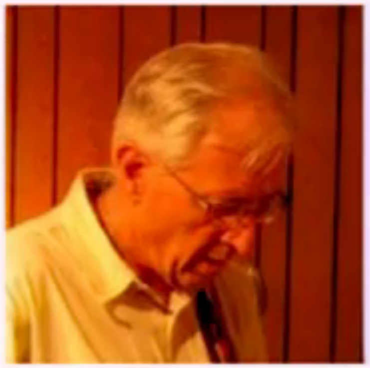
Symmetry Breaking and Synchronization Patterns in Networks of Coupled Oscillators

Francesco Sorrentino
Department of Mechanical Engineering
UNM

2015 SIAM Conference on Applications of Dynamical Sytems
Snowbird, Utah
May 18th 2015

Contributors and Co-Authors

- Lou Pecora (Naval Research Laboratory)
- Prof. Tom Murphy (UMD)
- Prof. Rajarshi Roy (UMD)
- Aaron Hagerstrom *(Graduate Research Assistant, Physics)*



UNIVERSITY OF MARYLAND

UNM

Contributors and Co-Authors

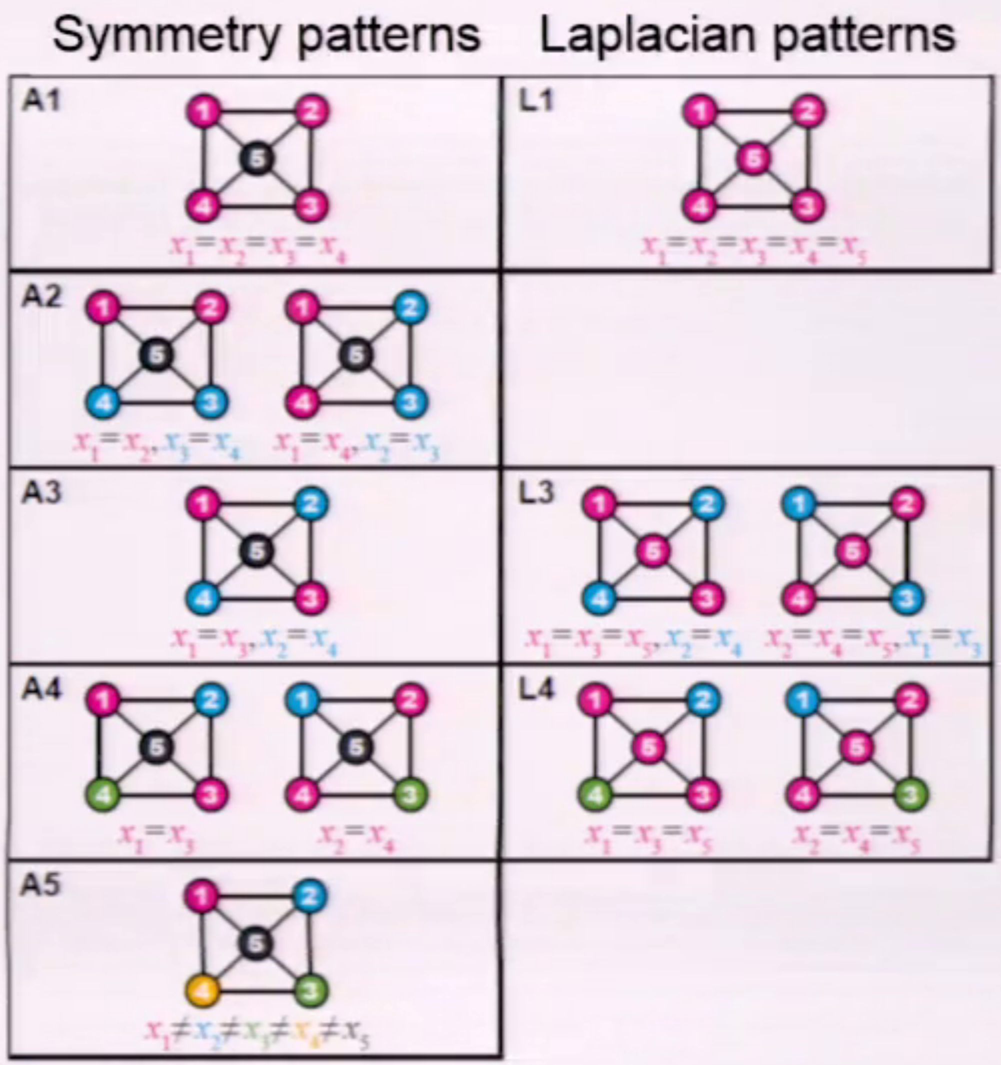
- Lou Pecora (Naval Research Laboratory)
- Prof. Tom Murphy (UMD)
- Prof. Rajarshi Roy (UMD)
- Aaron Hagerstrom *(Graduate Research Assistant, Physics)*

An example: a simple 5-node network

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

$$L = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix}$$

The figure on the right shows all of the synchronization patterns that may emerge

