Linearization and Domain Decomposition Methods for Non-Equilibrium Two-Phase Flow in Porous Media

Stephan Benjamin Lunowa[⋆], Iuliu Sorin Pop[⋆] and Barry Koren[⋄]

[⋆] Hasselt University (BE) [⋄] Eindhoven University of Technology (NL)

SIAM Geosciences, Houston, March 2019

UHASSELT
KNOWLEDGE IN ACTION

CMAT

Motivation

Flow in heterogeneous porous media

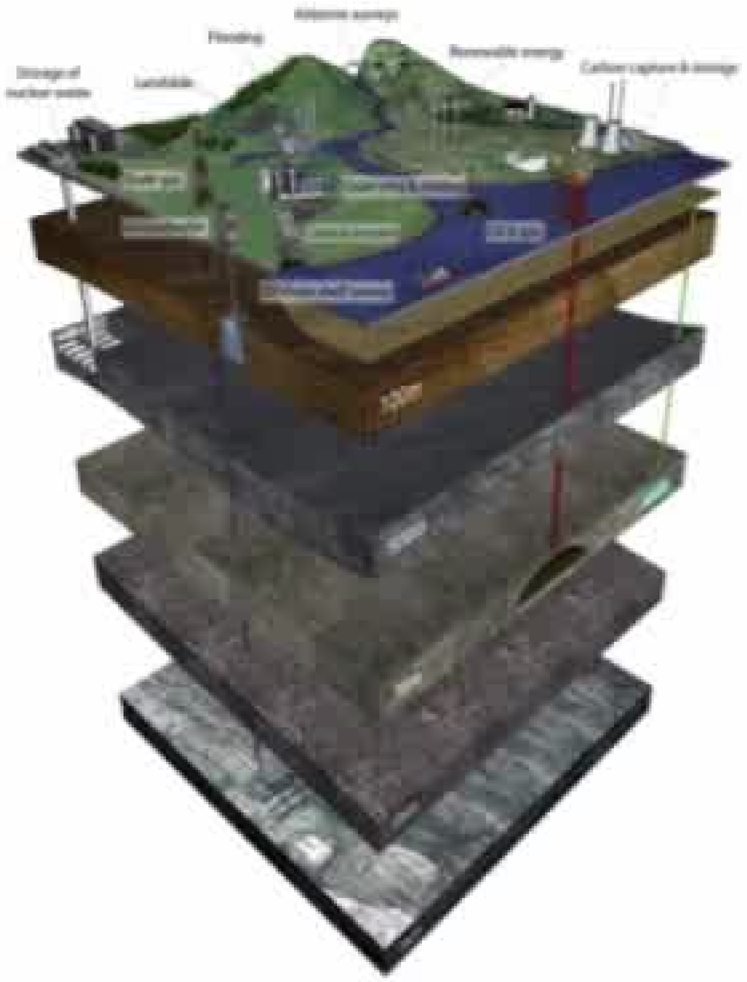
- (Ground)water pollution
- Enhanced oil recovery

- Highly nonlinear systems of equations

  ⇒ **Robust linearization methods**

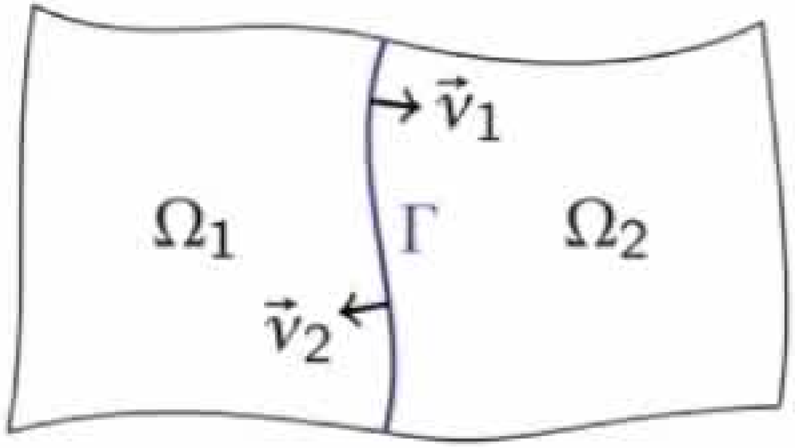
- Variations and jumps in the parameters

