Numerical Moments and the approximation of Fully Nonlinear Second Order PDEs

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INTRODUCTION

NUMERICAL MOMENTS AND A NARROW-STENCIL FINITE DIFFERENCE METHOD

LDG METHODS

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Introduction

Find $u \in X(\Omega) \subset B(\Omega)$ such that

$$\boxed{F[u] \equiv F(D^2u, \nabla u, u, x) = 0} \qquad x \in \Omega \subset \mathbb{R}^d, \qquad \text{(PDE)}$$

$$u = g \qquad x \in \partial\Omega. \qquad \text{(BC)}$$

$$\left[D^2 u\right]_{k\ell} \equiv \frac{\partial^2}{\partial x_k \partial x_\ell} u = u_{x_k x_\ell} \qquad \left[\nabla u\right]_k \equiv \frac{\partial}{\partial x_k} u = u_{x_k}$$

Application Areas of Fully Nonlinear PDEs:

- Differential geometry
- Mass transportation
- Astrophysics
- Optimal control
- Semigeostrophic flow
- Meteorology

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- $F[u]$ is said to be (degenerate) elliptic if

$$F(A, \mathbf{q}, v, x) \leq F(B, \mathbf{q}, v, x) \quad \forall A, B \in \mathrm{SL(n)}, \;\; A - B \geq 0,$$

where SL(n) denotes real, symmetric $n \times n$ matrices and $A - B \geq 0$ means $A - B$ is positive definite.

- $F[u]$ is said to be proper elliptic if, *in addition to being degenerate elliptic*,

$$F(A, \mathbf{q}, v, x) \leq F(A, \mathbf{q}, w, x) \quad \forall v, w \in \mathbb{R}, \;\; v - w \leq 0.$$

- The problem $F[u] = 0$ satisfies a *comparison principle* if for any upper semi-continuous function u and lower semi-continuous function v on $\overline{\Omega}$ such that u is a viscosity subsolution and v is a viscosity supersolution, then $u \leq v$ on $\overline{\Omega}$.

The comparison principle holds for proper elliptic equations and is often referred to as a *strong uniqueness property*.

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Assume F is elliptic in a function class $\mathcal{A} \subset C^0(\Omega)$,

(I) $u \in \mathcal{A}$ is called a **viscosity subsolution** of $F[u] = 0$ if $\forall \varphi \in C^2(\overline{\Omega})$, when $u - \varphi$ has a *local maximum* at $x_0 \in \overline{\Omega}$,

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \leq 0$$

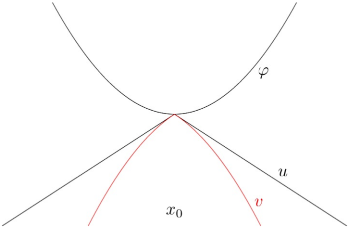
(II) $u \in \mathcal{A}$ is called a **viscosity supersolution** of $F[u] = 0$ if $\forall \varphi \in C^2(\overline{\Omega})$, when $u - \varphi$ has a *local minimum* at $x_0 \in \overline{\Omega}$,

$$F(D^2\varphi(x_0), \nabla\varphi(x_0), u(x_0), x_0) \geq 0$$

(III) $u \in \mathcal{A}$ is called a **viscosity solution** of $F[u] = 0$ if u is both a sub- and supersolution of $F[u] = 0$.

Suppose $v(x_0) = 0$ is a relative maximum.

- "$v_x(x_0) = 0$"
- "$v_{xx}(x_0) \leq 0$" by concavity.



Let $v = u - \varphi$ for $\varphi \in C^2(\Omega)$.

- "$u_x(x_0) = \varphi_x(x_0)$"
- "$u_{xx}(x_0) \leq \varphi_{xx}(x_0)$"

Ellipticity:

$$\boxed{F(\varphi_{xx}(x_0), \varphi_x(x_0), \varphi(x_0), x_0) \leq} \text{ ``}F(u_{xx}(x_0), u_x(x_0), u(x_0), x_0) \;= \text{''} \boxed{0}$$

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Directly approximate viscosity solutions in $C(\Omega)$ using a DG framework to allow for flexibility and increased accuracy.