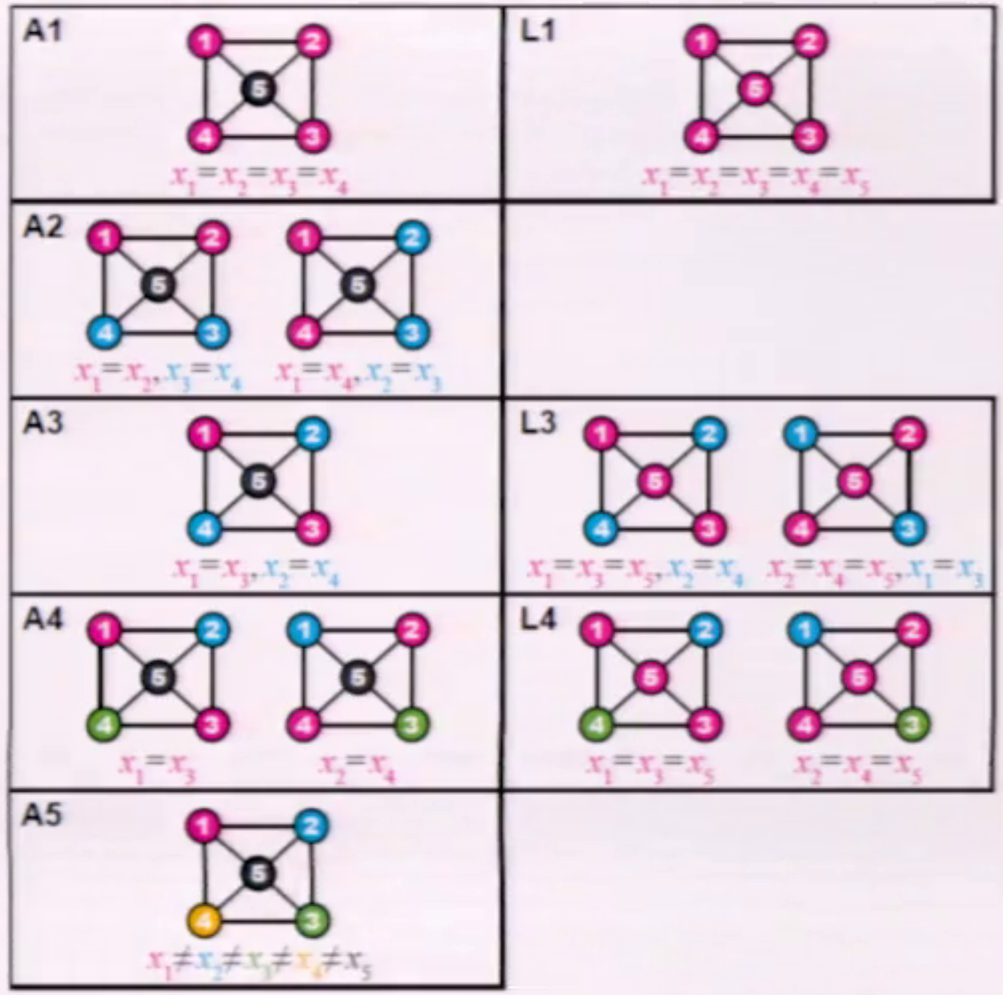


The L patterns

For each A pattern, several L patterns can be generated by merging the nodes in different clusters (merging not allowed by the symmetries) – a test is needed

In this particular example, the test reduces to checking whether node 5 can be included in an other existing cluster

Symmetry patterns Laplacian patterns



Cluster merges

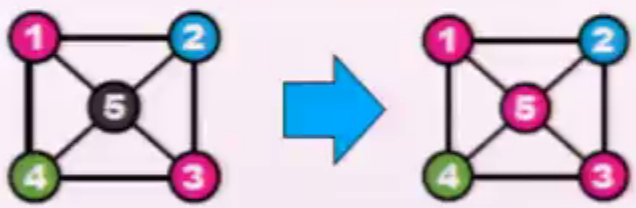
1) Observation: when clusters are synchronized, the diagonal feedback terms for each node cancel with the coupling terms from nodes in the same cluster.

2) To test for a merge, remove the inter-cluster couplings and adjust the diagonal entries accordingly → obtain a new coupling matrix

3) Compute the subgroups of the new coupling matrix.

4) If any of these subgroups includes nodes originally belonging to different clusters, then their dynamics is flow-invariant in the synchronized state and the cluster merging is possible.

Example: Pattern A4



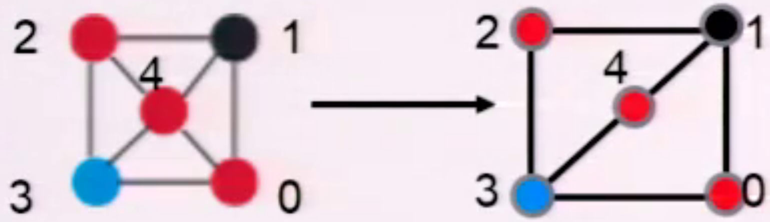
$$L = \begin{pmatrix} -3 & 1 & 0 & 1 & 1 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -3 & 1 & 1 \\ 1 & 0 & 1 & -3 & 1 \\ 1 & 1 & 1 & 1 & -4 \end{pmatrix} \Rightarrow L_{eq} = \begin{pmatrix} -2 & 1 & 0 & 1 & 0 \\ 1 & -3 & 1 & 0 & 1 \\ 0 & 1 & -2 & 1 & 0 \\ 1 & 0 & 1 & -3 & 1 \\ 0 & 1 & 0 & 1 & -2 \end{pmatrix}$$

Check to see if dynamics allow the new synchronized cluster

Original Laplacian

$$
\begin{pmatrix}
-3 & 1 & 0 & 1 & 1 \\
1 & -3 & 1 & 0 & 1 \\
0 & 1 & -3 & 1 & 1 \\
1 & 0 & 1 & -3 & 1 \\
1 & 1 & 1 & 1 & -4
\end{pmatrix}
$$

L



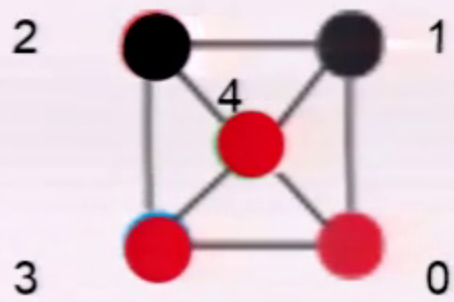
Dynamically equivalent Laplacian

$$
\begin{pmatrix}
-2 & 1 & 0 & 1 & 0 \\
1 & -3 & 1 & 0 & 1 \\
0 & 1 & -2 & 1 & 0 \\
1 & 0 & 1 & -3 & 1 \\
0 & 1 & 0 & 1 & -2
\end{pmatrix}
$$

L_{eq}

[0,2,4] is a symmetry cluster of L_{eq}

Counter example: Not all combined clusters will work.



