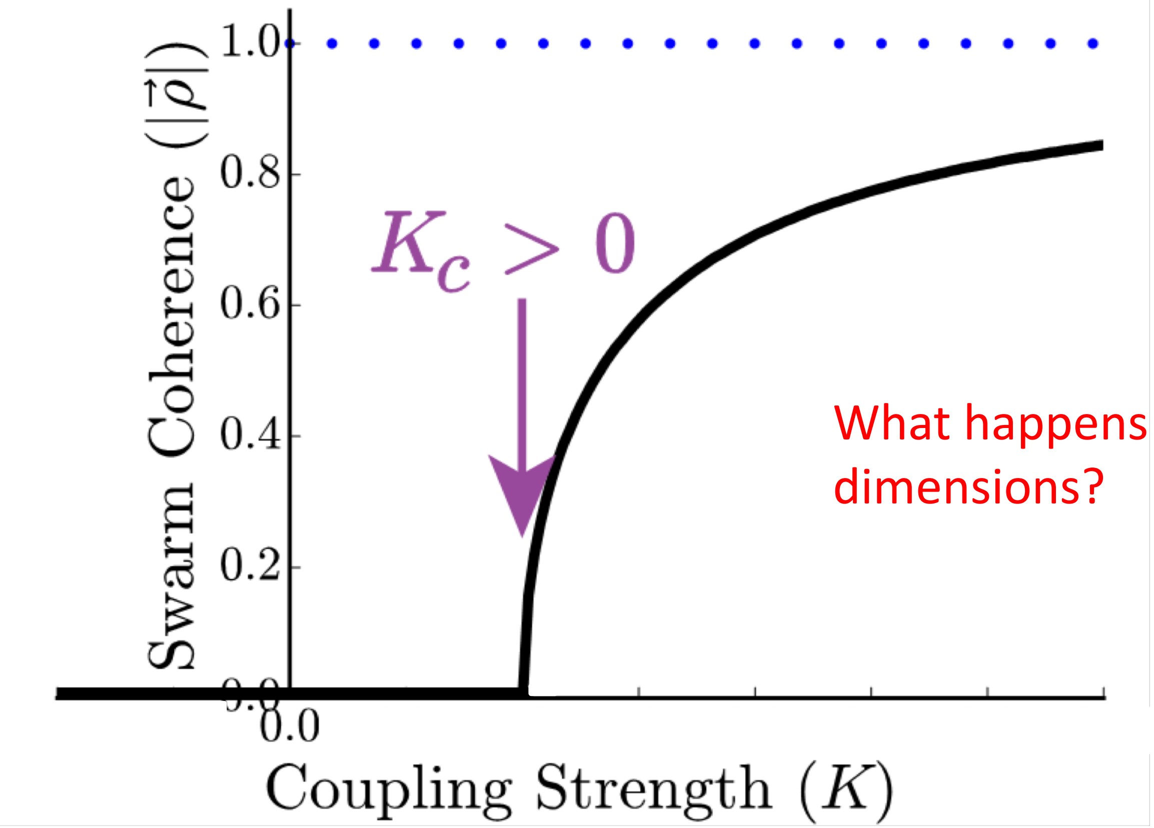
Can define an "order parameter", ρ

$$\rho = \frac{1}{N} \sum_{j=1}^{N} e^{\iota \theta_j} = r e^{\iota \psi}$$

$$\partial_t \theta_i = Kr \sin(\psi - \theta_i) + \omega_i$$



2D Kuramoto Model: Phase Transition



What happens in higher dimensions?

How do we generalize the Kuramoto Model?

$$\partial_t \vec{\sigma_i} = \frac{K}{N} \sum_{j=1}^{N} \left(\vec{\sigma_j} - (\vec{\sigma_j} \cdot \vec{\sigma_i}) \vec{\sigma_i} \right) + \boldsymbol{W}_i \vec{\sigma_i}$$

$$|\vec{\sigma_i}(t)| = 1$$

$$\boldsymbol{W}_i = -\boldsymbol{W}_i{}^T$$

- $\vec{\sigma_i}$ is a unit vector in D dimensions representing the state of each agent
- $\boldsymbol{W}_i$ represents the natural rotation of each agent

Olfati-Saber, IEEE (2006); Zhu, Phys. Lett. A (2013)

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- In 2 dimensions, if we set

$$\vec{\sigma_i} = \begin{pmatrix} \cos\theta_i \\ \sin\theta_i \end{pmatrix} \quad \text{and} \quad \boldsymbol{W}_i = \begin{pmatrix} 0 & \omega_i \\ -\omega_i & 0 \end{pmatrix}$$

then this reduces to the Kuramoto model as earlier

Olfati-Saber, IEEE (2006); Zhu, Phys. Lett. A (2013)

Generalized Kuramoto Model: $D = 3$

$$\partial_t \vec{\sigma_i} = \frac{K}{N}\sum_{j=1}^{N}\left(\vec{\sigma_j} - (\vec{\sigma_j}\cdot\vec{\sigma_i})\vec{\sigma_i}\right) + \underbrace{\boldsymbol{W}_i\vec{\sigma_i}}_{\vec{\omega_i}\times\vec{\sigma_i}}$$

$$|\vec{\sigma_i}(t)| = 1$$

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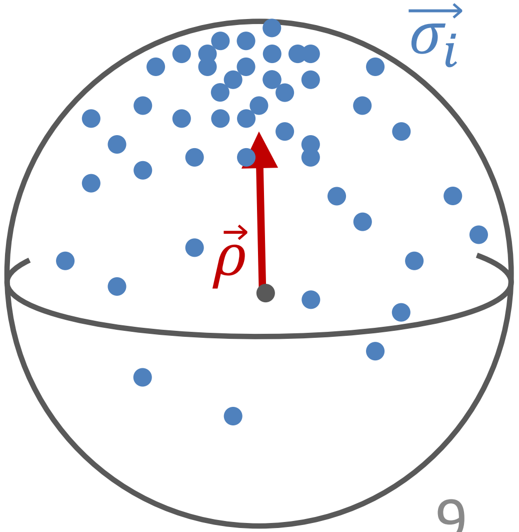
Can define an order parameter, $\vec{\rho}$ as earlier

$$\vec{\rho} = \frac{1}{N}\sum_{i=1}^{N}\vec{\sigma_i}$$



$$\partial_t \vec{\sigma_i} = K[\vec{\rho} - (\vec{\rho}\cdot\vec{\sigma_i})\vec{\sigma_i}] + \vec{\omega_i}\times\vec{\sigma_i}$$

