Time-Dependent Spatiotemporal Chaos in Pattern-Forming Systems with Two Length Scales

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R., Silber & Skeldon (2012), *Phys. Rev. Lett.*, **108** 074504
Catllá, McNamara & Topaz (2012), *Phys. Rev. E*, **85** 026215

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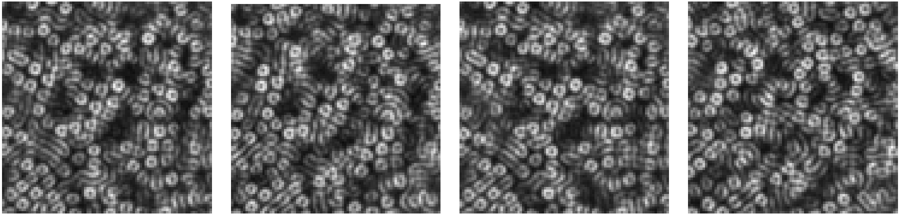
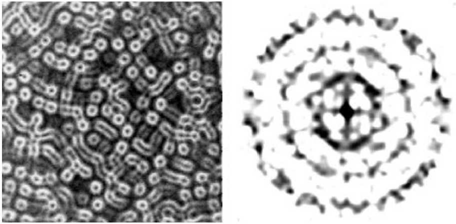
Patterns with two length scales I



Epstein & Fineberg (2005)

Spatiotemporal chaos: "... continually evolving irregular domains of patterns with differing spatial orientations."



Arbell & Fineberg (2002)

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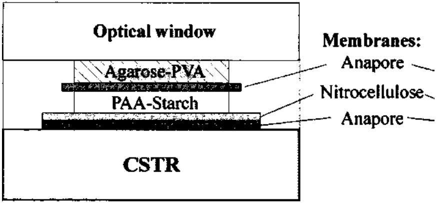
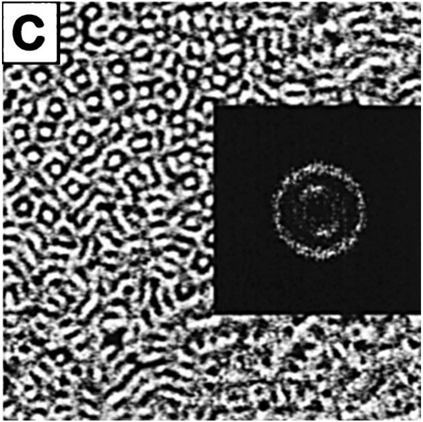
Patterns with two length scales II

Two-layer Turing (reaction–diffusion) patterns:



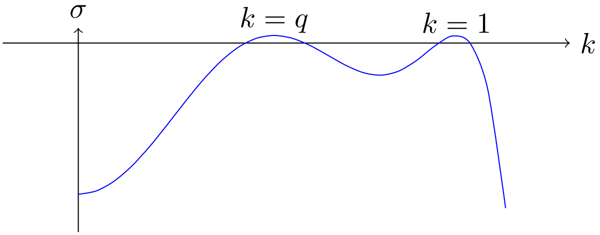
Patterns with different length-scales ($0.46\,\mathrm{mm}$ and $0.25\,\mathrm{mm}$) in the two layers are diffusively coupled



Berenstein et al. (2004)

Two length scales: linear theory I

Consider waves with wavenumbers $k = 1$ and $k = q$ $(q < 1)$ becoming unstable, with growth rates μ and ν respectively:

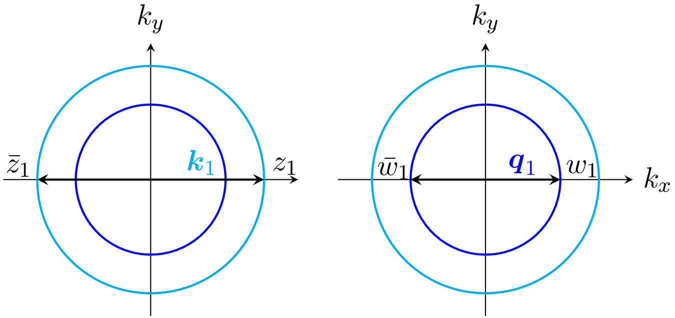


At onset, the pattern $U(x, y, t)$ will contain a combination of eigenfunctions: Fourier modes $e^{i\boldsymbol{k}\cdot\boldsymbol{x}}$ with $|\boldsymbol{k}| = q$ or $|\boldsymbol{k}| = 1$:

$$U = \sum_{\boldsymbol{q}_j} w_j(t) e^{i\boldsymbol{q}_j\cdot\boldsymbol{x}} + \sum_{\boldsymbol{k}_j} z_j(t) e^{i\boldsymbol{k}_j\cdot\boldsymbol{x}}$$