Combine clusters from different original clusters

dynamically equivalent network

Observations: beyond synchronization

Our routine, based on computational group theory, provides all of the possible patterns that can be observed for a given network topology

The routine's input is the network coupling matrix (either adjacency or Laplacian) and its output is a list of all the patterns that can emerge – *some patterns are allowed, while others are not!*

We are able to answer these questions: *is this pattern at all possible for this network? What are (all) the patterns that are compatible with a given network?*

Our work has immediate practical relevance: for example to technological networks, neuronal networks, genetic networks, for which certain patterns may be ***good*** and others may be ***bad***.

Up to this point, our approach is purely topological, not dynamical.

The role of dynamics: stability

In Pecora & Carroll "Master stability functions for synchronized coupled systems", PRL (1998), the stability problem for the fully synchronous pattern (L1) was studied

In Pecora, Sorrentino, Hagerstrom, Murphy, and Roy, "Cluster Synchronization and Isolated Desynchronization in Complex Networks with Symmetries", Nature Communications (2014), we studied the stability problem for the *maximal symmetry pattern* (A1)

We are currently studying stability of the remaining patterns in a low-dimensional form

Q1: Which ones of the topologically valid patterns are stable?

Q2: Can we reduce the dimensionality of these stability problems?

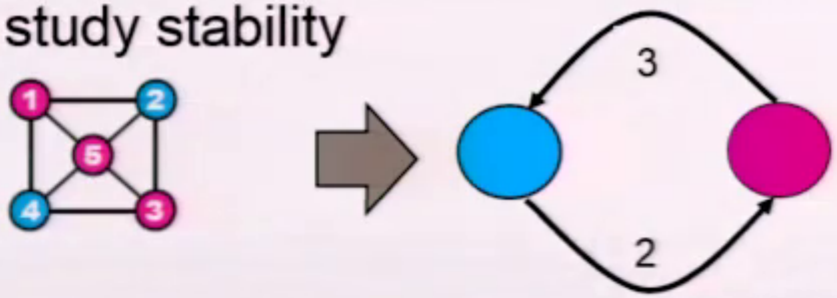
Quotient networks

To each pattern corresponds a quotient network

In the quotient network, each cluster is represented by only one node

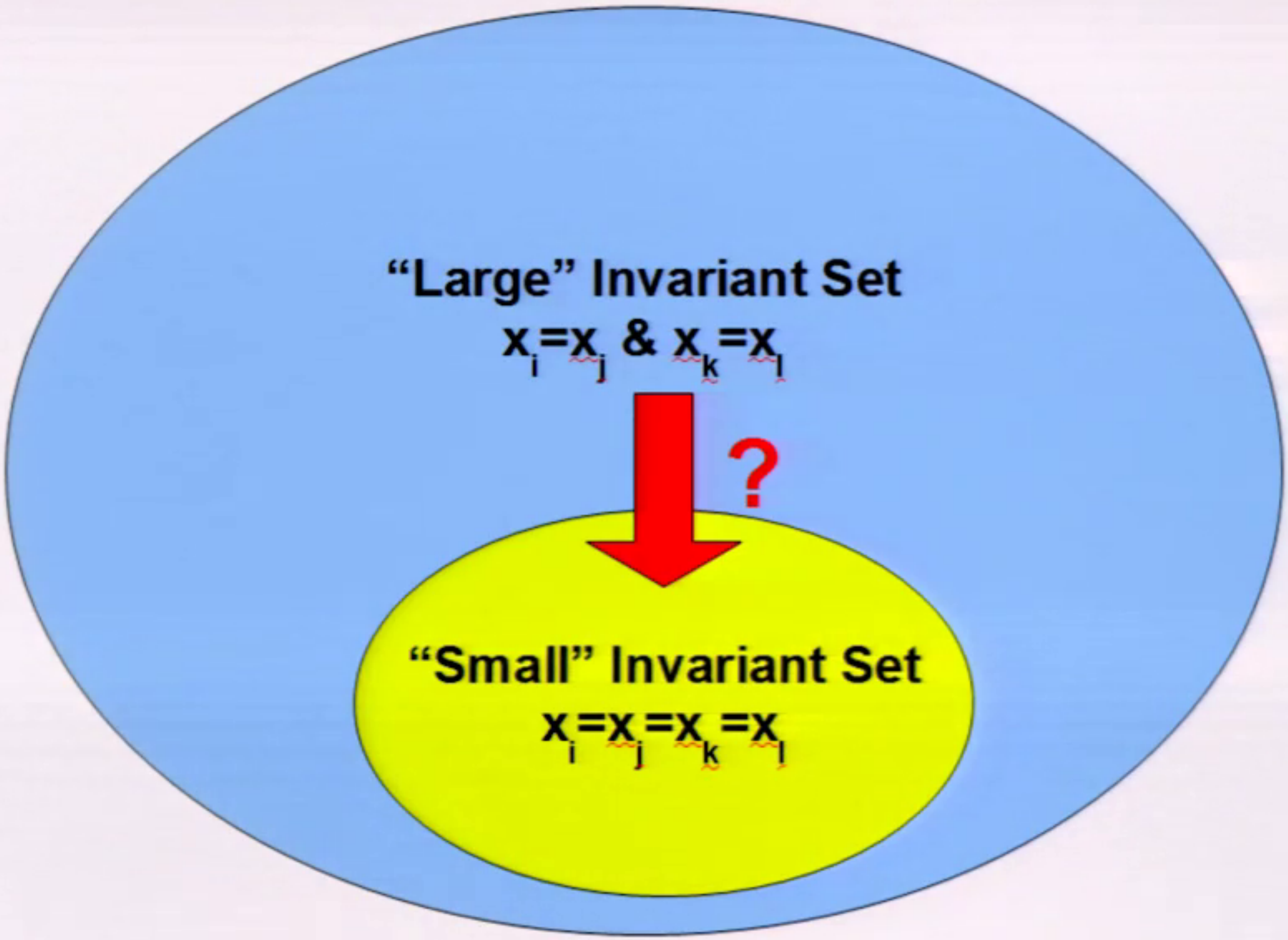
This is helpful because we can linearize about the quotient network dynamics and study stability

Example:

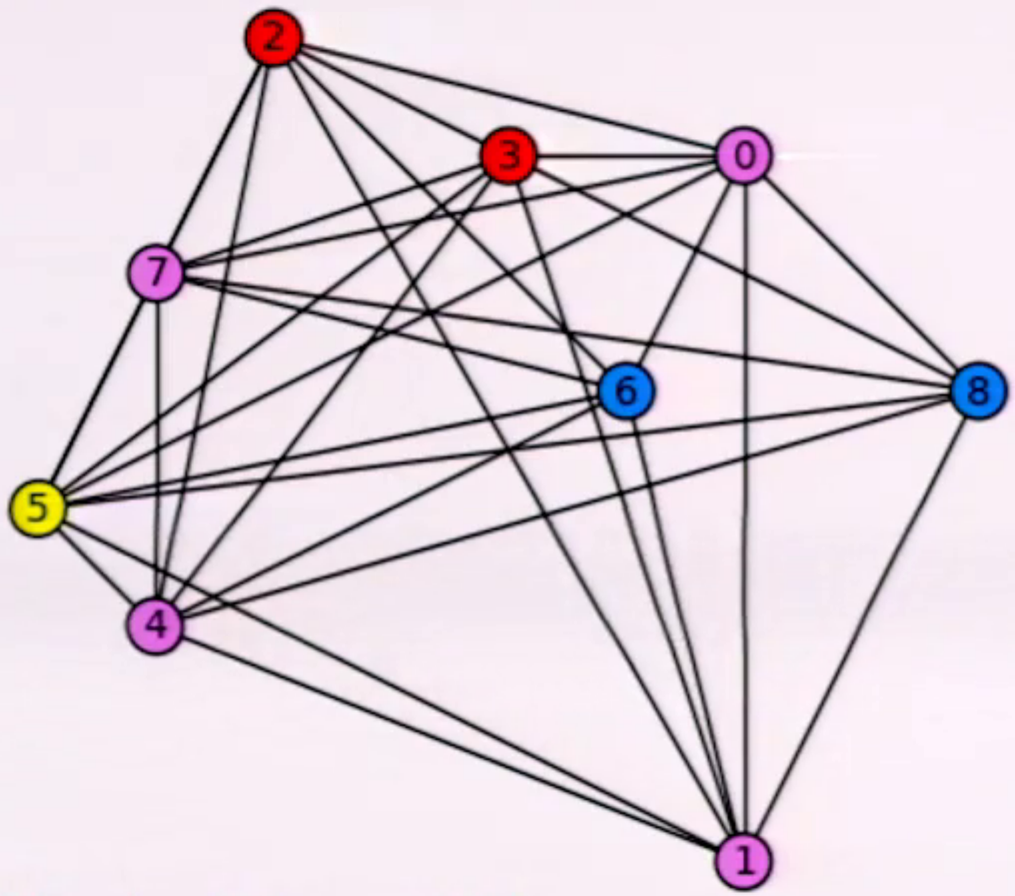


PROBLEM: not all the topologically valid patterns are also dynamically valid.
For example, under Laplacian coupling, it is possible that after a transient the blue and magenta clusters of the quotient network synchronize on the same time evolution.

NESTED INVARIANT SETS



Example: a 9-node “random” network



ORBITS:

Nodes 0,1,4,7. In what follows: 1,2,5,8.

Nodes 2,3. In what follows: 3,4

Nodes 6,8. In what follows: 7,9

RED AND BLUE CLUSTERS ARE INTERTWINED

Coupled map equations: Laplacian coupling

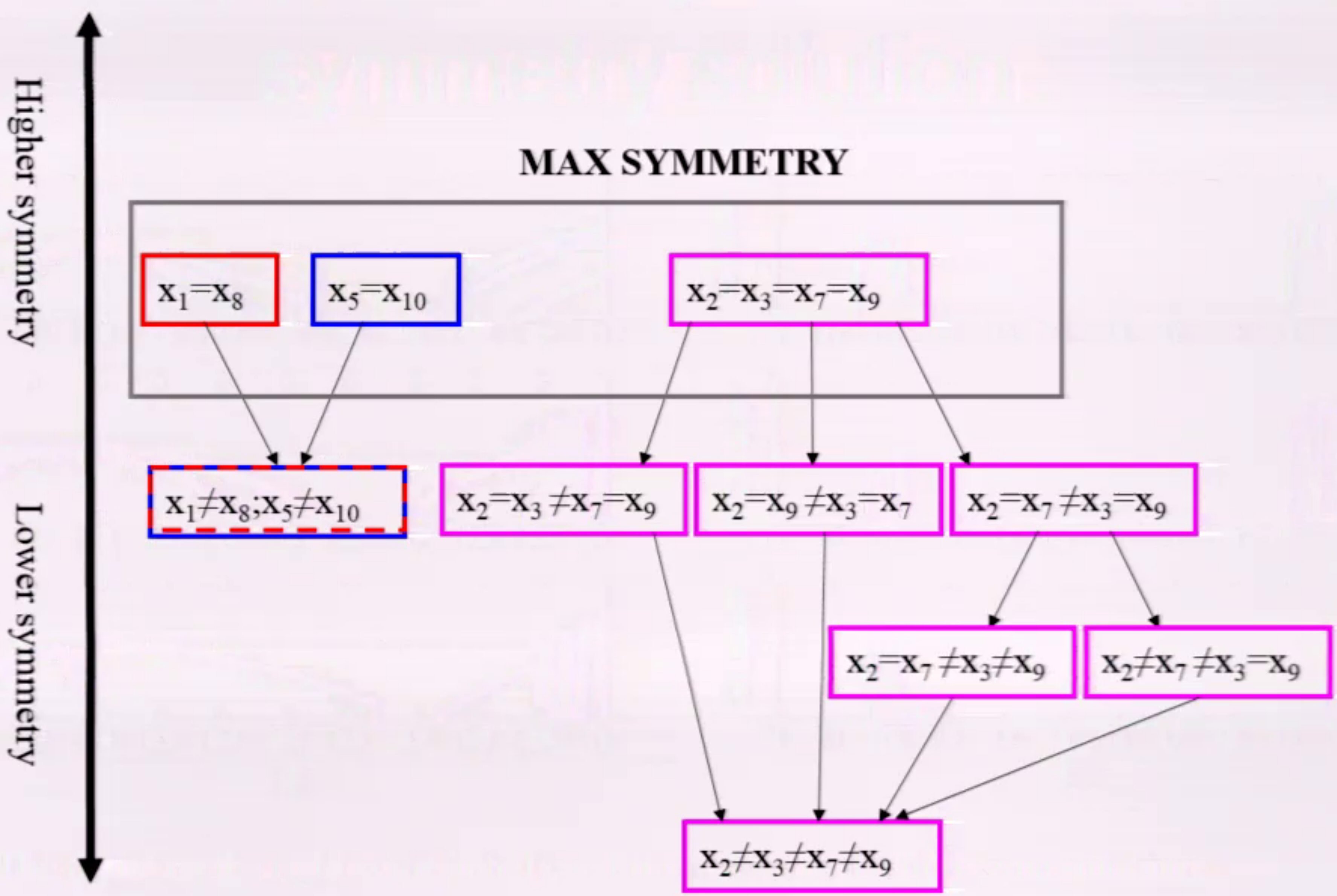
$$x_i[n+1] = \beta I\left(x_i[n]\right) + \sigma \sum_{j=1}^{N} C_{ij} I\left(x_j[n]\right) + \delta$$

