Non-linearizable Wave Equation: Nonlinear Sonic Vacuum

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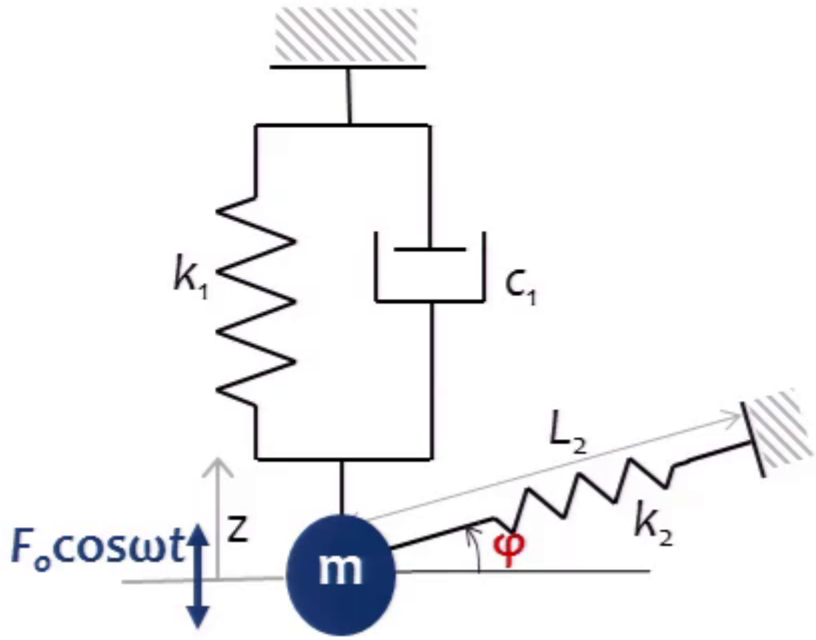
Geometric Nonlinearity



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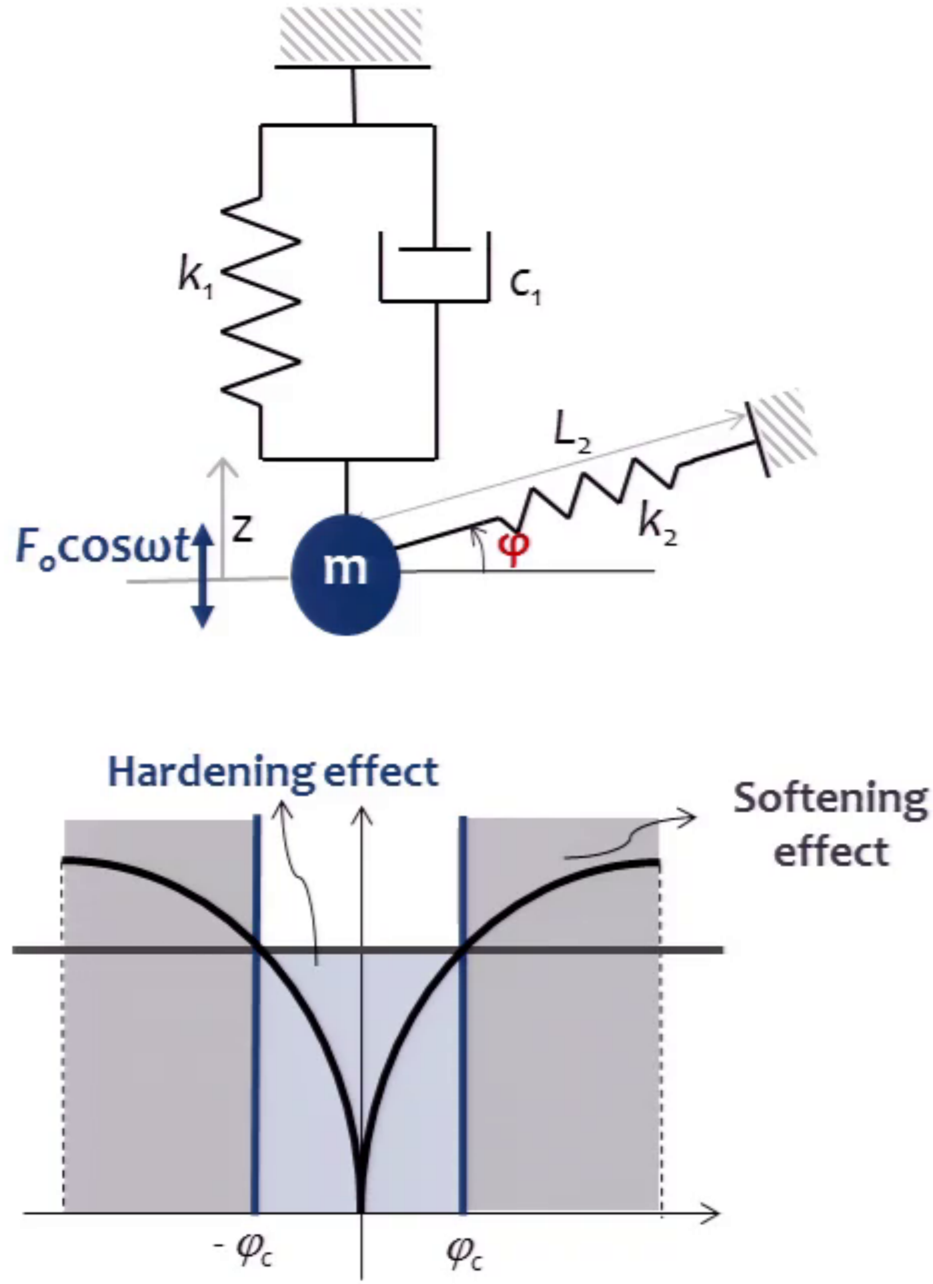