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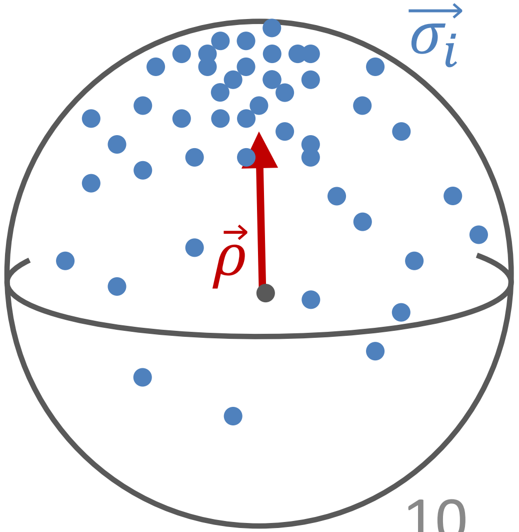

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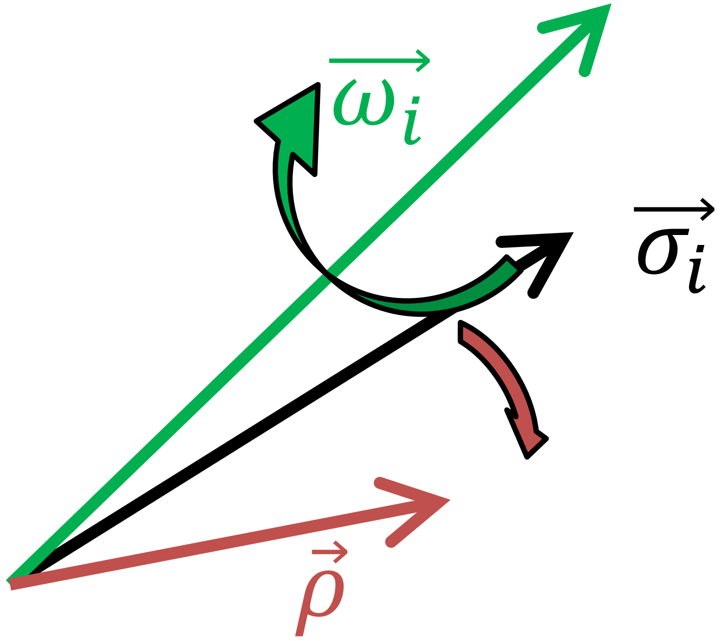
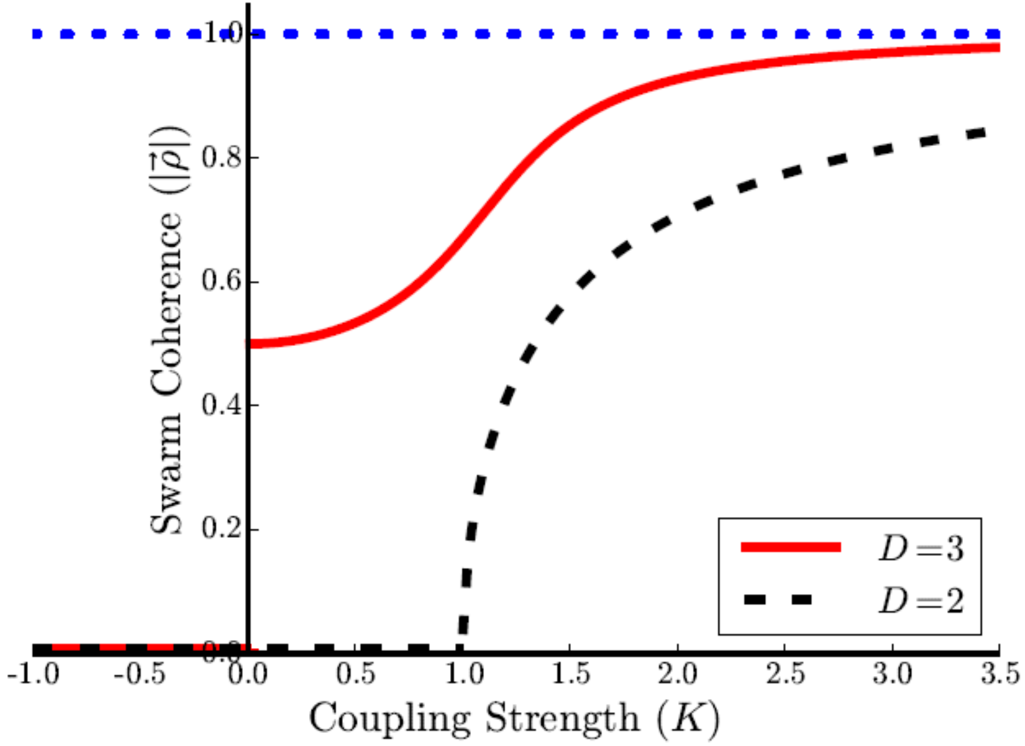


Distribution of Rotations

- Assume that the natural rotations $\overrightarrow{\omega_i}$ are sampled from a distribution $G(\vec{\omega})$
- Cannot shift the mean of $G(\vec{\omega})$ like in the case of $D = 2$

- We consider $G(\vec{\omega})$ such that
$$G(\vec{\omega}) = g(\omega_0)U[\hat{\omega}] \qquad (\vec{\omega} = \omega_0\hat{\omega})$$
where $U[\hat{\omega}]$ is the uniform distribution on the sphere, and $g(\omega_0)$ is a unimodal distribution

Discontinuous Phase Transition!