$$\beta = 5$$

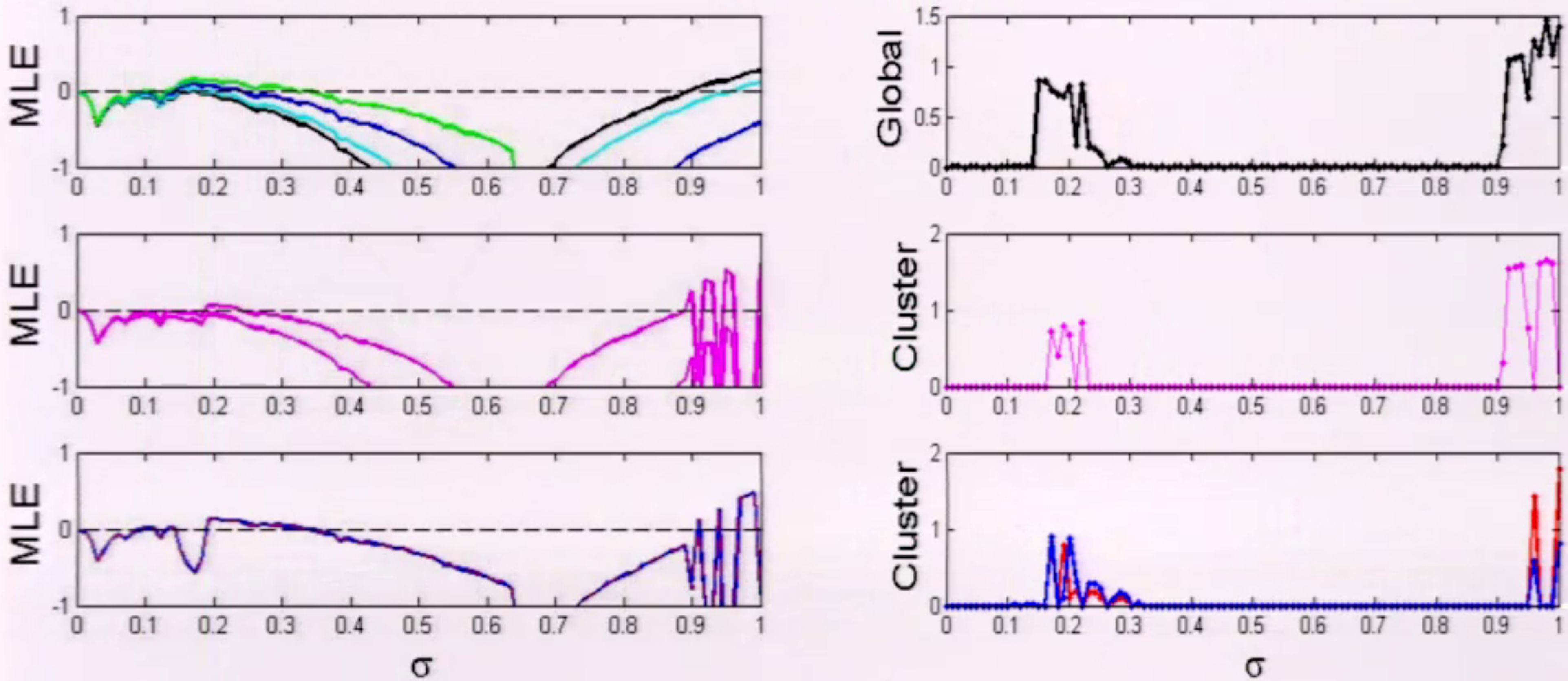
C is the Laplacian matrix corresponding to the N=9 node network in the previous slide, hence the fully synchronous pattern is a solution

We swipe σ while keeping β fixed.