Two length scales: linear theory II

From the multitude, focus on one wave from each of the two circles: $z_1 e^{i\boldsymbol{k}_1\cdot\boldsymbol{x}}$ and $w_1 e^{i\boldsymbol{q}_1\cdot\boldsymbol{x}}$, as well as complex conjugates:

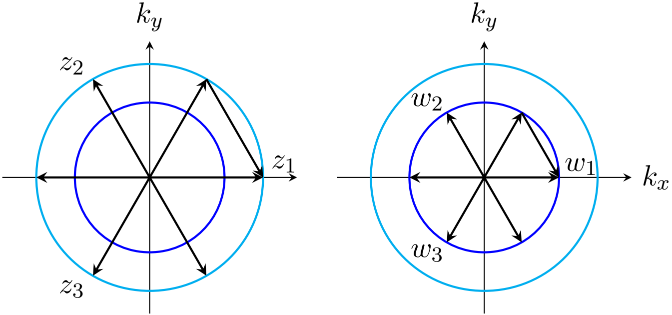


and the evolution of the amplitudes z_1 and w_1 will governed by:

$$\dot{z}_1 = \mu z_1, \qquad \dot{w}_1 = \nu w_1$$

Two length scales: nonlinear theory I

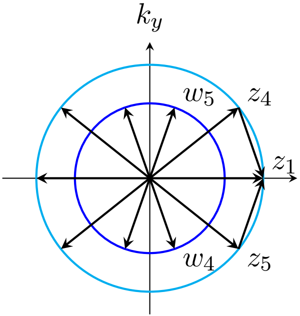
Products of waves lead to sums of wave vectors. Expanding in a power series in the small amplitude of the waves, at second order, there will be contributions from all possible three-wave interactions. The simplest interations involve modes at 60°:

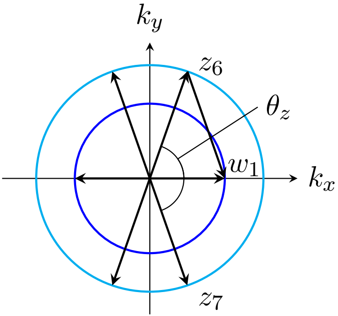


$$\dot{z}_1 = \mu z_1 + Q_{zh}\bar{z}_2\bar{z}_3, \qquad \dot{w}_1 = \nu w_1 + Q_{wh}\bar{w}_2\bar{w}_3$$

Two length scales: nonlinear theory II

Two waves on the outer circle can couple to a wave on the inner circle: $\boldsymbol{k}_6 + \boldsymbol{k}_7 = \boldsymbol{q}_1$, defining $\theta_z = 2\arccos(q/2)$.

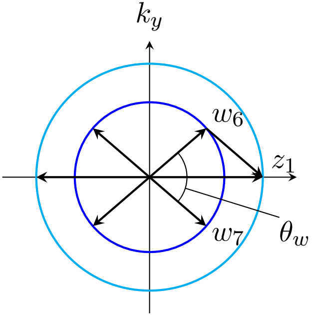




$$\dot{z}_1 = \cdots + Q_{zw}(z_4 w_4 + z_5 w_5), \qquad \dot{w}_1 = \cdots + Q_{zz} z_6 z_7$$

Two length scales: nonlinear theory III

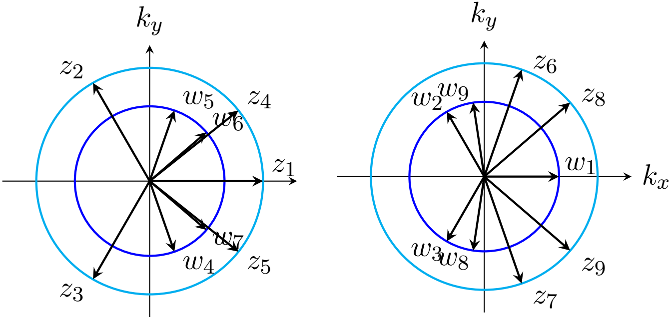
Two waves on the inner circle can couple to a wave on the outer, provided $q \geq \frac{1}{2}$: $\boldsymbol{q}_6 + \boldsymbol{q}_7 = \boldsymbol{k}_1$, defining $\theta_w = 2\arccos(1/2q)$.