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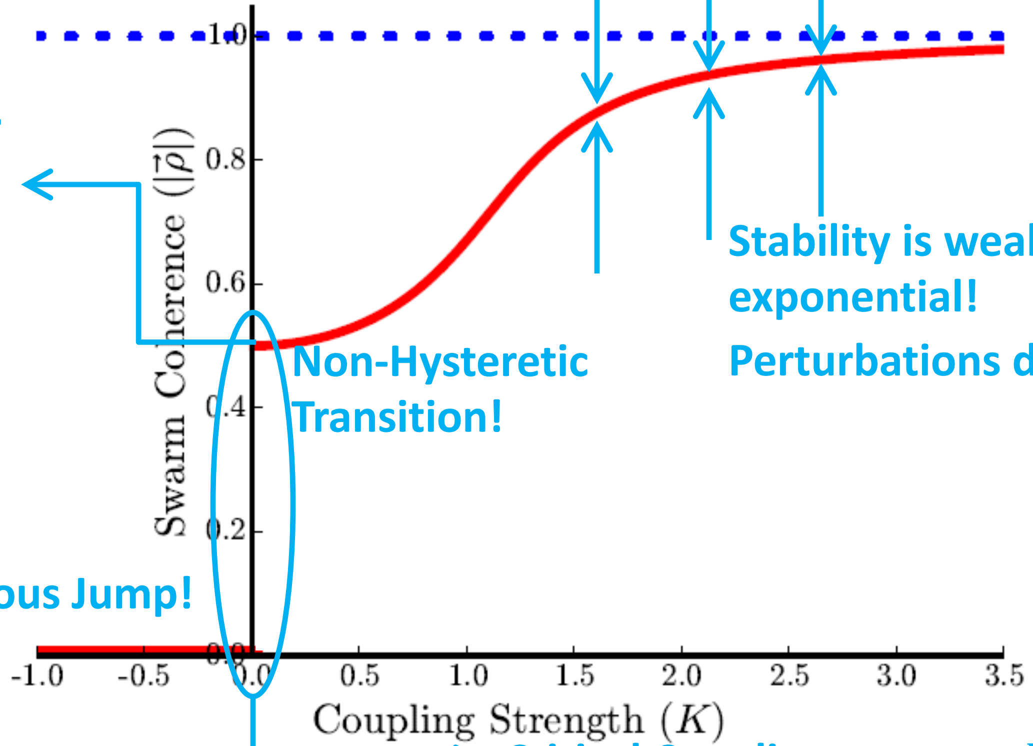
Discontinuous Phase Transition!

$|\vec{\rho}| = 0.5$ for $K \to 0^+$ for all $g(\omega_0)$



Stability is weaker than exponential!

Perturbations decay as $1/\sqrt{t}$

Non-Hysteretic Transition!

Discontinuous Jump!

Critical Coupling constant $K_c = 0$!

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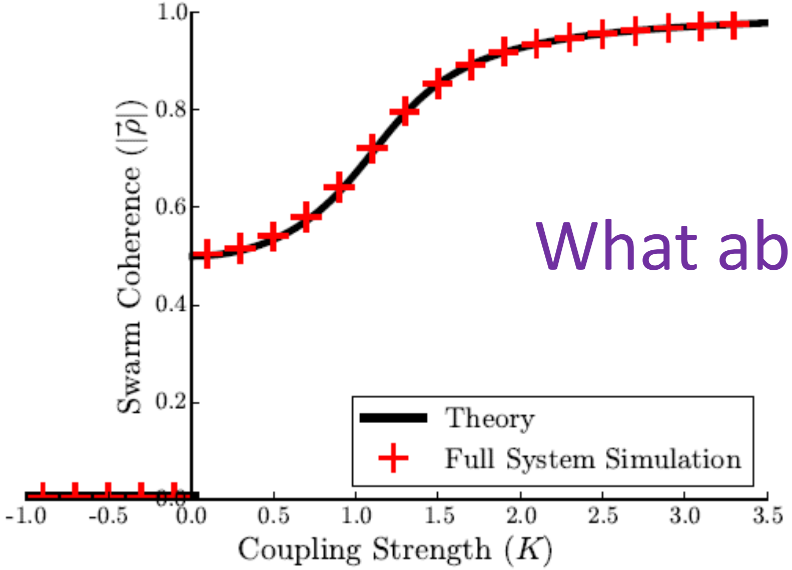
Theory Predicts Numerical Results Well

- We can derive a theory for the magnitude of coherence as a function of K based on arguments of fixed points for the agents



What about $D > 3$?