The discretization should either remove numerical artifacts or naturally pair with a solver that allows for selectivity.

Background on Numerical Methods:

- Most (non-problem-specific) methods are based on Finite Difference (FD) and the *monotonicity* framework of Barles-Souganidis.

  For nonlinear problems that depend on $u_{x_k x_\ell}$ with $k \neq \ell$, monotonicity typically implies wide-stencils.
- Wide-Stencil (semi-Lagrangian, meshless) schemes are less natural in the DG framework for higher-degree elements.
- Many methods require a higher regularity assumption on the viscosity solution.
- **Review Paper**: X. Feng, R. Glowinski, M. Neilan, "Recent developments in numerical methods for second order fully nonlinear PDEs", *SIAM Rev.*, 55(2):205–267, 2013.

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Numerical Moments and a Narrow-Stencil Finite Difference Method

$$\left[\nabla_h^{\pm}\right]_k \equiv \delta_{x_k,h}^{\pm} \qquad \left[D_h^{\mu\nu}\right]_{k\ell} \equiv \delta_{x_k,h}^{\mu}\delta_{x_\ell,h}^{\nu}$$

$$\boxed{\widehat{F}\left(D_h^{++}U_\alpha, D_h^{+-}U_\alpha, D_h^{-+}U_\alpha, D_h^{--}U_\alpha, \nabla_h^{+}U_\alpha, \nabla_h^{-}U_\alpha, U_\alpha, x_\alpha\right) = 0}$$

Criteria for $\widehat{F}$:

- Consistency: $\widehat{F}(P, P, P, P, q, q, u, x) = F(P, q, u, x)$,
- Generalized Monotonicity: $\widehat{F}(\uparrow, \downarrow, \downarrow, \uparrow, \downarrow, \uparrow, u, x)$.

Example: Lax-Friedrich's-like numerical operator

$$\widehat{F}[u] \equiv F(P, q, u, x) - \underbrace{\vec{b} \cdot (q^+ - q^-)}_{\text{Numerical Viscosity}} + \underbrace{A : (P^{++} - P^{+-} - P^{-+} + P^{--})}_{\text{Numerical Moment}}$$

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The Numerical Viscosity:

$$-\sum_{k=1}^{d}\left(\delta_{x_k,h}^{+}U_\alpha - \delta_{x_k,h}^{-}U_\alpha\right) = -\sum_{k=1}^{d} h_k\, \delta_{x_k,h}^{2} U_\alpha$$

The Numerical Moment:

$$\sum_{k,\ell=1}^{d}\left(\left[D_h^{++}U_\alpha\right]_{k\ell} - \left[D_h^{+-}U_\alpha\right]_{k\ell} - \left[D_h^{-+}U_\alpha\right]_{k\ell} + \left[D_h^{--}U_\alpha\right]_{k\ell}\right)$$

$$= \sum_{k,\ell=1}^{d} h_k h_\ell\, \delta_{x_k,h}^{2}\delta_{x_\ell,h}^{2} U_\alpha$$

- $b_k = -1$: Numerical viscosity approximates $h\Delta u$.
- $A_{k\ell} = 1$: Numerical moment approximates $h^2\Delta^2 u$.

The corresponding FD method is a direct realization of the vanishing moment method of Feng and Neilan.

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THEOREM

Suppose the PDE satisfies the comparison principle, has a unique continuous viscosity solution u, the Dirichlet boundary data is continuous, and the operator F is proper elliptic and Lipschitz. Let $u_{\mathbf{h}}$ be a piecewise constant extension of the FD solution U.

Then $u_{\mathbf{h}}$ converges to u locally uniformly as $h \to 0^+$ assuming the mesh and the coefficient matrices A and $\vec{\beta}$ are chosen appropriately.

- A will be strictly positive definite and diagonally dominant with negative off-diagonal entries.
- The Hamilton-Jacobi-Bellman equation satisfies the Lipschitz assumption.

  Without a Lipschitz assumption, the coefficients for the numerical moment would need to be chosen adaptively.
- The Lipschitz assumption and choice of numerical moment are sufficient for ensuring *consitency*, *admissibility*, and ℓ^∞ *stability* of the scheme.