Hierarchy of the symmetry patterns



Stability of the fully sync state versus stability of the max symmetry solution

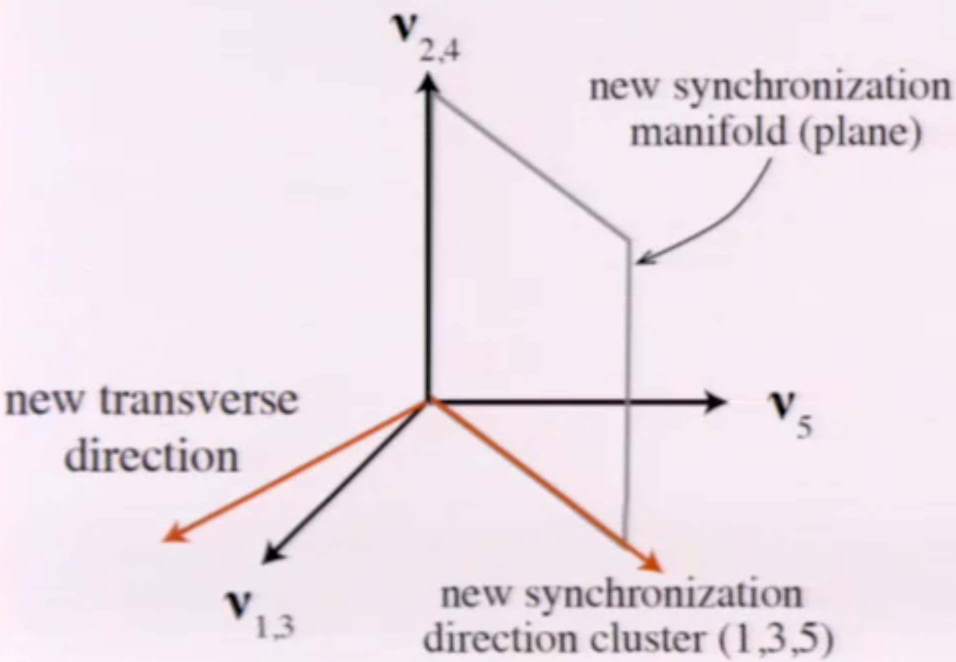


The figure on the right is for initial conditions close to the fully sync solution

Stability of the lower-symmetry patterns

The block-diagonalized form for a given symmetry pattern can be obtained from that of another symmetry pattern

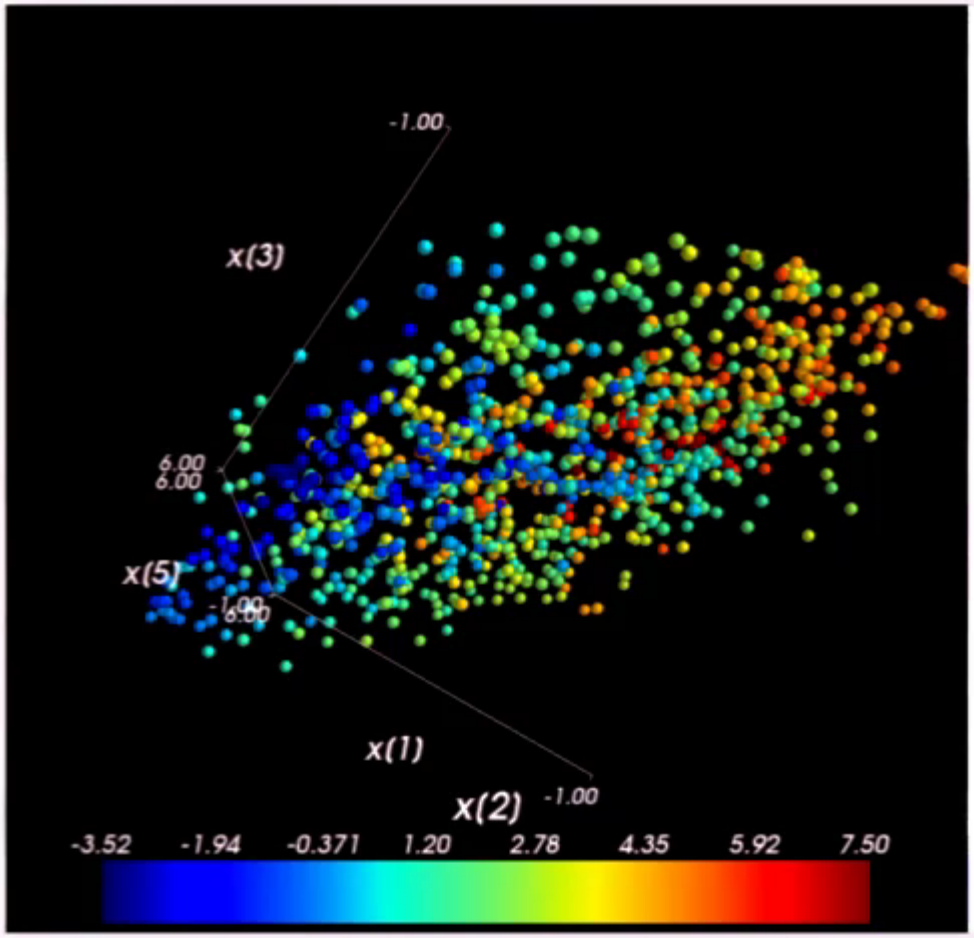
EXAMPLE: (A3 →L3)



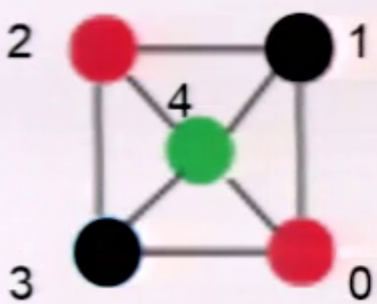
$$L' = \begin{pmatrix} -4.00 & -1.41 & 1.41 & 0.0 & 0.0 \\ -1.41 & -3.00 & -2.00 & 0.0 & 0.0 \\ 1.41 & -2.00 & -3.00 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -3.00 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3.00 \end{pmatrix}$$

$$\Downarrow$$

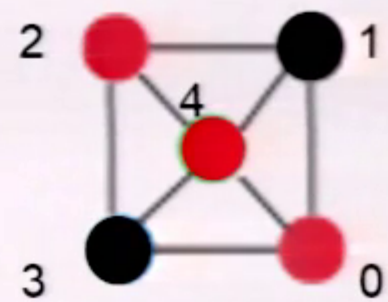
$$L'' = \begin{pmatrix} -3.00 & 2.45 & 0.0 & 0.0 & 0.0 \\ 2.45 & -2.00 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.0 & -5.00 & 0.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & -3.00 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 & -3.00 \end{pmatrix}$$