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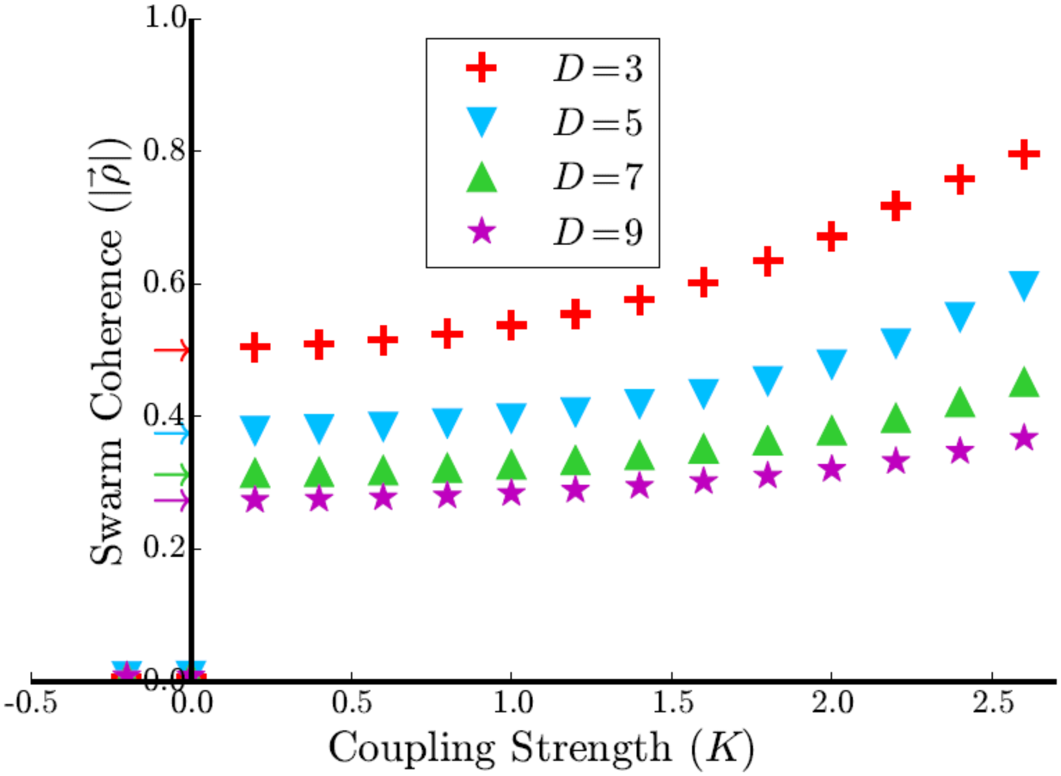
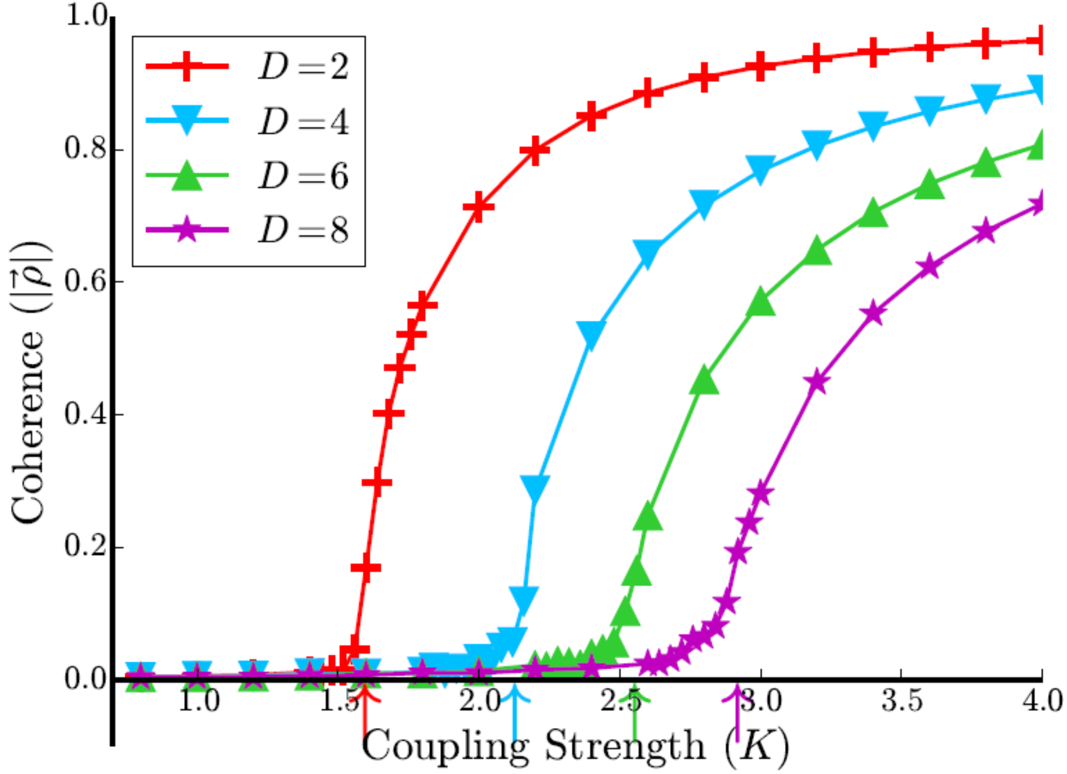
Discontinuities are Characteristic of Odd D