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Let $v \in USC(\Omega)$ have a relative maximum at $\mathbf{x}_0 \in \Omega$ and $i \in \{1, 2, \ldots, \infty\}$. Then

$$\liminf_{h_i \to 0^+} \left(\delta^{++}_{x_i,h_i} + \delta^{--}_{x_i,h_i} - \delta^{+-}_{x_i,h_i} - \delta^{-+}_{x_i,h_i} \right) v(\mathbf{x}_0) \geq 0.$$

The numerical moment acts as a positive stabilization term.

Let $v \in USC(\Omega)$ have a relative maximum at $\mathbf{x}_0 \in \Omega$ and $i \in \{1, 2, \ldots, \infty\}$.
If $\limsup_{h_i \to 0^+} \delta^2_{x_i,h_i} v(\mathbf{x}_0) = -\infty$, then

$$\liminf_{h_i \to 0^+} \left(C\delta^{++}_{x_i,h_i} + C\delta^{--}_{x_i,h_i} - \delta^{+-}_{x_i,h_i} - \delta^{-+}_{x_i,h_i} \right) v(\mathbf{x}_0) \geq 0$$

for any constant $1 \leq C \leq 2$.

The factor of C allows for control of mixed-derivatives without using a wide-stencil.

The numerical moment steers the approximation towards being a sub/supersolution when the underlying viscosity solution has low regularity.

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LDG Methods

$$\boxed{F\left(D^2 u, \nabla u, u, x\right) = 0 \qquad \text{in } \Omega}$$

$\mathcal{T}_h$ denotes a locally quasi-uniform triangularization of Ω.

$V_r^h \equiv \prod_{K \in \mathcal{T}_h} \mathcal{P}_r(K)$ for $r \geq 0$.

Standard Local Discontinuous Galerkin (LDG):

$$\left(\mathcal{F}\left(q_h, u_h, x\right), \phi_h\right)_{\mathcal{T}_h} = 0 \qquad \forall \phi_h \in V_r^h$$

$$\left([q_h]_k, \varphi_k^h\right)_{\mathcal{T}_h} + \left(u_h, \partial_{x_k} \varphi_k^h\right)_{\mathcal{T}_h} = \left\langle \widehat{u}_h, \varphi_k^h n_k \right\rangle_{\mathcal{E}_h} \qquad \forall \varphi_k^h \in V_r^h$$

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$$F\left(D^2 u, \nabla u, u, x\right) = 0 \qquad \text{in } \Omega$$

No integration by parts:

- Form a gradient and Hessian approximation.
- Use an L^2-projection of the nonlinear equation.

$$\left(F\left(P_h, q_h, u_h, x\right), \phi_h\right)_{\mathcal{T}_h} = 0 \qquad \forall \phi_h \in V_r^h$$

$$\left([q_h]_k, \varphi_k^h\right)_{\mathcal{T}_h} + \left(u_h, \partial_{x_k} \varphi_k^h\right)_{\mathcal{T}_h} = \left\langle \widehat{u_h}, \varphi_k^h n_k \right\rangle_{\mathcal{E}_h} \qquad \forall \varphi_k^h \in V_r^h$$

$$\left([P_h]_{k\ell}, \psi_{k\ell}^h\right)_{\mathcal{T}_h} + \left([q_h]_k, \partial_{x_\ell} \psi_{k\ell}^h\right)_{\mathcal{T}_h} = \left\langle \widehat{[q_h]_k}, \psi_{k\ell}^h n_\ell \right\rangle_{\mathcal{E}_h} \qquad \forall \psi_{k\ell}^h \in V_r^h$$

The individual components of the (discrete) gradient and Hessian form the auxiliary variables.

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$$\boxed{F\left(D^2 u, \nabla u, u, x\right) = 0 \qquad \text{in } \Omega}$$

Low regularity for u: Form *multiple* gradient and Hessian approximations.

$D_h^2 v(x_0) \leq 0$ is not guaranteed at a relative maximum when v has low regularity.