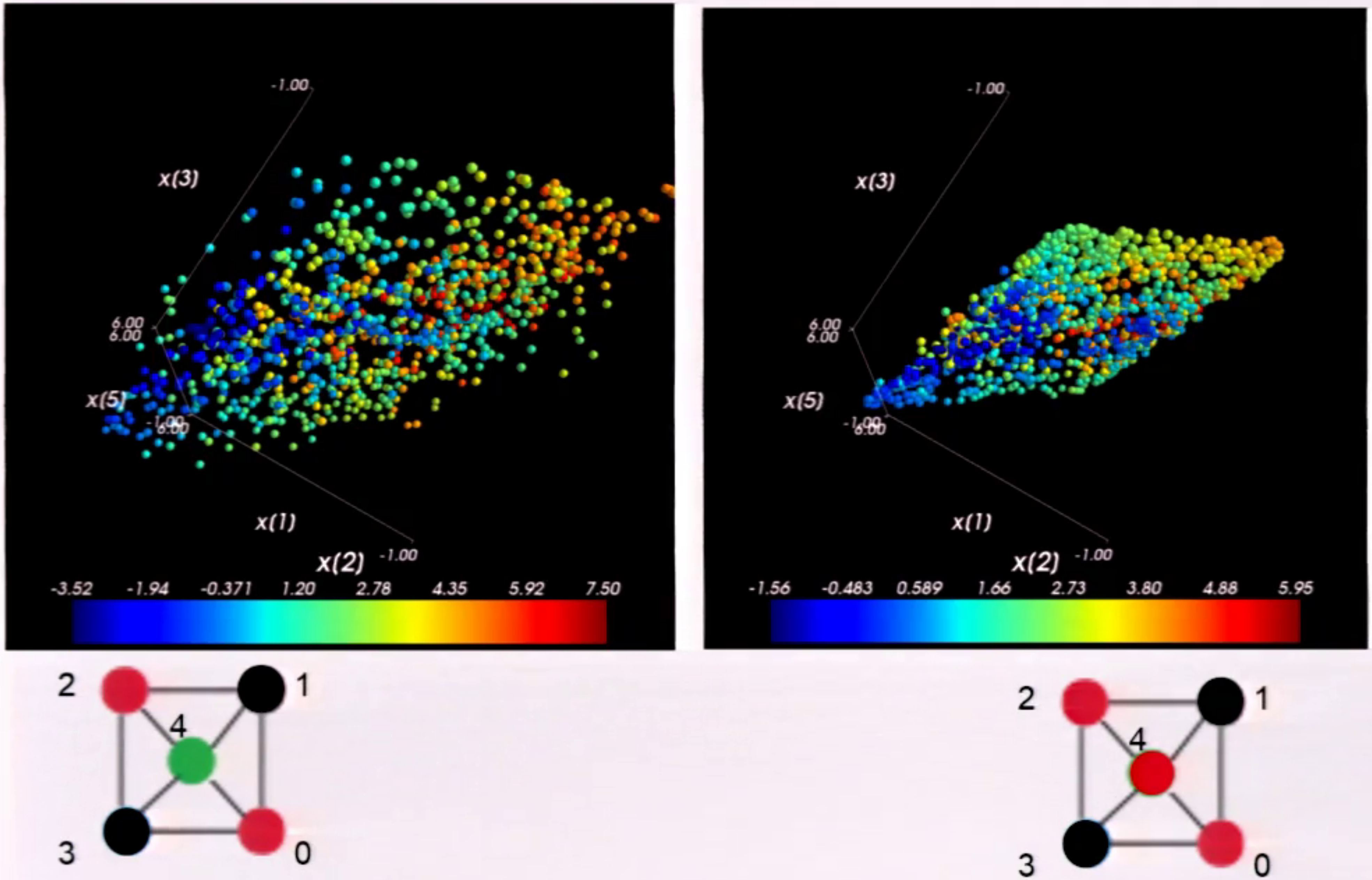
-1.00
x(3)
6.00
6.00
x(5)
-1.00
6.00
x(1)
x(2)
-1.00
-3.52
-1.94
-0.371
1.20
2.78
4.35
5.92
7.50