  ⇒ **Domain decomposition methods**



British Geological Survey
`https://www.bgs.ac.uk/research/images/manageTheSubsurface.jpg`

Two-Phase Flow Modelling (Darcy scale)



Variables $(\alpha = n, w)$

Saturations s_α, pressures p_α, velocities u_α

Physical relations in each subdomain

$$s_n + s_w = 1$$

Mass conservation $$\partial_t s_\alpha + \nabla \cdot \vec{u}_\alpha = 0$$

Extended Darcy's law $$\vec{u}_\alpha = -\lambda_\alpha(s_\alpha) K \nabla p_\alpha$$

Generalized capillary pressure $$p_n - p_w = F(s_w, \partial_t s_w)$$

Interface conditions at Γ

Normal flux continuity $$\sum_{l=1}^{2} \vec{u}_{\alpha,l} \cdot \vec{v}_l = 0$$

Pressure continuity $$p_{\alpha,1} = p_{\alpha,2}$$

Linearization and Domain Decomposition Method

θ-scheme + Linearization & Stabilization

$$-\frac{s_l^{k,i}-s_l^{k-1}}{\Delta t} + \nabla\cdot\left(\theta\vec{u}_{n,l}^{k,i} + (1-\theta)\vec{u}_{n,l}^{k-1}\right) = -\mathscr{L}_{p,l}\left(p_{n,l}^{k,i} - p_{n,l}^{k,i-1} - p_{w,l}^{k,i} + p_{w,l}^{k,i-1}\right)$$

$$\frac{s_l^{k,i}-s_l^{k-1}}{\Delta t} + \nabla\cdot\left(\theta\vec{u}_{w,l}^{k,i} + (1-\theta)\vec{u}_{w,l}^{k-1}\right) = \mathscr{L}_{p,l}\left(p_{n,l}^{k,i} - p_{n,l}^{k,i-1} - p_{w,l}^{k,i} + p_{w,l}^{k,i-1}\right)$$

$$\vec{u}_{\alpha,l}^{k,i} = -\lambda_{\alpha,l}(s_l^{k,i})K_l\nabla p_{\alpha,l}^{k,i}$$

$$\frac{s_l^{k,i}-s_l^{k-1}}{\Delta t} = \theta\Psi_l\left(p_{n,l}^{k,i-1} - p_{w,l}^{k,i-1} - p_c(s_l^{k,i})\right) + (1-\theta)\Psi_l^{k-1}$$

Interface conditions + Reformulation + Domain decomposition[4]

$$g_{\alpha,l}^{i} = \vec{u}_{\alpha,l}^{k,i}\cdot\vec{\nu}_l - \mathscr{L}_\Gamma p_{\alpha,l}^{k,i}$$

$$g_{\alpha,l}^{i} = -2\mathscr{L}_\Gamma p_{\alpha,3-l}^{k,i-1} - g_{\alpha,3-l}^{i-1}$$

---

[4] Lions, 1988, 1st international symposium on DD methods for PDE

Existence and Convergence Analysis

Theorem (Existence, uniqueness and convergence of the solutions)

Under some mild assumptions, and if

$$\mathscr{L}_{p,l} > 2\theta L_{\Psi,l}, \qquad \mathscr{L}_{\Gamma} > 0, \qquad \Delta t \leq C, \qquad (C \text{ mesh independent})$$

the LDD-scheme has a unique sequences solutions, which converge, for any initial guesses.

- Fixed point argument
- Series of error norms is bounded

Numerical Experiments

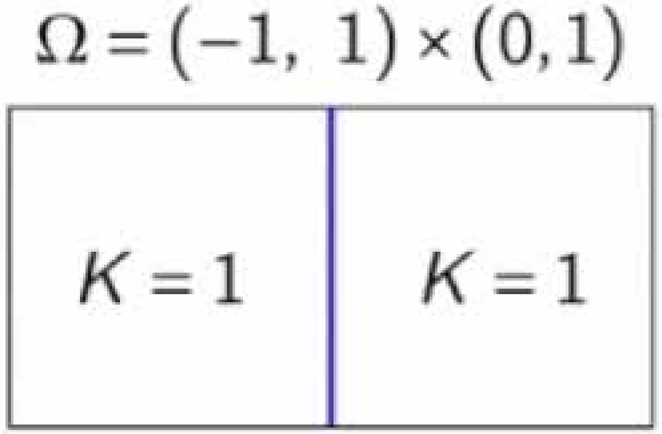
- Crank-Nicolson method in time
- Q-FEM with uniform mesh in space