Find $u_h \in V_r^h$, $q_h^+, q_h^- \in [V_r^h]^d$, $P_h^{++}, P_h^{+-}, P_h^{-+}, P_h^{--} \in [V_r^h]^{d\times d}$ such that

$$\left(\widehat{F}\left(P_h^{++}, P_h^{+-}, P_h^{-+}, P_h^{--}, q_h^+, q_h^-, u_h, x\right), \phi_h\right)_{\mathcal{T}_h} = 0 \qquad \forall \phi_h \in V_r^h$$

$$\left(\left[q_h^\mu\right]_k, \varphi_k^\mu\right)_{\mathcal{T}_h} + \left(u_h, \partial_{x_k}\varphi_k^\mu\right)_{\mathcal{T}_h} = \left\langle \widehat{u_h}^\mu, \varphi_k^\mu n_k\right\rangle_{\mathcal{E}_h} \qquad \forall \varphi_k^\mu \in V_r^h$$

$$\left(\left[P_h^{\mu\nu}\right]_{k\ell}, \psi_{k\ell}^{\mu\nu}\right)_{\mathcal{T}_h} + \left(\left[q_h^\mu\right]_k, \partial_{x_\ell}\psi_{k\ell}^{\mu\nu}\right)_{\mathcal{T}_h} = \left\langle \widehat{\left[q_h^\mu\right]_k}^\nu, \psi_{k\ell}^{\mu\nu} n_\ell\right\rangle_{\mathcal{E}_h} \qquad \forall \psi_{k\ell}^{\mu\nu} \in V_r^h$$

for all $k, \ell = 1, 2, \ldots, d$, for all $\mu, \nu \in \{+, -\}$

Use upwind/downwind fluxes to define the interior traces.

Corresponding DG differential operators can be formed to eliminate the auxiliary equations and reduce the number of unknowns to those for u_h.

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For $r = 0$ on uniform Cartesian meshes the formulation is equivalent to the (narrow-stencil) generalized monotone finite difference methods of Feng, L.

Heuristically, for $r > 0$, the convergence properties are maintained while accuracy is increased.

The methods naturally extend the direct LDG methods of Yan and Osher for first order fully nonlinear equations / Hamilton-Jacobi equations.

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The Numerical Viscosity:

$$-\vec{b}\cdot\left(q_h^+ - q_h^-, \varphi_h\right)_{\mathcal{T}_h} = \sum_{k=1}^{d} \beta_k \Big\langle [u_h], [\varphi_h]\, n_k \Big\rangle_{\mathcal{E}_h^I}$$

The Numerical Moment:

$$A : \left(P_h^{++} - P_h^{+-} - P_h^{-+} + P_h^{--}, \varphi_h\right)_{\mathcal{T}_h} = \sum_{k,\ell=1}^{d} a_{k\ell} \Big\langle [q_k^- - q_k^+], [\varphi_h]\, n_\ell \Big\rangle_{\mathcal{E}_h^I}$$

The numerical moment is a new type of jump-stabilization term based on the difference of two different gradient approximations.

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IDEA: Iterate over the Laplacian of u_h to take advantage of the elliptic structure and the strong monotonicity guaranteed by the numerical moment.

1. Pick an initial guess for u_h.
2. Let the auxiliary variables be defined by $q_h^{\pm} = \nabla_h^{\pm} u_h$ and $P_h^{\mu\nu} = D_h^{\mu\nu} u_h$.
3. Set

$$[G]_k \equiv F\left(P_h, q_h, u_h, x\right) - \vec{\beta} \cdot \left(q_h^+ - q_h^-\right) + \sum_{\substack{i,j=1 \\ i \neq j}}^{d} A_{ij} \left[P_h^{--} - P_h^{-+} - P_h^{+-} + P_h^{++}\right]_{ij}$$

$$+\gamma \left[P_h^{--} - P_h^{-+} - P_h^{+-} + P_h^{++}\right]_{kk}$$

for some fixed constant $\gamma > 0$ and solve

$$\left([G]_k, \phi_k\right)_{\mathcal{T}_h} = 0 \qquad \forall \phi_k \in V_r^h$$

for $\left[\left(P_h^{-+} + P_h^{+-}\right)/2\right]_{kk}$ for all $k = 1, 2, \ldots, d$.
For sufficiently large γ and a differentiable operator F, the above set of equations has a negative definite Jacobian.