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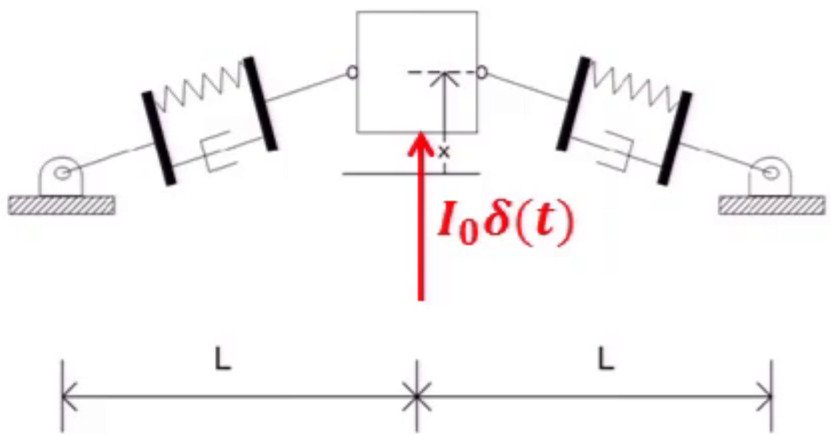

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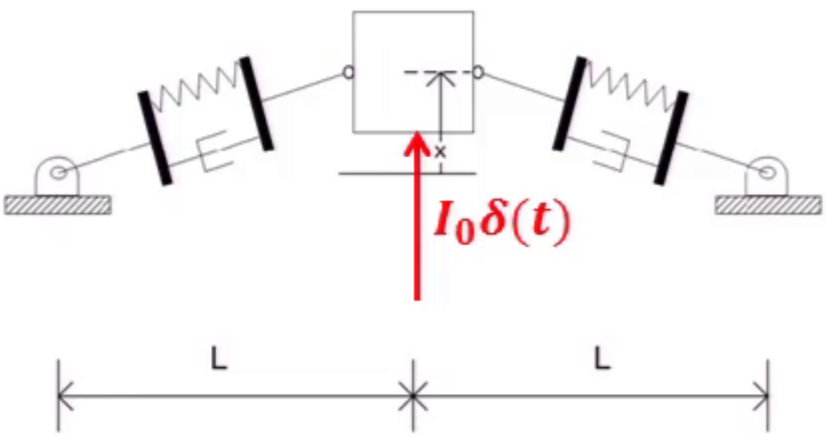
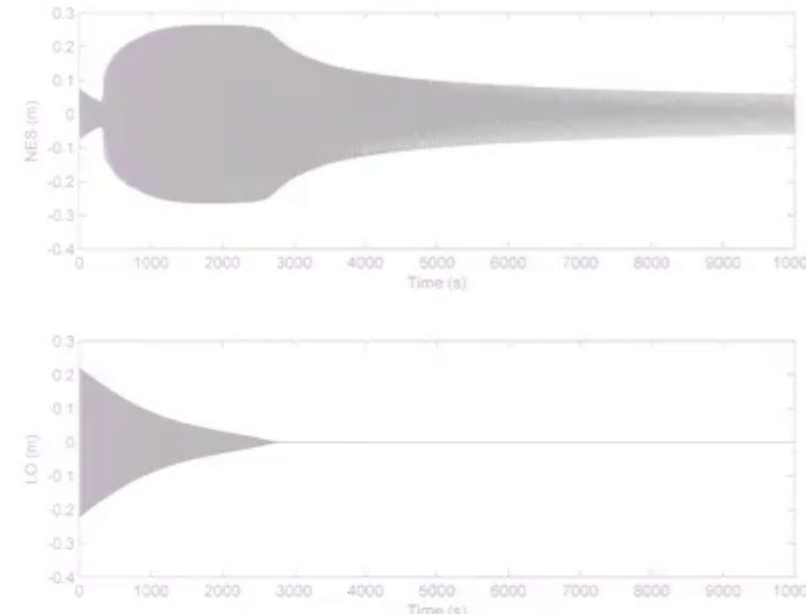


$I_0 = 0.65$



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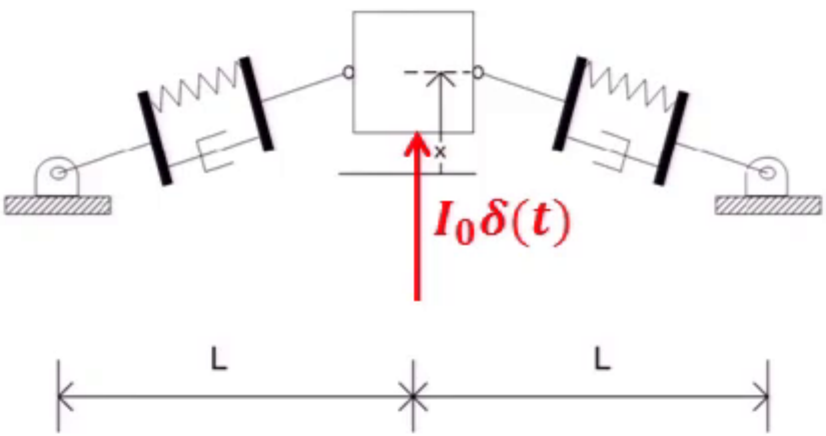
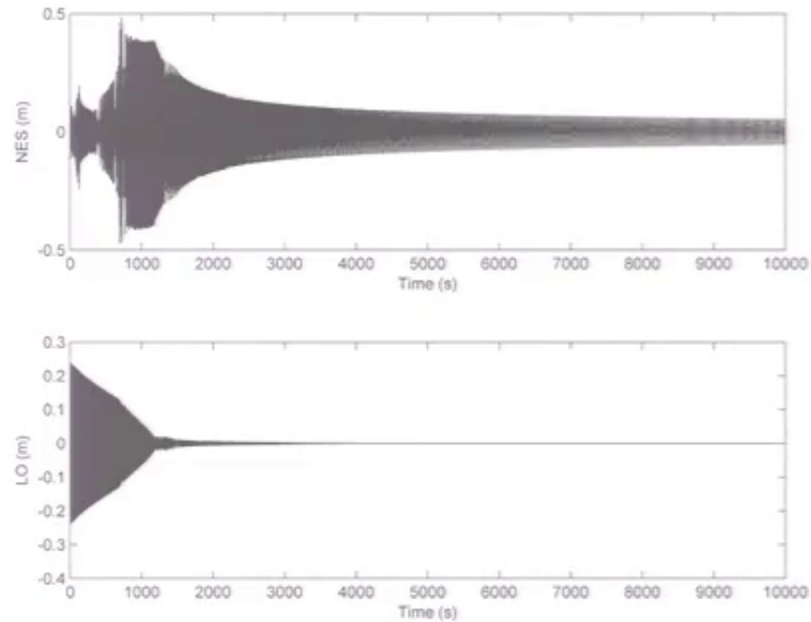