$$\dot{z}_1 = \cdots + Q_{ww} w_6 w_7, \qquad \dot{w}_1 = \cdots + Q_{wz}(w_8 z_8 + w_9 z_9)$$

Two length scales: nonlinear theory IV

Putting it all together: there are 8 modes that couple to each of z_1 and w_1:

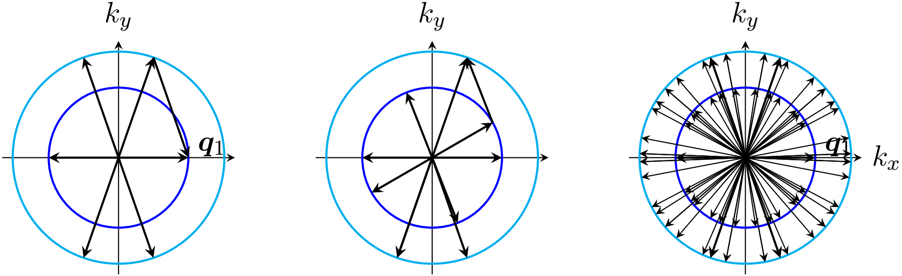


$$\dot{z}_1 = \mu z_1 + Q_{zh}\bar{z}_2\bar{z}_3 + Q_{zw}(z_4 w_4 + z_5 w_5) + Q_{ww} w_6 w_7,$$

$$\dot{w}_1 = \nu w_1 + Q_{wh}\bar{w}_2\bar{w}_3 + Q_{zz} z_6 z_7 + Q_{wz}(w_8 z_8 + w_9 z_9)$$

Two length scales: nonlinear theory V

However, each z mode we've introduced couples to 8 other modes, and each w mode we've introduced couples to 8 other modes, and so on: an infinite number of modes can be generated:



Here, $q = 0.66$, $\theta_z = 141.4°$, $\theta_w = 81.5°$.

At cubic order, all modes couple to all other modes.

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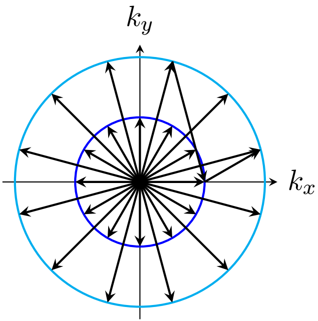
Two length scales: nonlinear theory VI



For $q = \frac{1}{2}(\sqrt{6} - \sqrt{2}) = 0.5176$ ($\theta_z = 150^\circ$, $\theta_w = 30^\circ$), these interactions lead to a finite number of waves, 12 on each circle.

This is the only q for which a finite number of waves will form a closed set under three-wave interaction in two dimensions, suggesting why 12-fold quasipatterns are the most common in 2D.

Three-wave interactions I

How to make progress? Pull out one of the basic three-wave interactions, two outer vectors coupling to an inner:

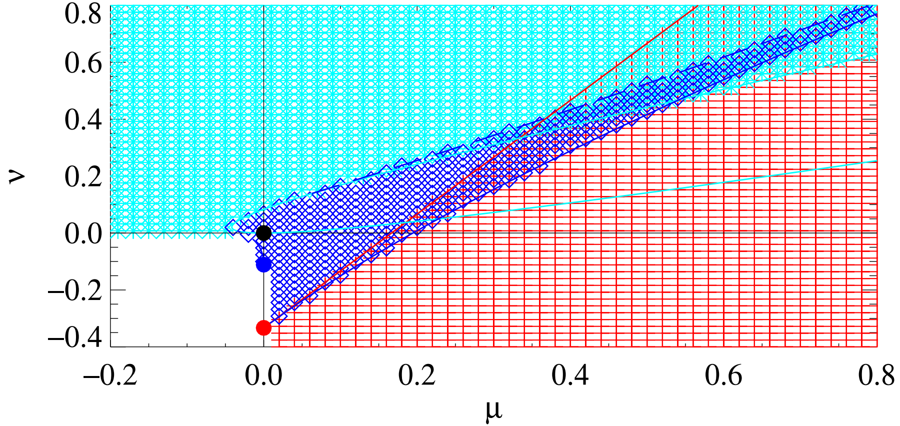
We illustrate using:

$$\begin{aligned}
\dot{z}_1 &= \mu z_1 + Q_{zw}\bar{z}_2 w_1 - (3|z_1|^2 + 6|z_2|^2 + 6|w_1|^2)z_1 \\
\dot{z}_2 &= \mu z_2 + Q_{zw}\bar{z}_1 w_1 - (6|z_1|^2 + 3|z_2|^2 + 6|w_1|^2)z_2 \\
\dot{w}_1 &= \nu w_1 + Q_{zz} z_1 z_2 - (6|z_1|^2 + 6|z_2|^2 + 3|w_1|^2)w_1
\end{aligned}$$