4. Solve $\Delta_h u_h = \text{tr}\ \left(P_h^{-+} + P_h^{+-}\right)/2$ using the DWDG method (L., Neilan).
5. Repeat steps 2 - 4.

$$F[u] \equiv 1 - u_{xx}^2 = 0, \qquad \text{on } (0,1),$$
$$u(0) = 0, \qquad u(1) = 1/2.$$

- Two classical solutions:

$$u^+(x) = \frac{1}{2}x^2, \qquad\qquad u^-(x) = -\frac{1}{2}x^2 + x.$$

- Infinitely many $C^1 \cap H^2$ artifacts (a.e. solutions) such as:

$$\mu(x) = \begin{cases} \frac{1}{2}x^2 + \frac{1}{4}x, & \text{for } x < \frac{1}{2}, \\ -\frac{1}{2}x^2 + \frac{5}{4}x - \frac{1}{4}, & \text{for } x \geq \frac{1}{2}. \end{cases}$$

- Unique viscosity solution: u^+:

$$F_{u_{xx}}\Big|_{u^+} = -2u_{xx}\Big|_{u^+} = -1 < 0 \implies \text{elliptic}$$

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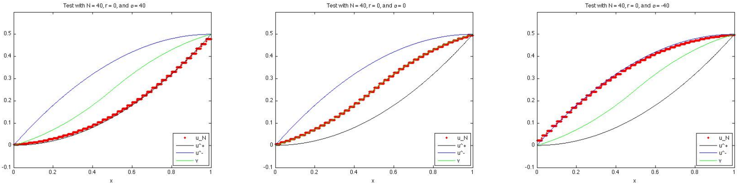
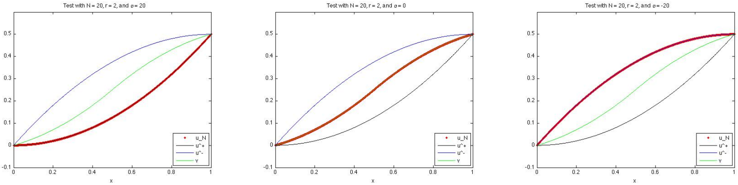


(a) $r = 0$ and $A = \frac{1}{h}$. **(b)** $r = 0$ and $A = 0$. **(c)** $r = 0$ and $A = -\frac{1}{h}$.



(d) $r = 2$ and $A = \frac{1}{h}$. **(e)** $r = 2$ and $A = 0$. **(f)** $r = 2$ and $A = -\frac{1}{h}$.

$$u_h^{(0)} = \frac{3}{4}\mu + \frac{1}{4}\overline{u}, \qquad \text{Split Solver}$$

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$u_h^{(0)} = \overline{u}$, Newton Solver

r	Norm	$h = 1/4$	$h = 1/8$		$h = 1/16$		$h = 1/32$	
		Error	Error	Order	Error	Order	Error	Order
0	L^2	7.1e-02	3.5e-02	1.02	1.4e-02	1.30	7.5e-03	0.92
	L^∞	1.3e-01	8.7e-02	0.57	5.3e-02	0.73	2.9e-02	0.87
1	L^2	1.6e-02	5.0e-03	1.67	1.3e-03	1.90	3.4e-04	1.95
	L^∞	2.2e-02	6.3e-03	1.84	1.6e-03	2.00	3.9e-04	2.00
2	L^2	3.1e-13	3.0e-13		3.0e-13		3.1e-13	
	L^∞	7.4e-13	6.1e-13		6.7e-13		7.1e-13	

Figure: LDG with $A = 10$.

μ is a C^1 solution of the problem in the test space V_2^h. The split solver steered away from the artifact while the Newton solver converged to an artifact when the initial guess was not convex with $r \geq 2$.

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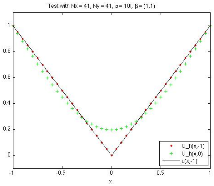
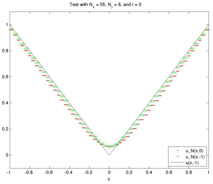
$$F[u] \equiv -\det D^2 u = -u_{xx}\,u_{yy} + u_{xy}\,u_{yx} = f, \qquad \Omega = (0,1)\times(0,1),$$
$$u = g, \qquad \partial\Omega$$

Viscosity solution: $u(x,y) = |x| \in H^1(\Omega)$.