$I_0 = 0.70$



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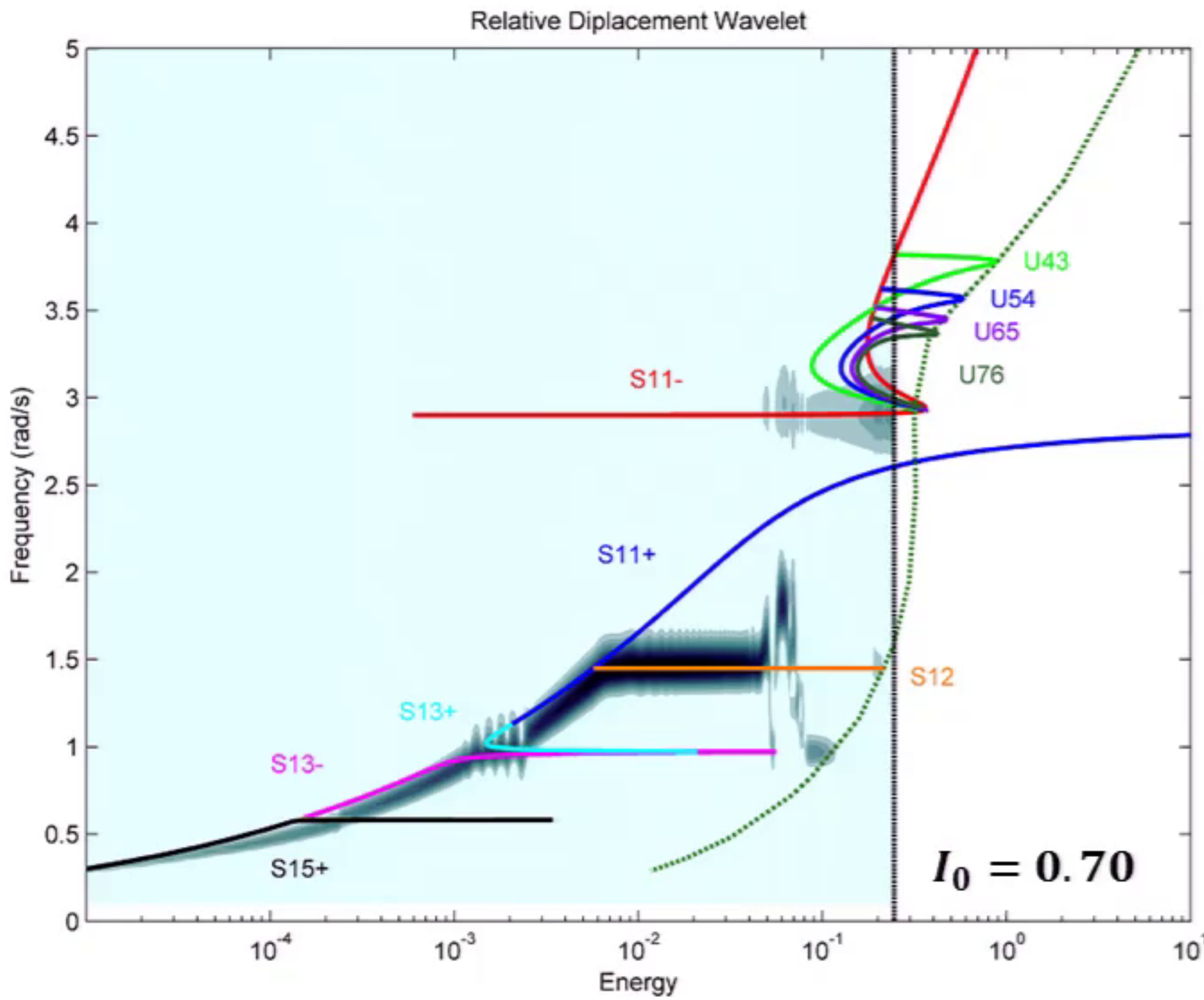
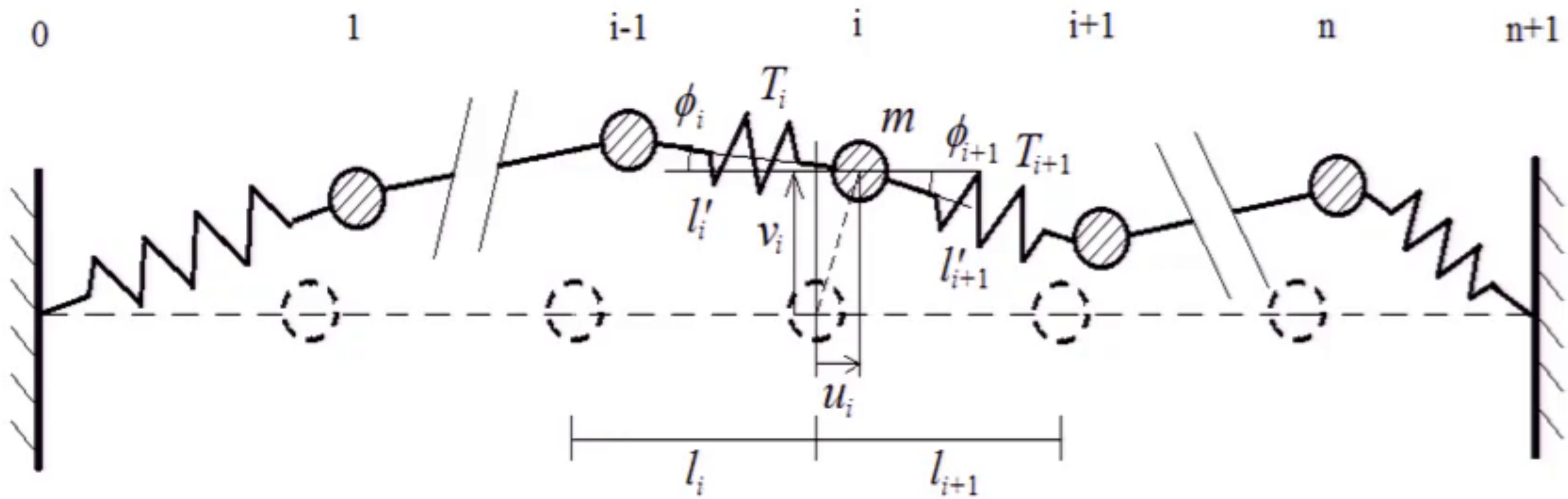