Even D appear similar to $D = 2$



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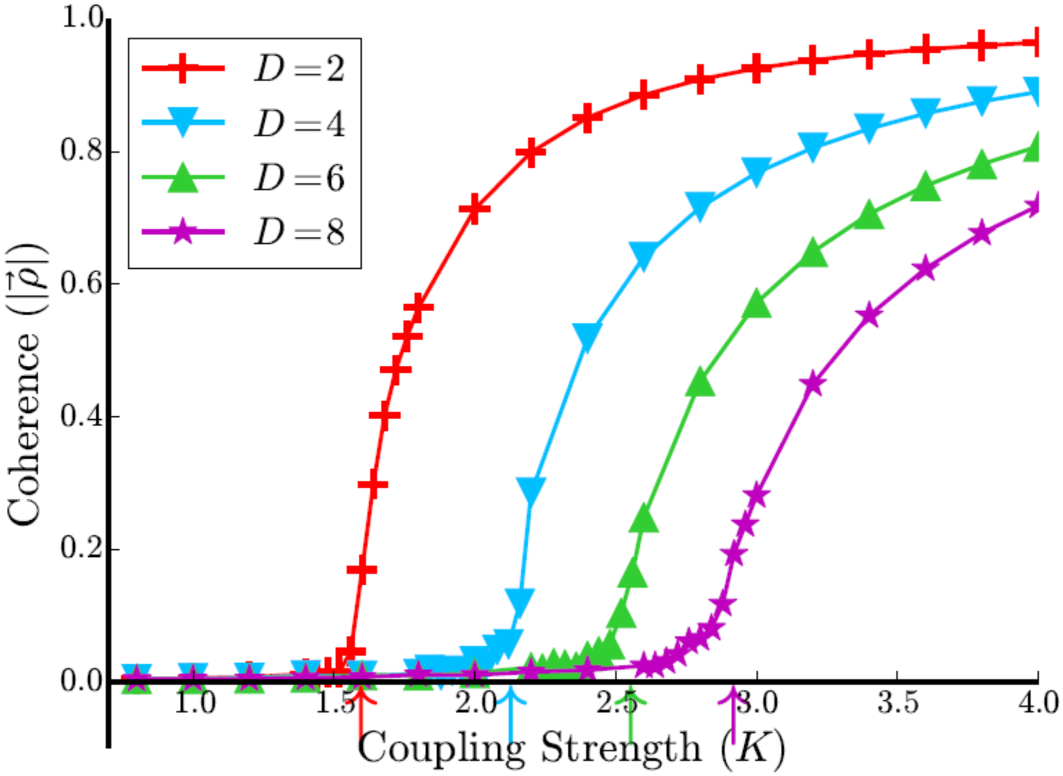
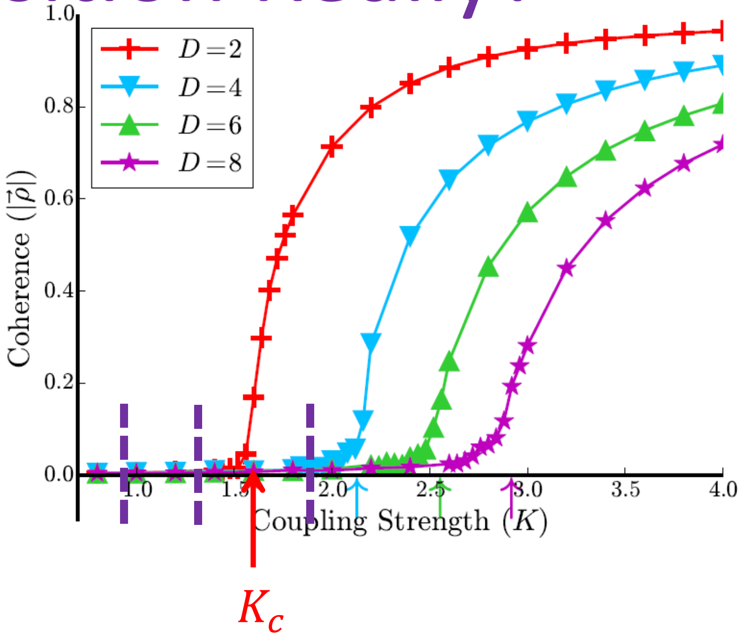
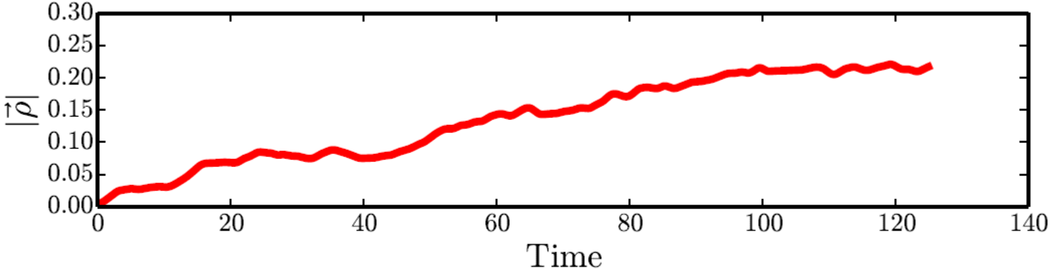


What is a Phase Transition Really?

- Initialize a $|\vec{\rho}| = 0$ state with uniformly random $\overrightarrow{\sigma_i}$ on sphere
- Evolve system with a given K and note the final equilibrium value of $|\vec{\rho}|$



$D = 2$



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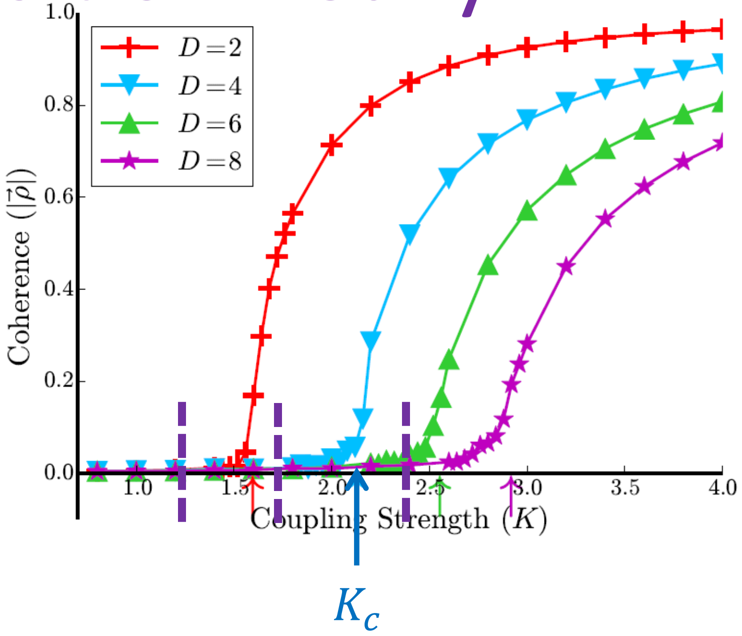
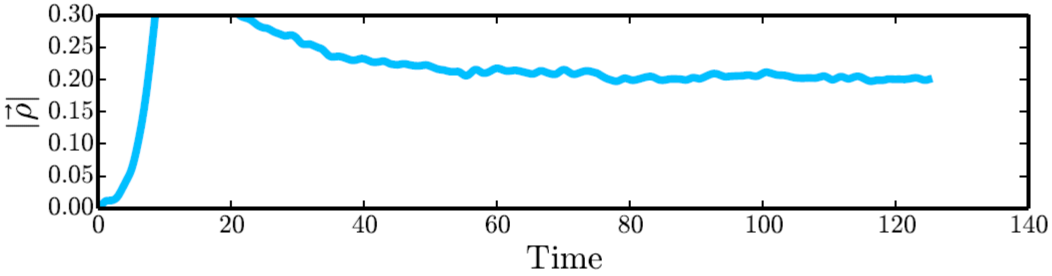