Academic test case

- Linear coefficients

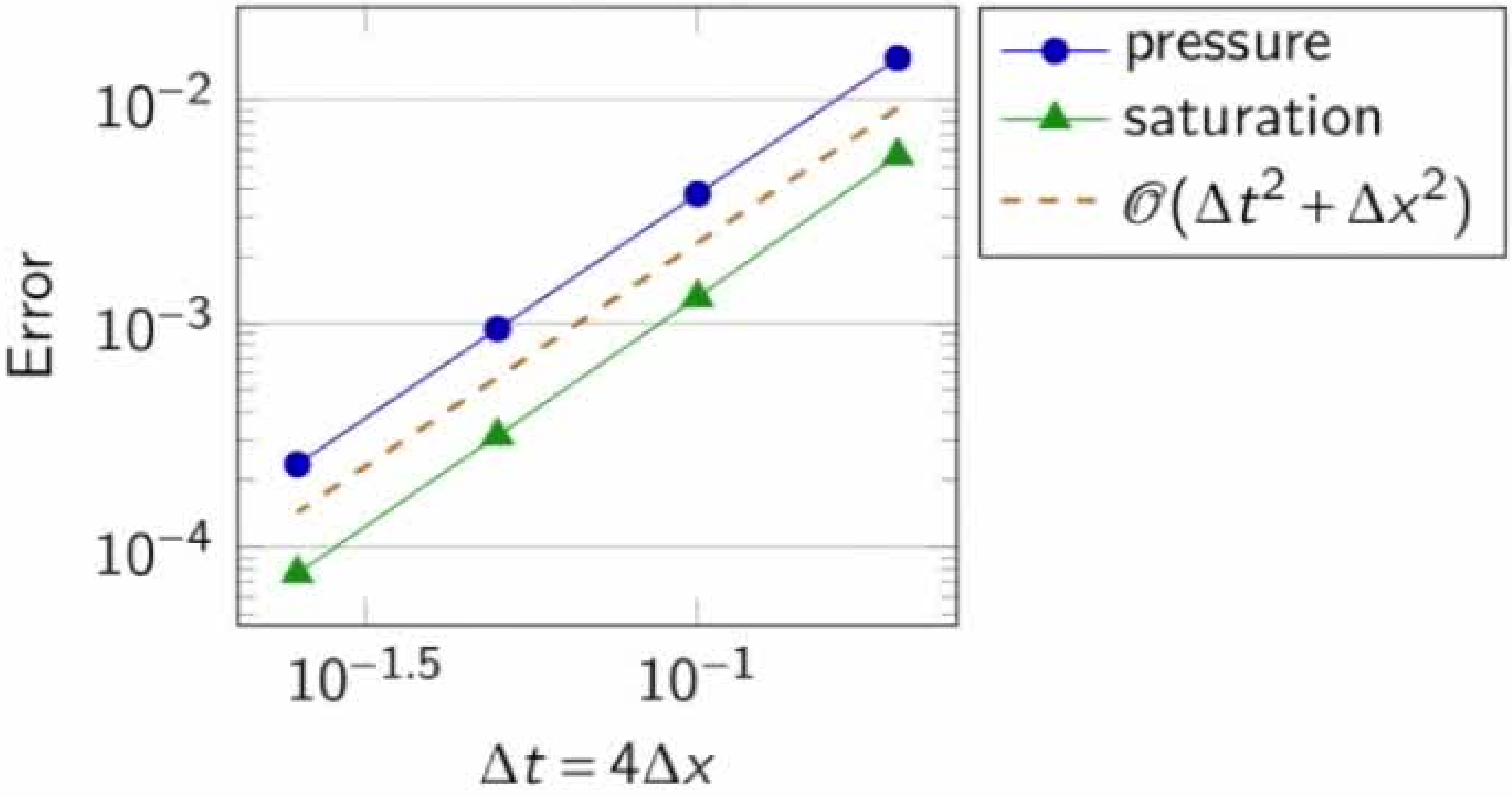
$$\lambda_n(s) = 1 - s, \qquad \lambda_w(s) = s,$$
$$p_c(s) = -s, \qquad \tau = 1, \qquad \gamma = 1$$