Lattice

Finite lattice of particles oscillating in the plane

Next-neighbor interactions through linear springs, $l_i = l = 1/(n+1)$



$$m\ddot{u}_i + T_i cos\phi_i - T_{i+1} cos\phi_{i+1} = 0$$
$$m\ddot{v}_i + T_i sin\phi_i - T_{i+1} sin\phi_{i+1} = 0$$

Lattice

In the limit of small-energy oscillations and small angles, we expand the geometrically nonlinear terms in Taylor series in terms of the small differences $(u_i - u_{i-1})$ and $(v_i - v_{i-1})$ to get,

$$T_i = k l_i \varepsilon_i \ , \quad \varepsilon_i = \frac{1}{l_i}\left[(u_i - u_{i-1}) + \frac{1}{2l_i}(v_i - v_{i-1})^2\right] + \cdots$$

$$cos\phi_i = 1 + \mathcal{O}[(u_i - u_{i-1})^2, (v_i - v_{i-1})^2]$$

$$sin\phi_i = \frac{1}{l_i}(v_i - v_{i-1}) + \mathcal{O}[(u_i - u_{i-1}), (v_i - v_{i-1})]$$

Finally we introduce the re-scalings,

$$\tau = \varepsilon \left(\frac{k}{m}\right)^{1/2} t, \qquad u_i \longrightarrow \frac{\varepsilon^2 u_i}{l}, \qquad v_i \longrightarrow \frac{\varepsilon v_i}{l}$$

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Lattice

Then the equations governing the axial oscillations are,

$$\varepsilon^2 u_i'' - (u_{i+1} + u_{i-1} - 2u_i) - \frac{1}{2}(v_{i+1} - v_i)^2 + \frac{1}{2}(v_i - v_{i-1})^2 + \cdots = 0, \quad u_0 = u_{n+1} \equiv 0$$

or in terms of the re-scaled axial tensions,

$$\varepsilon^2 u_i'' + \bar{T}_i - \bar{T}_{i+1} + \cdots = 0$$

The equations governing the transverse oscillations are:

$$v_i'' - (u_{i+1} - u_i)(v_{i+1} - v_i) - \frac{1}{2}(v_{i+1} - v_i)^3 + (u_i - u_{i-1})(v_i - v_{i-1}) + \frac{1}{2}(v_i - v_{i-1})^3 + \cdots = 0,$$

$$v_0 = v_{n+1} \equiv 0$$

This permits the partition of the axial dynamics in terms of slow and fast parts and the asymptotic treatment of the dynamics.

Lattice

At the leading order slow approximation with $\varepsilon = 0$ we neglect the axial inertia effects to get $\bar{T}_1 = \bar{T}_2 = \cdots = \bar{T}_{n+1} \equiv \bar{T}$. Since,

$$\bar{T} = \frac{1}{n+1} \sum_{p=1}^{n+1} \bar{T}_p = \frac{1}{2(n+1)} \sum_{q=0}^{n} \left(v_{q+1} - v_q\right)^2$$

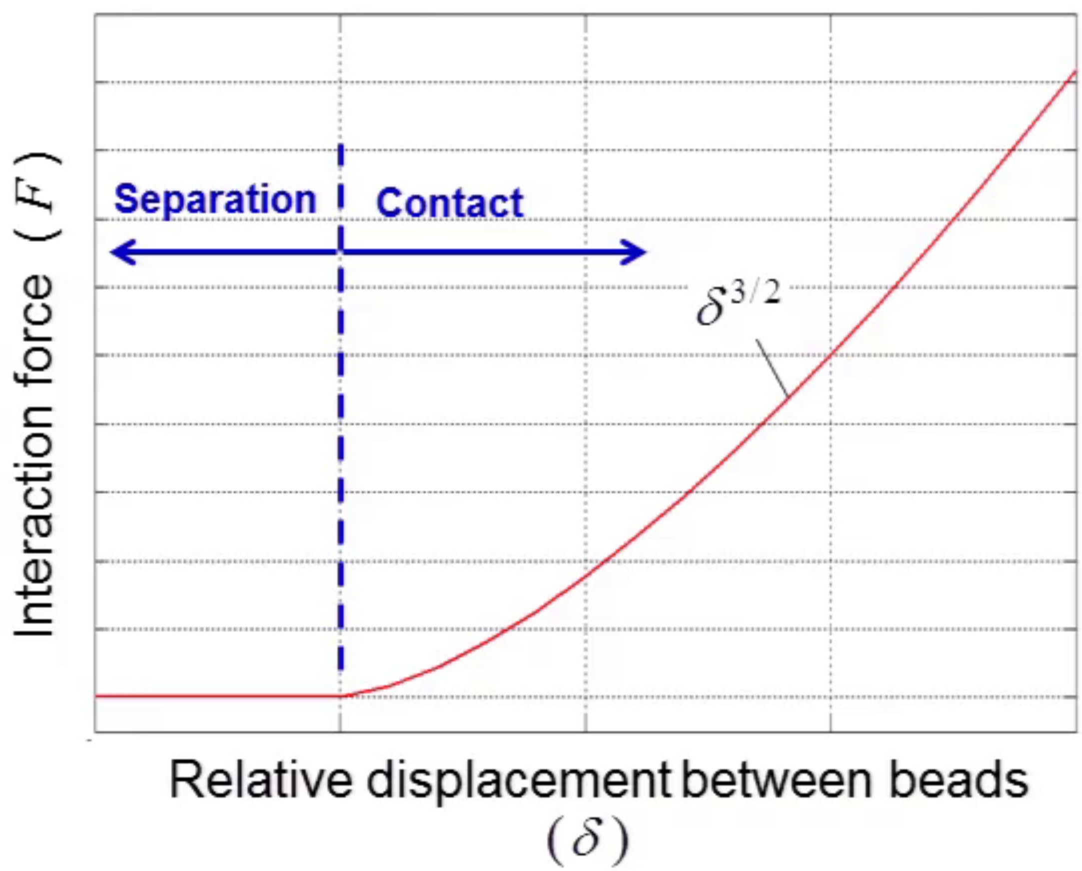
at leading order the tension is constant in space but varies in time. When higher-order terms are taken into account the tension varies slowly in space.