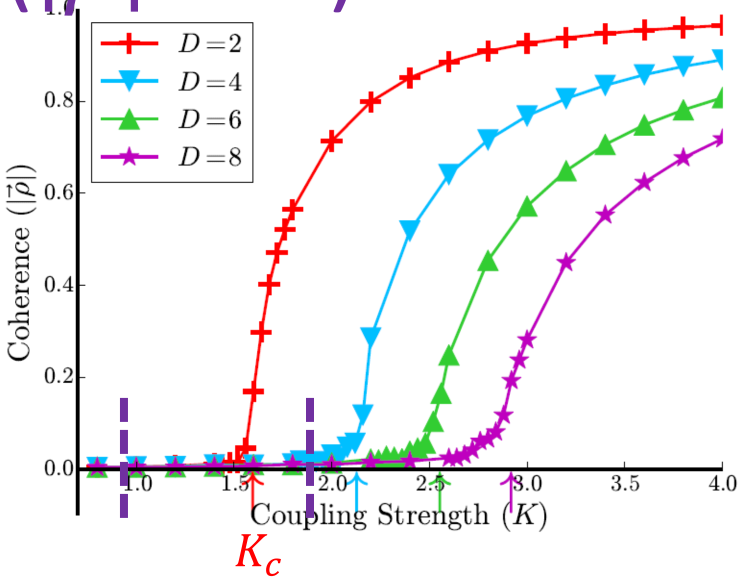
$D = 4$

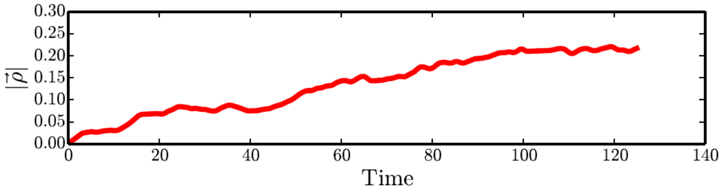


So what's going on?

The Tale of Incoherent ($|\vec{\rho}| = 0$) States

- In $D = 2$,
  - For $K < K_c$ single stable $|\vec{\rho}| = 0$ state
  - $|\vec{\rho}| = 0$ state loses stability at K_c
  - Stable $|\vec{\rho}| > 0$ state only for $K > K_c$





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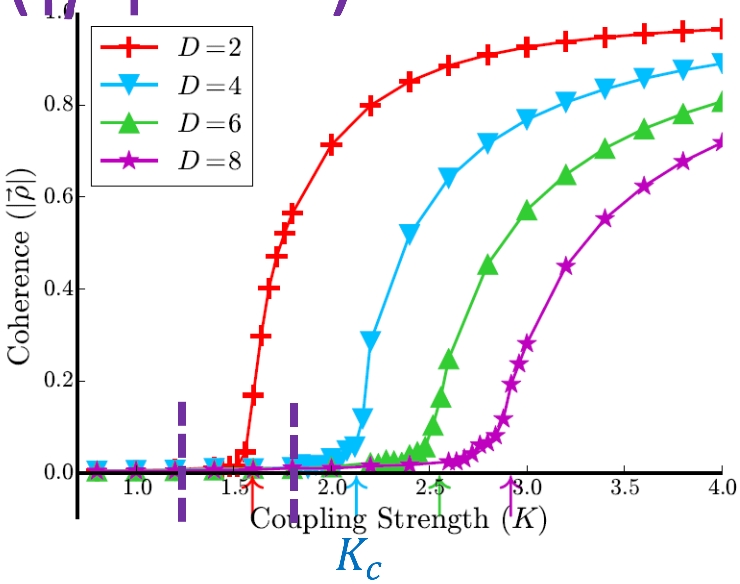
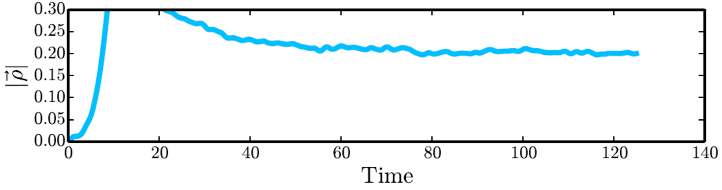
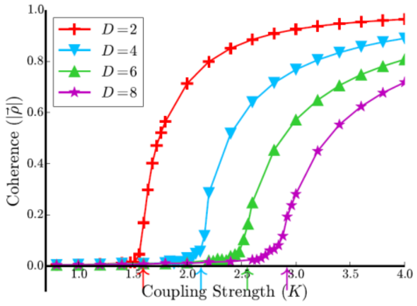
- In even $D > 2$,
  - For $K < K_c$ multiple $|\vec{\rho}| = 0$ states
  - Each loses stability at different $K < K_c$
  - When a state loses it's stability the system is pushed away from $|\vec{\rho}| = 0$, only to fall back to *another* $|\vec{\rho}| = 0$ state which is stable for that K
  - All of them lose stability by K_c
  - Stable $|\vec{\rho}| > 0$ state only for $K > K_c$





The Tale of Incoherent States: Summary

- In even dimensions larger than $D = 2$ there are multiple (an infinite number!) stable incoherent ($|\vec{\rho}| = 0$) states
- Despite remaining in the regime $K < K_c$ there are multiple transitions among incoherent states which leave *signatures in <u>transient dynamics</u>*
- After transition, the new incoherent state is stable for the given value of K
- Macroscopic phase transition to coherent state ($|\vec{\rho}| > 0$) occurs at K_c



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Infinite Size Limit?

Infinite Size Limit?

$$\partial_t \overrightarrow{\sigma_i} = \frac{K}{N}\sum_{j=1}^{N}\left(\overrightarrow{\sigma_j} - (\overrightarrow{\sigma_j}\cdot\overrightarrow{\sigma_i})\overrightarrow{\sigma_i}\right) + \boldsymbol{W}_i\overrightarrow{\sigma_i}$$

- Most cases of interest have $N \gg 1$
- Methods to analyze the system?
- Assume that the agents can be represented by a distribution $f(\vec{\sigma}, \boldsymbol{W}, t)$
- Represent dynamics of individual agents as flow of this distribution

$$\frac{\partial f}{\partial t} + \nabla_S \cdot (f(\vec{\sigma}, \boldsymbol{W}, t)\partial_t \vec{\sigma}) = 0$$

Ott-Antonsen Ansatz

- Ott & Antonsen (2008) derived a method to analyze the *two-dimensional* Kuramoto model in the infinite size limit
- We use a generalization of their method to arbitrary dimensions