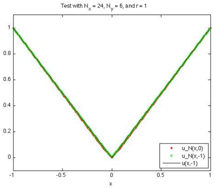
Initial guess $u_h^{(0)} = 0$.



(a) FD with $A = 10I$, $\vec{b} = \mathbf{1}$.



(b) LDG with $r = 0$ and $A = I$.



(c) LDG with $r = 1$ and $A = I$.

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	$\beta = \mathbf{1}$		$\beta = \mathbf{0}$	
h	ℓ^∞ Error	Order	ℓ^∞ Error	Order
1.01e-01	2.80e-01		2.18e-01	
8.84e-02	2.44e-01	1.01	1.87e-01	1.14
7.86e-02	2.18e-01	0.98	1.65e-01	1.06
7.07e-02	1.97e-01	0.95	1.49e-01	1.00

FD with $A = 10I$.

h_x	L^∞ norm	order
1.33E-01	1.87E-01	
8.00E-02	1.30E-01	0.71
5.71E-02	1.02E-01	0.72
4.44E-02	8.51E-02	0.74
3.64E-02	7.33E-02	0.74

LDG with $r = 0$, $A = 24I$, and $h_y = \frac{1}{4}$ fixed.

h_x	L^∞ norm	order	L^2 norm	order
2.50E-01	3.86E-02		3.42E-02	
1.25E-01	2.08E-02	0.89	1.85E-02	0.88
8.33E-02	1.38E-02	1.02	1.24E-02	0.99

$r = 1$, $A = I$, and $h_y = \frac{1}{3}$ fixed. Odd number of intervals in the x-direction.

IP-DG Methods

$$\boxed{F\left(D^2 u, \nabla u, u, x\right) = 0 \qquad \text{in } \Omega}$$

Find u, P^+, P, P^- such that

$$\begin{aligned} \widehat{F}(P^+, P, P^-, \nabla u, u, x) &= 0, \\ P^+(x) - D^2 u(x^+) &= 0, \\ P(x) - D^2 u(x^a) &= 0, \\ P^-(x) - D^2 u(x^-) &= 0 \end{aligned}$$

for all $x \in \Omega$, where $D^2 u(x^a)$ can be thought of as the arithmetic average of $D^2 u(x^+)$ and $D^2 u(x^-)$.

- The nonlinear equation is simply projected into the DG space using an L^2 projection.
- The three different Hessians correspond to the three jump formulas:

$$\begin{aligned} [v\, w] &= v^- [w] + [v] w^+, \\ [v\, w] &= \{v\}[w] + [v]\{w\}, \\ [v\, w] &= v^+ [w] + [v] w^- \end{aligned}$$

- The gradient is the piecewise gradient operator.

$$\boxed{\left(P^{*}_{k\ell}, \phi^{*h}_{k\ell}\right)_{\mathcal{T}_h} + a^{*}_{k\ell}\left(u_h, \phi^{*h}_{k\ell}\right) = f^{*}_{k\ell}\left(\phi^{*h}_{k\ell}\right) \qquad \forall \phi^{*h}_{k\ell} \in V^h}$$

for all $k, \ell = 1, 2, \ldots, d$ and $*$ takes $+$, $-$, and empty value.

$$a^{+}_{k\ell}(u, \phi) = b^{+}_{k\ell}(u, \phi) - \left\langle u^{+}_{x_k}, [\phi] n_\ell \right\rangle_{\mathcal{E}^I_h} + \epsilon^{+}_{k\ell}\left\langle [u], \phi^{+}_{x_k} n_\ell \right\rangle_{\mathcal{E}^I_h},$$

$$a_{k\ell}(u, \phi) = b_{k\ell}(u, \phi) - \left\langle \{u_{x_k}\}, [\phi] n_\ell \right\rangle_{\mathcal{E}^I_h} + \epsilon_{k\ell}\left\langle [u], \{\phi_{x_k}\} n_\ell \right\rangle_{\mathcal{E}^I_h},$$