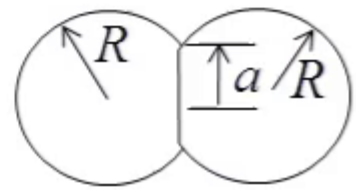
Then the leading-order approximation for the transverse oscillations is an essentially nonlinear sonic vacuum with strong non-locality:

$$v_i''(\tau) + \frac{1}{2(n+1)} \sum_{q=0}^{n} \left(v_{q+1} - v_q\right)^2 \left(2v_i(\tau) - v_{i+1}(\tau) - v_{i-1}(\tau)\right) + \cdots = 0, \qquad v_0 = v_{n+1} \equiv 0$$

Aside: Granular Sonic Vacuum

A different nonlinear sonic vacuum is found in the acoustics of ordered uncompressed granular media.



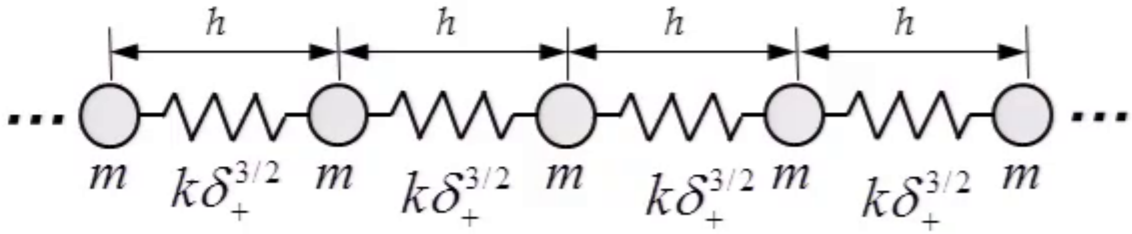


$$F \cong \begin{cases} k\,\delta^{3/2}, & \delta \geq 0 \\ 0, & \delta < 0 \end{cases} \Rightarrow$$

$$F \cong k\,\delta_{+}^{3/2}$$

Aside: Granular Sonic Vacuum

Considering a one-dim granular lattice



with equations of motion

$$m\ddot{u}_i = k\left[\left(u_{i-1} - u_i\right)_+^{3/2} - \left(u_i - u_{i+1}\right)_+^{3/2}\right]$$

oscillations or waves with wavelengths much longer than the lattice distance h (long wavelength approximation – LWA) correspond to:

$$\left.\begin{array}{c} u_i(t) \to u(i,t) \\ ih \to x \end{array}\right\} \Rightarrow \; u_i(t) \to u(x,t)$$

$$u_{i\pm1}(t) \to u\left[(i\pm1)h,t\right] = u(ih \pm h,t) =$$

$$u(x \pm h,t) = u(x,t) \pm h\frac{\partial u(x,t)}{\partial x} + \frac{h^2}{2}\frac{\partial^2 u(x,t)}{\partial x^2} \pm \ldots$$

Aside: Granular Sonic Vacuum

Correct up to O(h^3) obtain the essentially nonlinear wave equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{3kh^{5/2}}{2m}\left(-\frac{\partial u}{\partial x}\right)_+^{1/2}\left(\frac{\partial^2 u}{\partial x^2}\right) + \ldots$$

Compared to the classical wave equation,

$$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2}\right)$$

the LWA leads to an essentially nonlinear wave equation, with zero speed of sound (as defined in classical acoustics)

There is complete absence of linear acoustics ⇒ 'Sonic vacuum' (Nesterenko, 2001)

1:1 Resonance Interactions

Returning to the discrete nonlinear sonic vacuum, it can be proven that it has exactly n orthogonal NNMs with spatial dependencies identical to the normal modes of the linear lattice,

$$v_i'' + \frac{1}{2(n+1)}(2v_i - v_{i+1} - v_{i-1}) = 0, \quad v_0 = v_{n+1} = 0$$

and amplitudes governed by nonlinear modal oscillators,

$$C_p''(\tau) + \frac{1}{4}\omega_p^4 C_p^3(\tau) = 0, \quad \omega_p^2 = 2\left[1 - cos\left(\frac{p\pi}{n+1}\right)\right]$$

We wish to study 1:1 internal resonances between two arbitrary NNMs, say the $k-th$ and $p-th$ NNMs, and express the vector of transverse deformations of the lattice as,

$$\underline{v}(\tau) = C_k(\tau)\underline{\phi}_k + C_p(\tau)\underline{\phi}_p, \quad p, k \in [1, \dots, n]$$

This restricts the nonlinear dynamics of the lattice to the invariant manifold defined by the two NNMs.