E. Ott, T.M. Antonsen Chaos 2008

Generalized Ott-Antonsen Ansatz

$$\frac{\partial f}{\partial t} + \nabla_S \cdot \left(f(\vec{\sigma}, \boldsymbol{W}, t)\partial_t \vec{\sigma}\right) = 0$$

Assume the following form for $f(\vec{\sigma}, \boldsymbol{W}, t)$

$$f(\vec{\sigma}, \boldsymbol{W}, t) = C_D \frac{\left(1 - |\vec{\alpha}(\boldsymbol{W}, t)|^2\right)^{D-1}}{|\vec{\sigma} - \vec{\alpha}(\boldsymbol{W}, t)|^{2(D-1)}}$$

- The assumed form is shown to describe an invariant manifold, which is numerically found to be attracting
- Obtain a reduced set of equations for $\vec{\alpha}(\boldsymbol{W}, t)$

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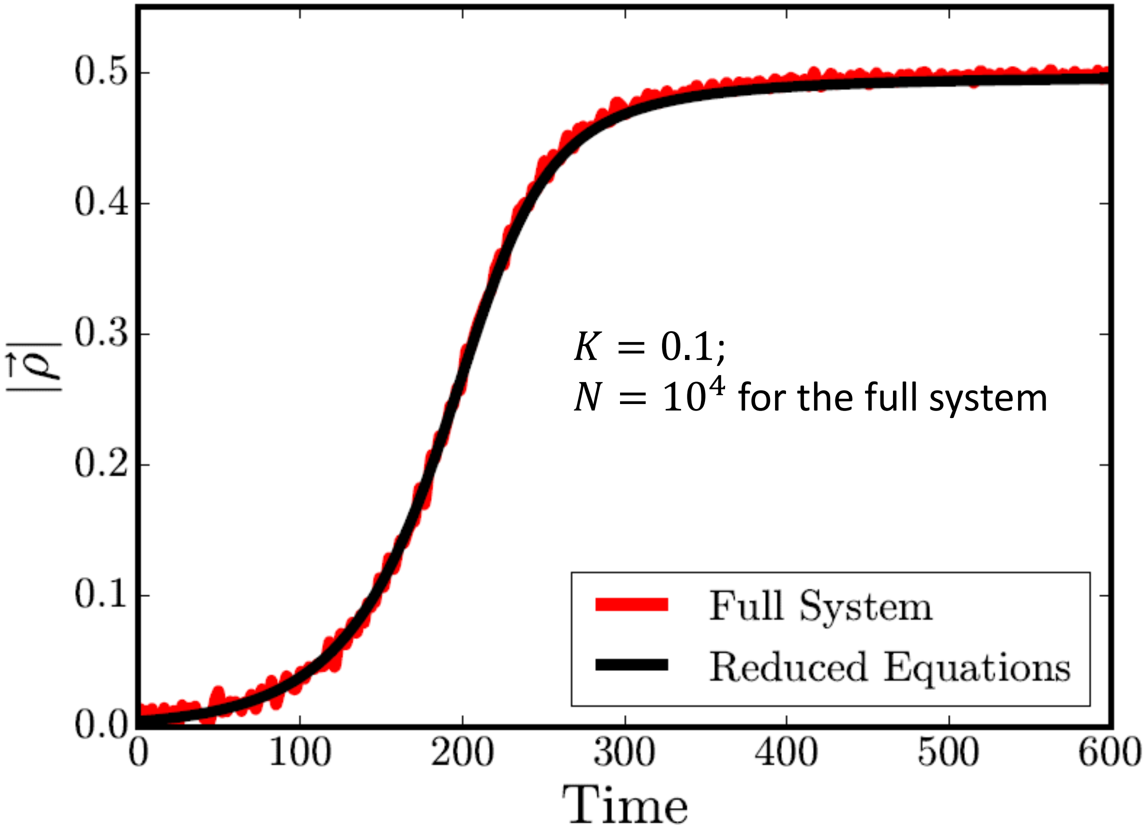
Reduced Equations Fit Numerics Well



Generalized Ott-Antonsen Ansatz

- What else does the Generalized Ott-Antonsen Ansatz apply to?
  - Communities of interacting agents
  - Network based coupling
  - Generalizations of related models (e.g., Kuramoto-Sakaguchi model, time-delayed Kuramoto model, etc.)

  - Very large class of problems involving interacting agents in D dimensions

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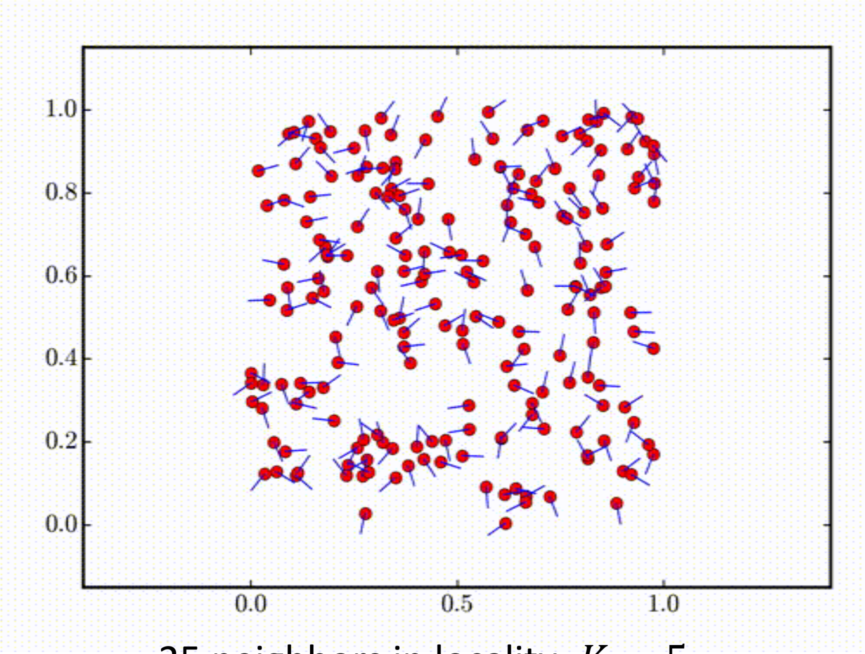
Do These Results Actually Apply Anywhere?