$$a^{-}_{k\ell}(u, \phi) = b^{-}_{k\ell}(u, \phi) - \left\langle u^{-}_{x_k}, [\phi] n_\ell \right\rangle_{\mathcal{E}^I_h} + \epsilon^{-}_{k\ell}\left\langle [u], \phi^{-}_{x_k} n_\ell \right\rangle_{\mathcal{E}^I_h},$$

$$f^{*}_{k\ell}(\phi) = \epsilon^{*}_{k\ell} \sum_{e \in \mathcal{E}^B_H} \left\langle g, \phi_{x_k} n_\ell \right\rangle_{\tilde{e}} + \sum_{e \in \mathcal{E}^B_h} \frac{\gamma^{0*}_{k\ell}}{h_e} \left\langle g, \phi \right\rangle_e,$$

with $b^{*}_{k\ell} : H^1(\mathcal{T}_h) \times H^1(\mathcal{T}_h) \to \mathbb{R}$ defined by

$$b^{*}_{k\ell}(v, w) = \left(v_{x_k}, w_{x_\ell}\right)_{\mathcal{T}_h} - \left\langle v_{x_k}, w\, n_\ell \right\rangle_{\mathcal{E}^B_h} + \epsilon^{*}_{k\ell}\left\langle v, w_{x_k} n_\ell \right\rangle_{\mathcal{E}^B_h}$$
$$+ \sum_{e \in \mathcal{E}^I_h} \frac{\gamma^{0*}_{k\ell}}{h_e} \left\langle [v], [w] \right\rangle_e + \sum_{e \in \mathcal{E}^B_h} \frac{\gamma^{0*}_{k\ell}}{h_e} \left\langle v, w \right\rangle_e$$

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For simplicity, we assume the three symmetrization constants are the same, i.e., $\epsilon^{+} = \epsilon = \epsilon^{-}$. Then,

$$\begin{aligned}\left(P_{k\ell}^{+h} - 2P_{k\ell}^{h} + P_{k\ell}^{-h}, \phi\right)_{\mathcal{T}_h} &= \sum_{e \in \mathcal{E}_h^B} \frac{\gamma_{k\ell}^{0+} - 2\gamma_{k\ell}^{0} + \gamma_{k\ell}^{0-}}{h_e} \left\langle g - u_h, \phi \right\rangle_e \\ &\quad - \sum_{e \in \mathcal{E}_h^I} \frac{\gamma_{k\ell}^{0+} - 2\gamma_{k\ell}^{0} + \gamma_{k\ell}^{0-}}{h_e} \left\langle [u_h], [\phi] \right\rangle_e\end{aligned}$$

for all $\phi \in V^h$.

The numerical moment corresponds to a jump penalization term that weakly enforces the boundary condition when the coefficient matrix is chosen appropriately.

For simplicity we choose $\gamma^{0+} = \gamma^{0-} < \gamma^{0}$ in which case the moment is the difference of two "central" Hessians with different penalty constants.

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1. Pick initial guesses for u_h, P_h^+, P_h, and P_h^-.
2. Set

$$[G]_k \equiv F\left(P_h, \nabla u_h, u_h, x\right) + \sum_{\substack{i,j=1 \\ i\neq j}}^{d} a_{ij}\left[P_h^+ - 2P_h + P_h^-\right]_{ij} + \gamma\left[P_h^+ - 2P_h + P_h^-\right]_{kk}$$

   for a fixed constant $\gamma > 0$, and solve

$$\left(\left[G\right]_k, \phi_k\right)_{\mathcal{T}_h} = 0 \qquad \forall \phi_\ell \in V^h$$

   for $[P_h]_{kk}$ for all $k = 1, 2, \ldots, d$.
3. Set $\Lambda_h = \sum_{k=1}^{d} [P_h]_{kk}$. Find u_h by solving Poisson's equation for the given value of Λ_h as a source using the IP-DG formulation corresponding to P_h.
4. Update $P_h^\pm$ using u_h.
5. Repeat Steps 2 - 4.