1:1 Resonance Interactions

Using the orthogonality conditions of the NNMs we reduce the dynamics to the system of coupled oscillators,

$$C_k''(\tau) + \frac{1}{4}\left[\, C_k^2(\tau)\omega_k^2 + C_p^2(\tau)\omega_p^2 \,\right]\omega_k^2\, C_k(\tau) = 0$$

$$C_p''(\tau) + \frac{1}{4}\left[\, C_k^2(\tau)\omega_k^2 + C_p^2(\tau)\omega_p^2 \,\right]\omega_p^2\, C_p(\tau) = 0$$

or, by setting $A_i(\tau) = \ C_i(\tau)\sin\left(\frac{\pi i}{2(n+1)}\right)$, to:

$$A_k''(\tau) + \left[\, A_k^2(\tau) + A_p^2\,(\tau) \,\right] \omega_k^2\, A_k(\tau) = 0$$

$$A_p''(\tau) + \left[\, A_k^2(\tau) + A_p^2\,(\tau) \,\right] \omega_p^2\, A_p(\tau) = 0$$

To impose the condition of 1:1 resonance we assume that the frequencies of the two interacting NNMs are close. If their amplitudes are of the same order this implies that $\left(\omega_p^2 - \omega_k^2\right)/\omega_k^2 \equiv \varepsilon_1 \ll 1$, which introduces a small parameter in the problem.

1:1 Resonance Interactions

Then we introduce the transformations,

$$\psi_1(\tau) = A_k'(\tau) + j\Omega A_k(\tau) \equiv \varphi_1(\tau_1)e^{j\Omega\tau}$$

$$\psi_2(\tau) = A_p'(\tau) + j\Omega A_p(\tau) \equiv \varphi_2(\tau_1)e^{j\Omega\tau}, \quad \tau_1 = \varepsilon_1\tau$$

and perform slow/fast partitions of the dynamics on the 1:1 resonance manifold. Letting,

$$\varphi_i = a_i e^{j\beta_i},\ i = 1,2,$$

$$a_1 = \left(\frac{\rho}{\omega_p}\right) sin\theta,\ a_2 = \left(\frac{\rho}{\omega_k}\right) cos\theta,\ \ \Delta = \beta_2 - \beta_1$$

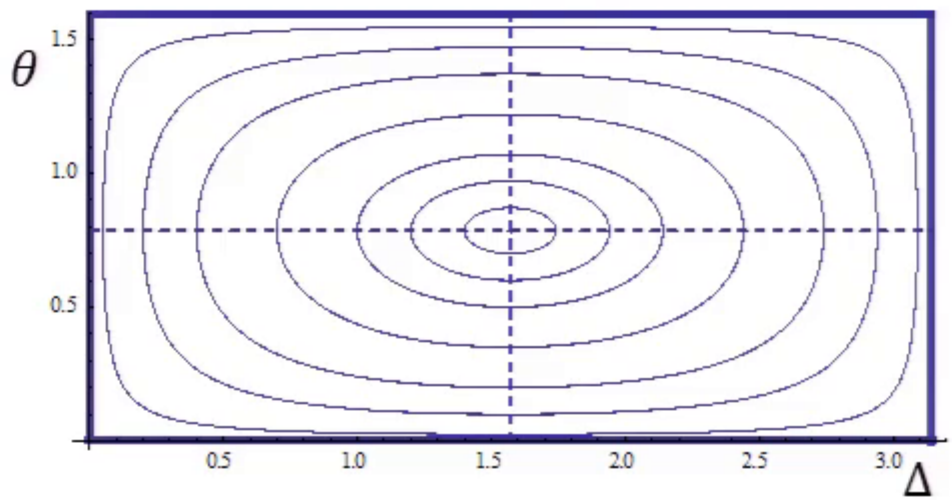
we reduce the dynamics on an iso-energetic two-torus $(\theta, \Delta) \in \left[0, \frac{\pi}{2}\right] \times [0, \pi]$.

Not all combinations of $(k, p) \in [n \times n]$ lead to nontrivial dynamics on the torus, so not all pairs of NNMs can engage in 1:1 resonance.

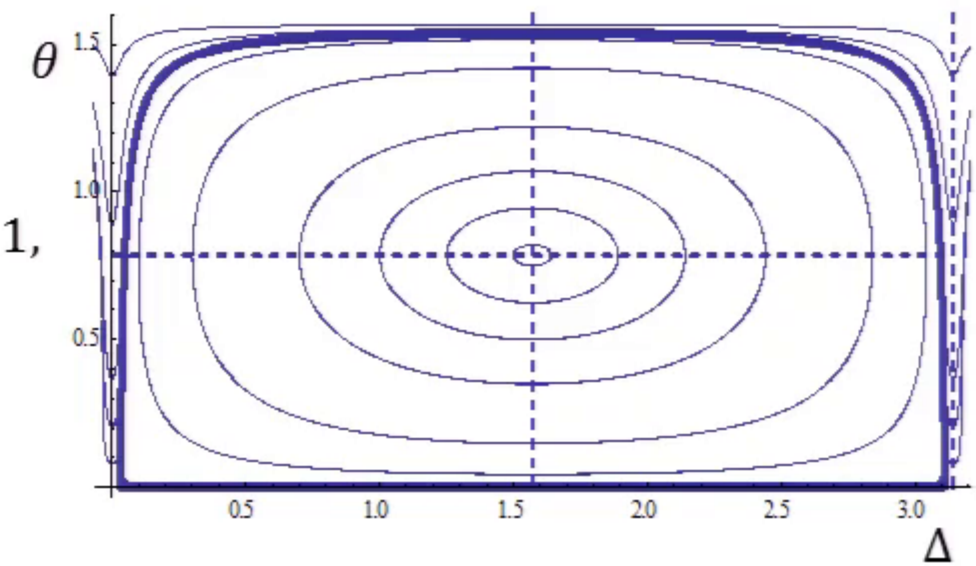
1:1 Resonance Interactions

For the two highest-order NNMs, $k = n - 1$ and $p = n$, the dynamics on the torus are shown below.