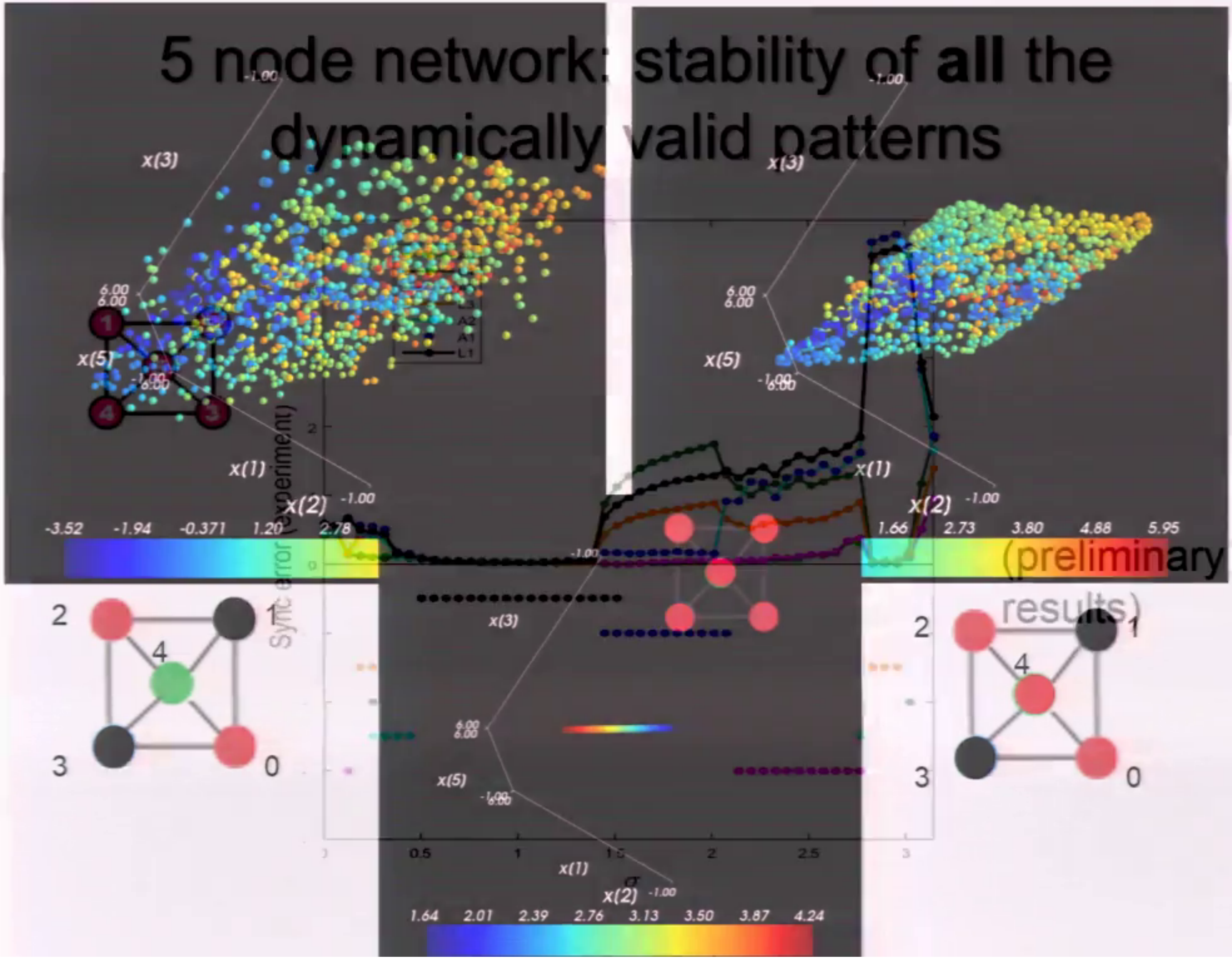
2
1
4
3
0



2
1
4
3
0

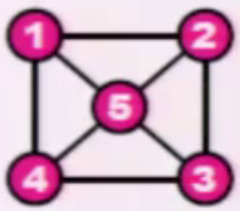

x(3)
x(5)
x(1)
x(2)
-1.00
6.00
-3.52 -1.94 -0.371 1.20 2.78 4.35 5.92 7.50
-1.56 -0.483 0.589 1.66 2.73 3.80 4.88 5.95
0
1
2
3
4

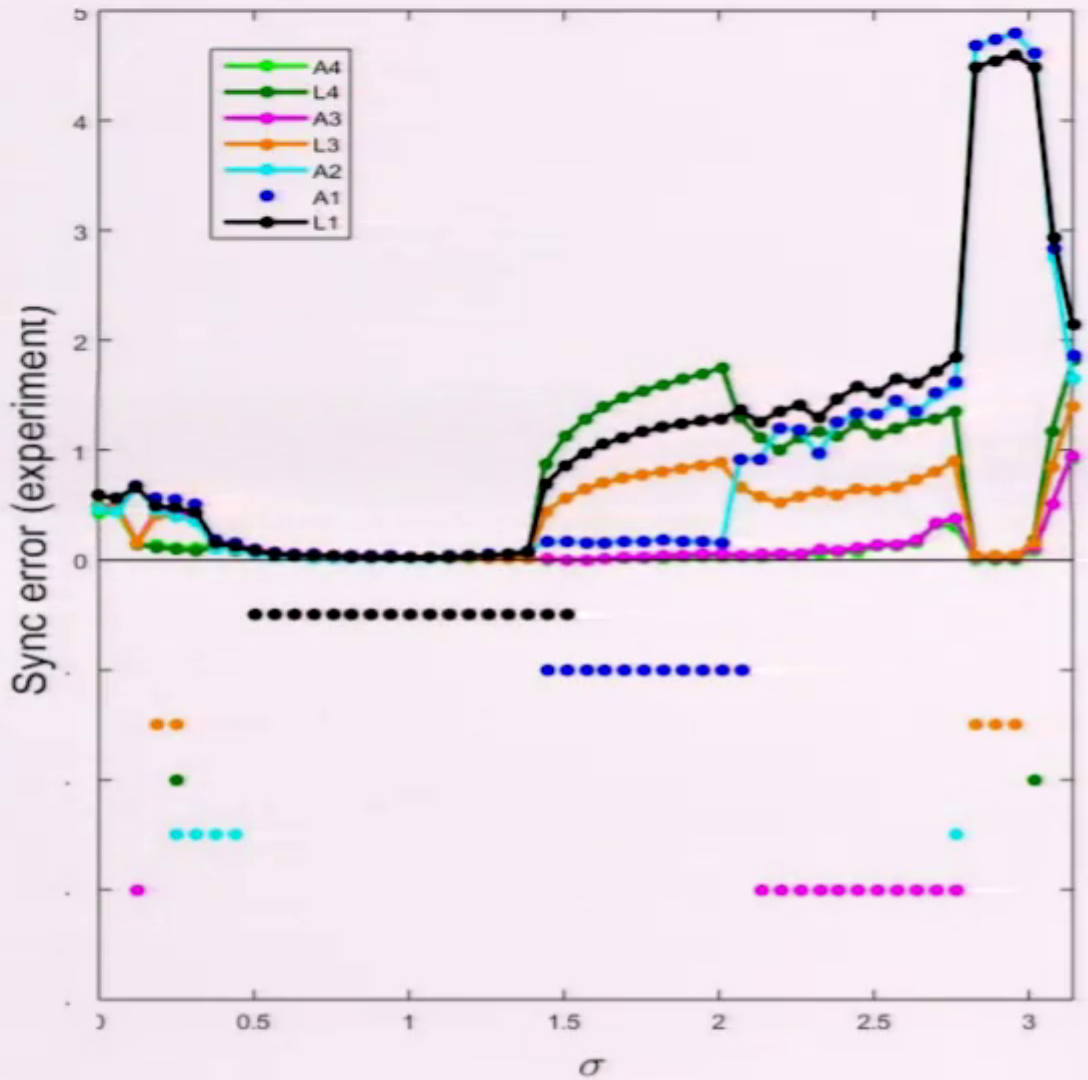
5 node network: stability of **all** the dynamically valid patterns



(preliminary results)

5 node network: stability of **all** the dynamically valid patterns





(preliminary results)