- We have unexpected results, but do they hold closer to application?
- Let's take a brief look at the collective behavior of animals:
  - Agents move in the direction of their $\vec{\sigma_i}$
  - Coupling is only local

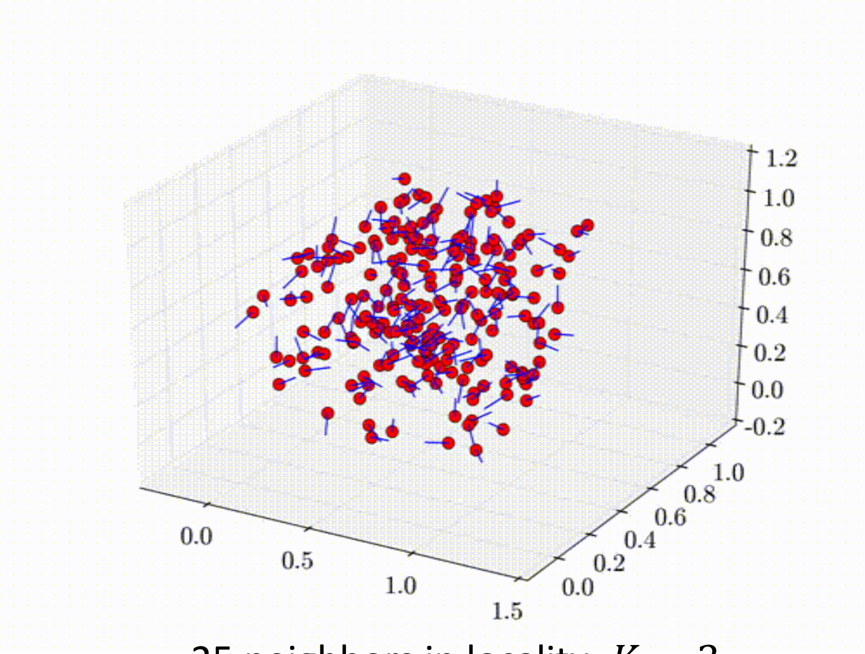
Herding and Flocking Animals

$D = 2$



25 neighbors in locality; $K = 5$

$D = 3$

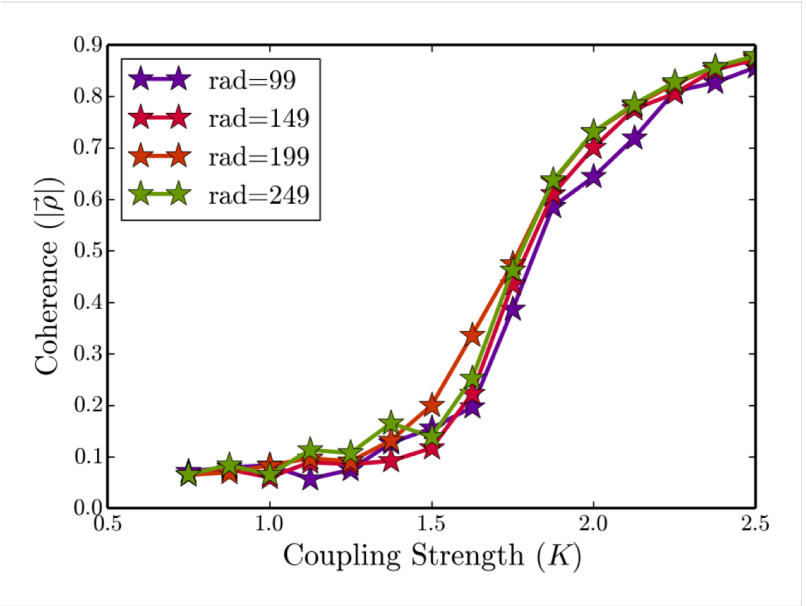


25 neighbors in locality; $K = 3$

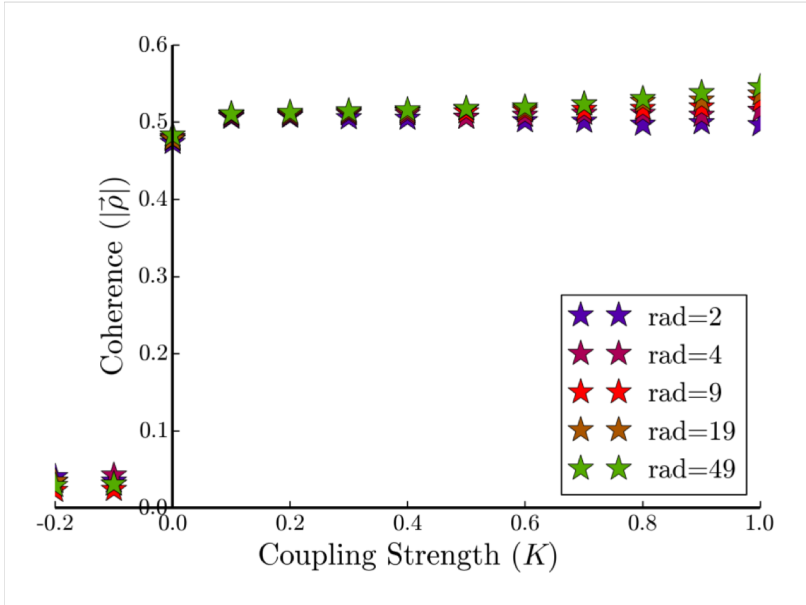
Herding and Flocking Animals

Still continue to see different behavior in odd and even dimensions!

$D = 2$



$D = 3$



(rad indicates the number of neighbors used for locality)

Conclusions

- The Kuramoto model shows remarkably different behavior in different dimensions
- The Kuramoto model in **odd dimensions** shows a discontinuous, non-hysteretic phase transition at a critical coupling $K_c = 0$
- The Kuramoto model in **even dimensions** appears to demonstrate similar behavior to the standard Kuramoto model; However, it has remarkably rich dynamics hidden in it's *transient* dynamics
- Ott-Antonsen methods generalize to higher dimensions
- The Generalized Kuramoto model gives good intuition for systems with additional complexities

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