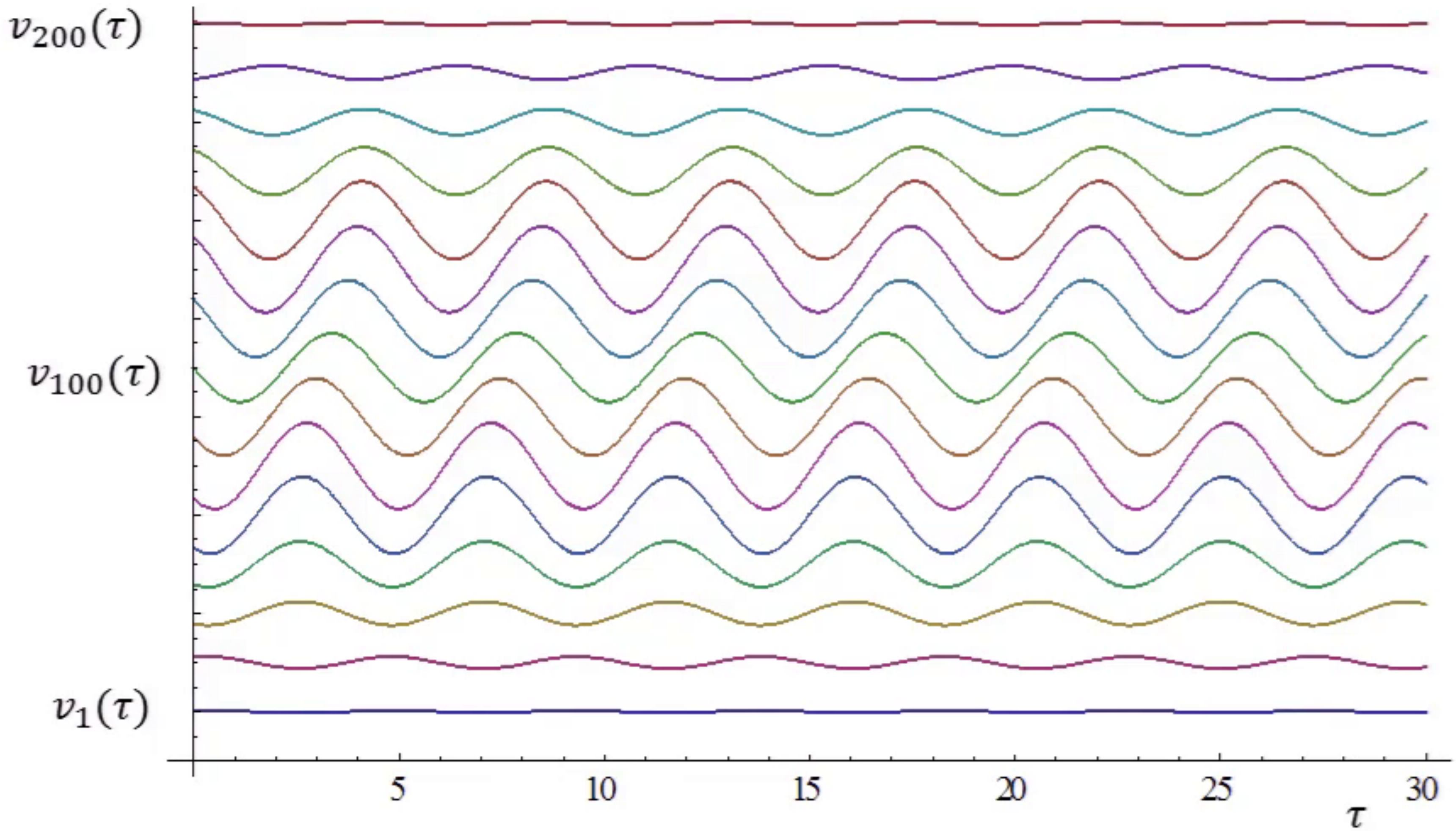
$\varepsilon_1 = 0, n \longrightarrow \infty$



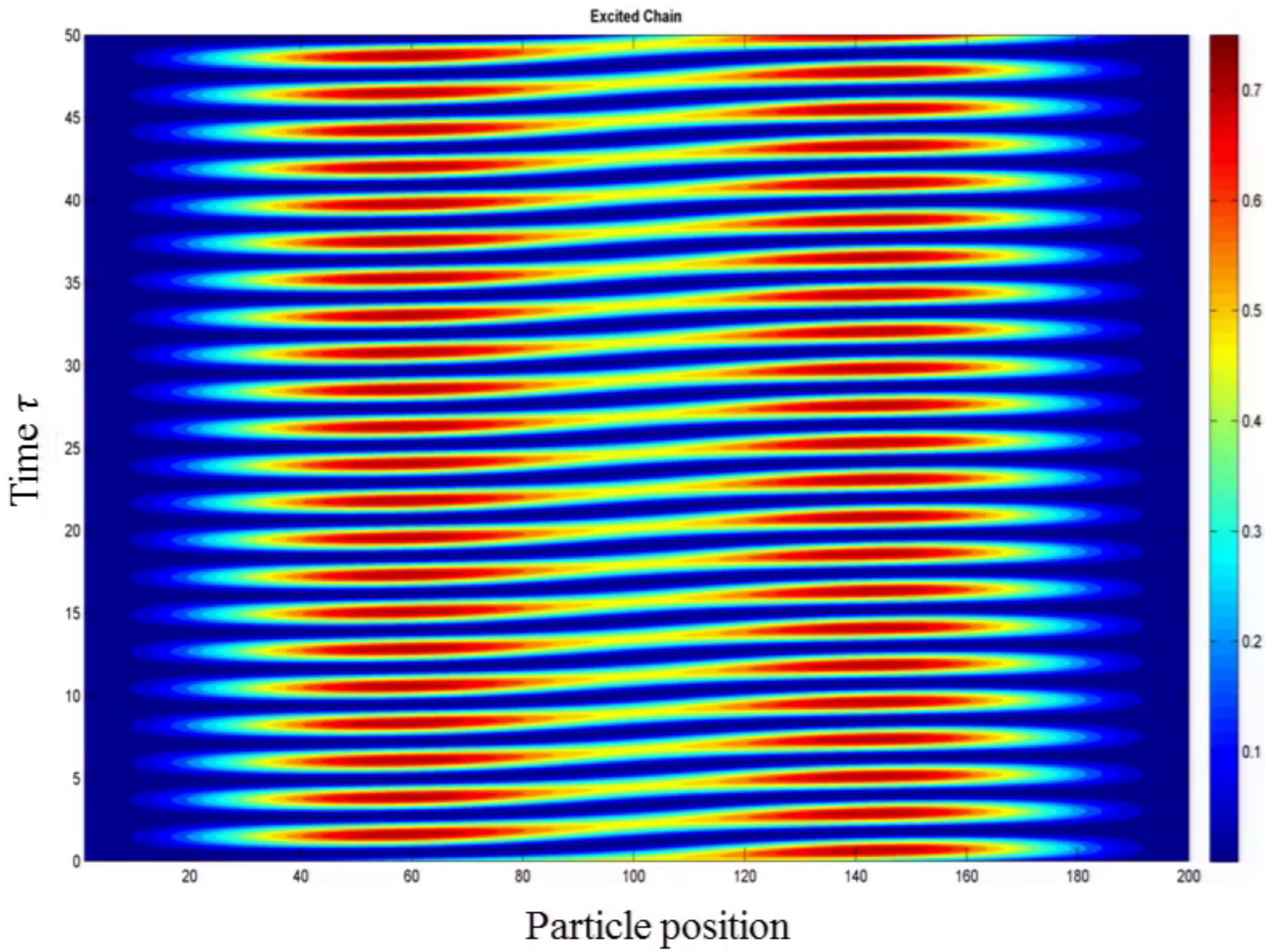
$\varepsilon_1 \approx (3/4)\left(\frac{\pi}{n}\right)^2 \ll 1,$
$n \gg 1$