- Solution models imbibition and drainage cycle

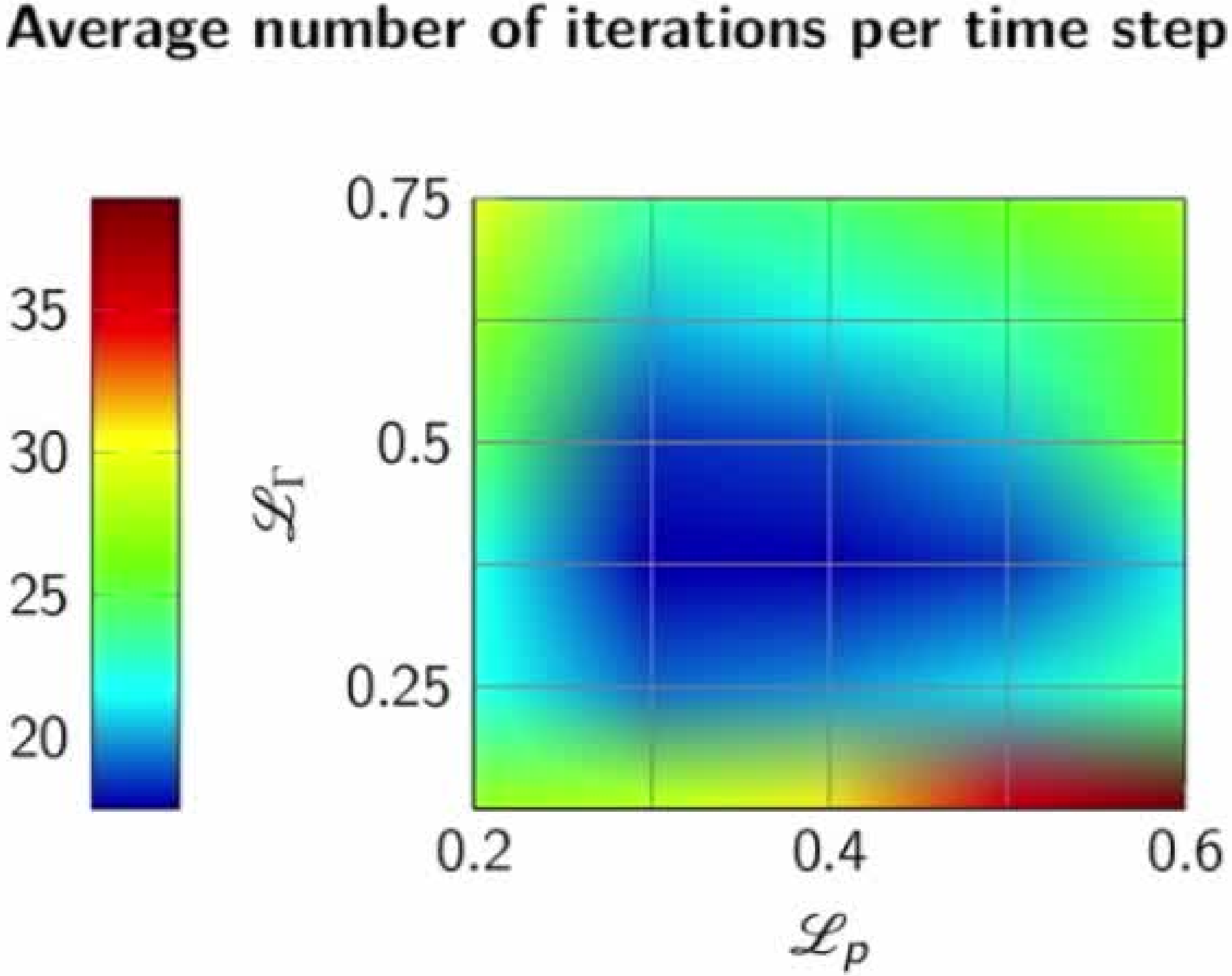

$\Omega = (-1,\ 1) \times (0,1)$

Academic Test Case

Convergence study in $L^2(0, T; H^1(\Omega))$

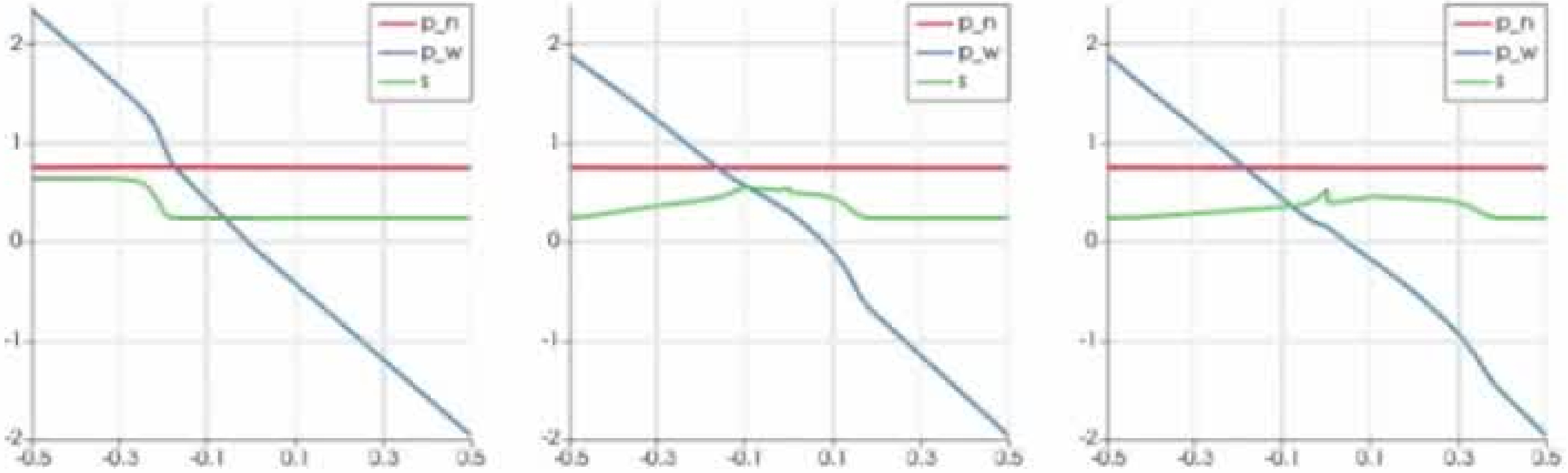


Academic Test Case



Realistic Test Case

Solution ($x_2 = 0.5$)



$t = 60s$ $t = 150s$ $t = 240s$

$\mathscr{L}_p$	$\mathscr{L}_\Gamma$	Avg.-Iter.	Time (h)
1.0	0.5	123.6	3.78
1.0	1.5	214.4	6.06

Discussion

- Linearization & domain decomposition scheme for non-equilibrium models of two-phase flow
- Rigorous proof of global convergence
- Mild restriction on the time step
- More than 2 subdomains possible

- Parameter optimization, other schemes
- Generalized models & interface conditions (degeneration, trapping)

Work based on

- S. B. Lunowa, 2018, Master thesis, *Linearization and domain decomposition methods for two-phase flow in porous media*, `https://research.tue.nl/en/studentTheses/linearization-and-domain-decomposition-methods`

Thanks

fwo Research Foundation Flanders
Opening new horizons

FACULTY OF SCIENCES
UHASSELT

Studienstiftung
des deutschen Volkes

Baden-Württemberg Stiftung
WIR STIFTEN ZUKUNFT

TU/e
EINDHOVEN
UNIVERSITY OF
TECHNOLOGY

Discussion

- Linearization & domain decomposition scheme for non-equilibrium models of two-phase flow
- Rigorous proof of global convergence
- Mild restriction on the time step
- More than 2 subdomains possible

- Parameter optimization, other schemes
- Generalized models & interface conditions (degeneration, trapping)

Work based on

- S. B. Lunowa, 2018, Master thesis,
  Linearization and domain decomposition methods for two-phase flow in porous media,
  `https://research.tue.nl/en/studentTheses/linearization-and-domain-decomposition-methods`