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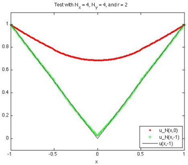
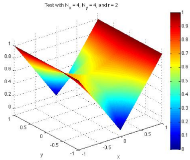
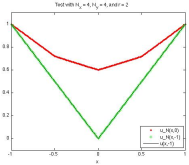


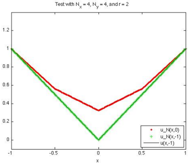
(d) 1 iteration



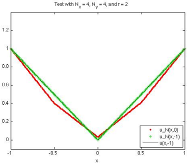
(e) 20 iterations 2D



(f) 20 iterations



(g) 100 iterations



(h) 150 iterations

Figure: $r = 2$, $\alpha = 100\, I$, $\gamma^{0+} = \gamma^{0-} = 20\,\mathbf{1}$, $\gamma^{0} = 40\,\mathbf{1}$, $\epsilon^* = 0$, h = 7.071068e-01 with initial guess $u_h^{(0)} = 0$.

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Conclusion

- A numerical moment requires having two distinct Hessian approximations.
- The numerical moment is a key tool in designing *narrow-stencil* convergent finite difference methods that can naturally be extended to LDG methods.
- The numerical moment acts as a stabilization / positive penalty term for finite difference methods with regards to the "differentiation-by-parts" definition of a viscosity solution.
- The numerical moment is a direct realization of the vanishing moment method for finite difference methods.
- The numerical moment acts as a penalization term when using DG methods.

  For LDG methods it penalizes the difference of two gradient approximations.

  For IP-DG methods it penalizes the continuity of the underlying approximation.
- The numerical moment can be used to design solvers that are more selective for a given discretization.

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Thank You!

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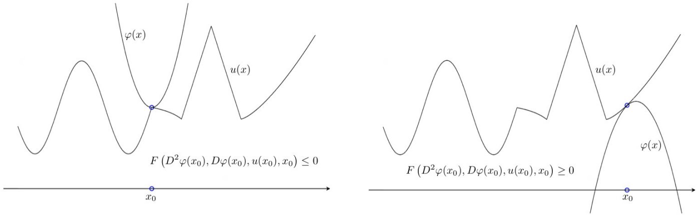
$u - \varphi$ local maximum at x_0

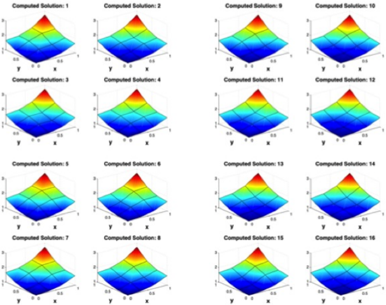
$u - \varphi$ local minimum at x_0



- The concept was introduced by Crandall, Lions, and Evans in the early 1980's.
- The concept is *nonvariational*. It is based on a local "differentiation by parts" approach.
- Viscosity solutions may only be unique in a restrictive function class: conditional uniqueness

Test:

- $\Omega = (0,1)^2$
- Monge-Ampère equation
- Solution $u = e^{(x^2+y^2)/2} \in C^\infty(\Omega)$
- Standard 9-point FD scheme
- Newton based solver
- Vary the initial guess



ALL 16 Possible Solutions Computed for $N = 4$

For a $N \times N$ grid, there are $2^{(N-2)^2}$ algebraic solutions for the standard 9-point FD scheme.

*Test data generated by Michael Neilan.

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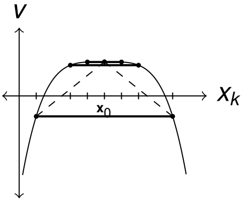
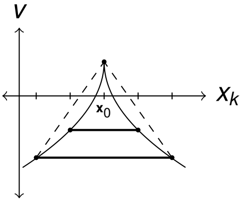


$$\delta^{\overline{2}}_{x_k,h_k} v(\mathbf{x}_0) - \delta^{2}_{x_k,h_k} v(\mathbf{x}_0) < 0$$



$$\delta^{\overline{2}}_{x_k,h_k} v(\mathbf{x}_0) - \delta^{2}_{x_k,h_k} v(\mathbf{x}_0) > 0$$