I:I Resonance Interactions – Pseudo Traveling Wave



1:1 Resonance Interactions – Pseudo Traveling Wave



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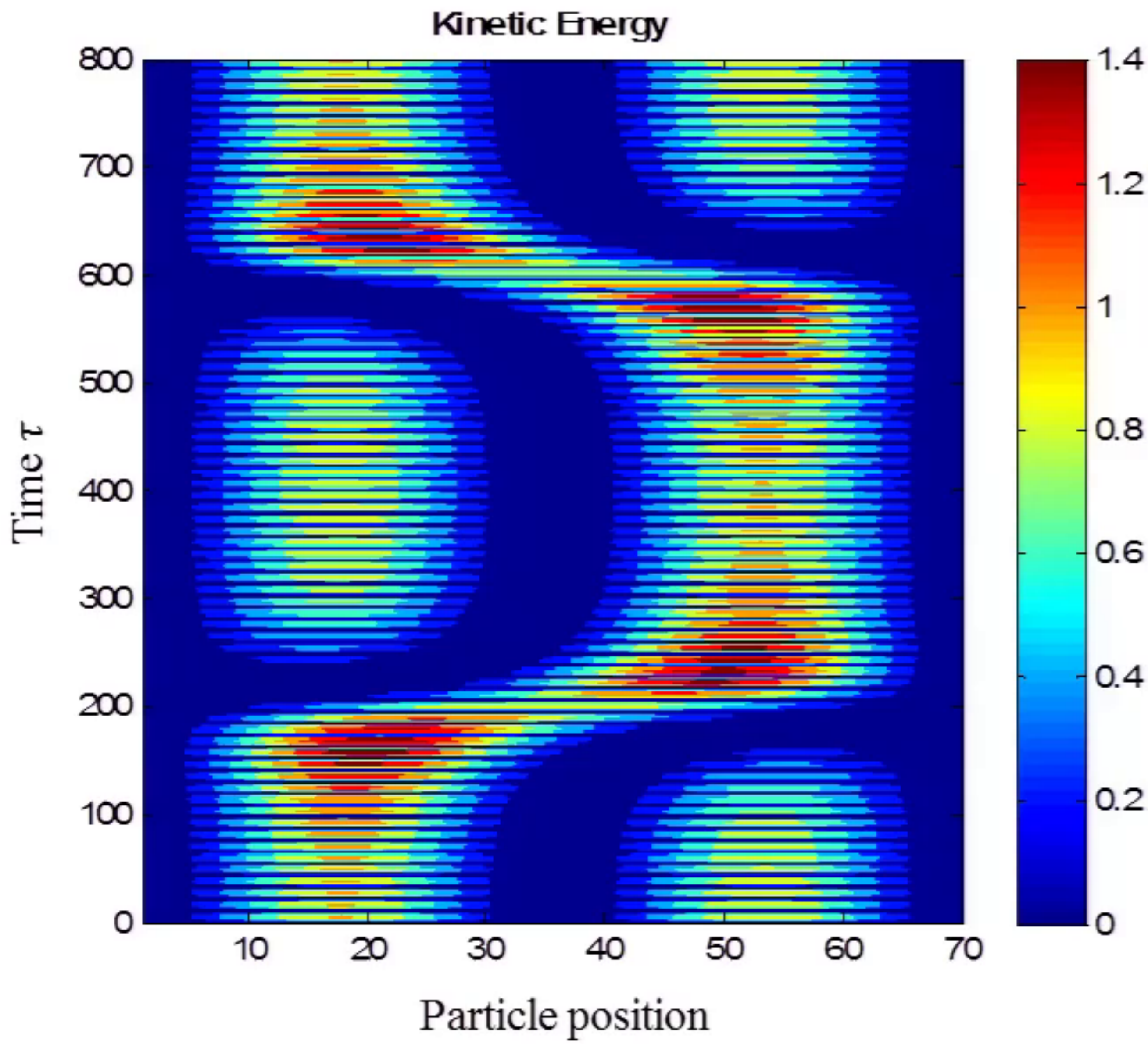
1:1 Resonance Interactions – Strong Energy Exchanges between NNMs close to LPT



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1:1 Resonance Interactions Between Pairs of NNMs