The outcome depends on the product of quadratic coefficients $Q_{zw}Q_{zz}$. Typically (Cf Porter & Silber 2004):

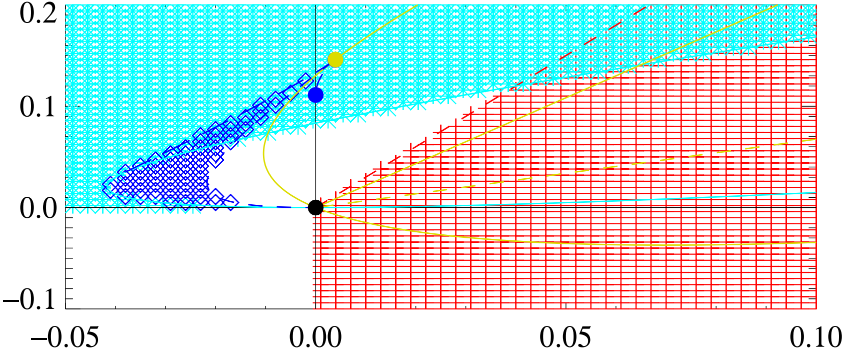
- Positive $Q_{zw}Q_{zz}$: stable steady stripes, or stable rhombs (mixed z and w);
- Negative $Q_{zw}Q_{zz}$: stable steady stripes, or time-dependent competition between z and w modes.
- Same conclusion for any of the three-wave interactions.

Three-wave interactions II



Positive $Q_{zw}Q_{zz}$: stable steady z (red) or w (cyan) stripes, or stable rhombs (blue), which are mixed z and w.

Three-wave interactions III



Negative $Q_{zw}Q_{zz}$: stable steady z or w stripes, some stable rhombs (blue), or time-dependent competition between z and w modes (empty area). (Cf Porter & Silber 2004.)

Three-wave interactions IV

With multiple three-wave interactions, we hypothesise (wth $q > \frac{1}{2}$):

- We expect to find steady complex patterns or spatiotemporal chaos, according to the signs of $Q_{zw}Q_{zz}$ and $Q_{wz}Q_{zz}$.
- If $Q_{zw}Q_{zz}$ and $Q_{wz}Q_{zz}$ are both negative, we expect to see greater time dependence.
- These effects will be more pronounced for larger values of the products.
- With $q = \frac{1}{2}(\sqrt{6} - \sqrt{2}) = 0.5176$ we may find steady or time-dependent 12-fold quasipatterns, according to the signs of $Q_{zw}Q_{zz}$ and $Q_{wz}Q_{zz}$.

Coupled Turing I

The Brusselator is a simple example of a Turing (reaction–diffusion) system:

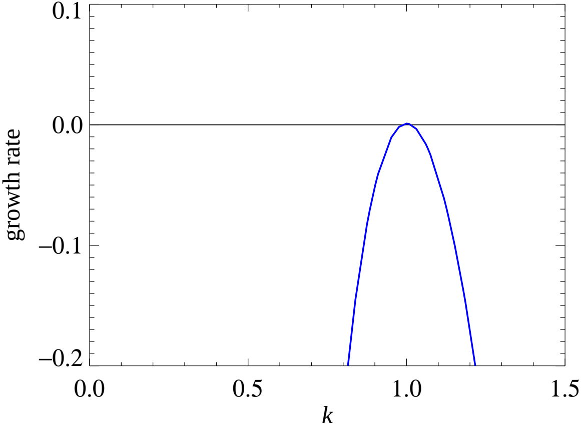
$$\frac{\partial U}{\partial t} = (B-1)U + A^2V + D_U\nabla^2U + \frac{B}{A}U^2 + 2AUV + U^2V,$$
$$\frac{\partial V}{\partial t} = -BU - A^2V + D_V\nabla^2V - \frac{B}{A}U^2 - 2AUV - U^2V,$$

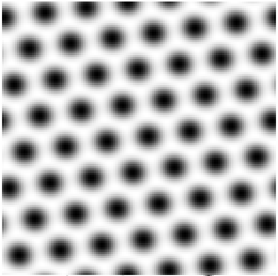
where:

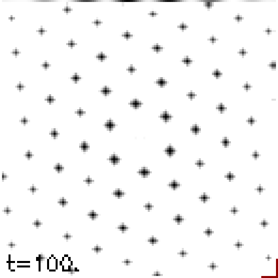
- $U(x,y,t)$ and $V(x,y,t)$ represent chemical concentrations
- A and B are parameters ($A=3$ and $B=9$)
- D_U and D_V are diffusion constants
- Hopf ($k=0$) and pitchfork ($k\neq 0$) instabilities are possible
- The usual nontrivial equilibrium has been moved to the origin

Coupled Turing II

Typical Turing pattern: $D_U = 1.99833$ and $D_V = 4.50875$, 8×8 box







Coupled Turing III

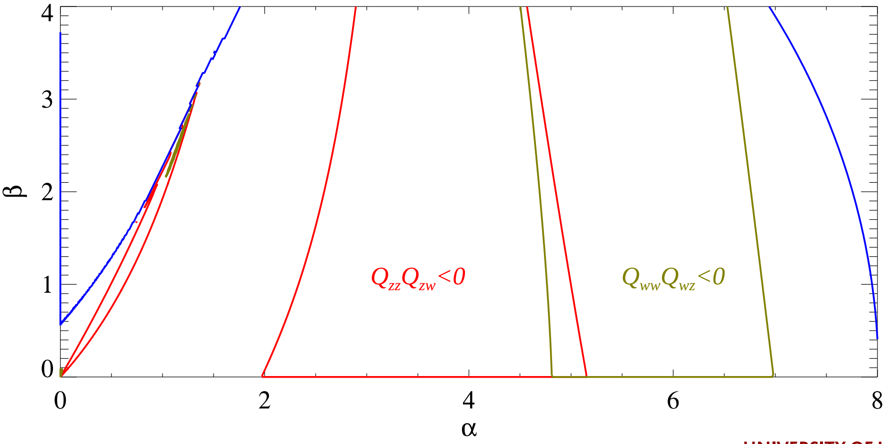
Two layer model (Yang et al 2002, Catllá et al 2012):

$$\frac{\partial U_1}{\partial t} = (B-1)U_1 + A^2 V_1 + D_{U_1}\nabla^2 U_1 + \alpha(U_2 - U_1) + \mathsf{NLT},$$
$$\frac{\partial V_1}{\partial t} = -BU_1 - A^2 V_1 + D_{V_1}\nabla^2 V_1 + \beta(V_2 - V_1) + \mathsf{NLT},$$
$$\frac{\partial U_2}{\partial t} = (B-1)U_2 + A^2 V_2 + D_{U_2}\nabla^2 U_2 + \alpha(U_2 - U_1) + \mathsf{NLT},$$
$$\frac{\partial V_2}{\partial t} = -BU_2 - A^2 V_2 + D_{V_2}\nabla^2 V_2 + \beta(V_2 - V_1) + \mathsf{NLT},$$

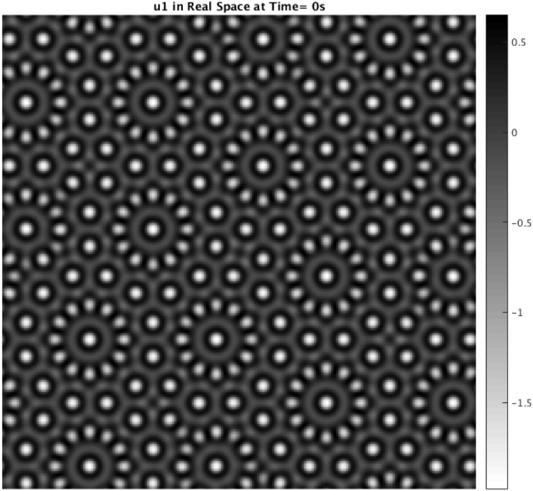
- $U_{1,2}$ and $V_{1,2}$ are concentrations in each layer
- Same A and B and nonlinear terms (NLT) as before
- The diffusion coefficients are not the same in each layer
- The α and β terms couple the two layers

Coupled Turing IV

For $q = 0.5176$ and for a range of α and β, we solve for the four values D_{U_1}, D_{U_2}, D_{V_1} and D_{V_2} at the codimension-two point, and compute the quadratic coefficients Q_{zz}, Q_{zw}, Q_{ww} and Q_{wz}:



Coupled Turing V



$\alpha = 1$, $\beta = 1.0$, $\mu = -0.0115$, $\nu = 0.0277$, 30×30, $D_{U_1} = 1.6108$, $D_{V_1} = 4.6687$, $D_{U_2} = 9.9397$, $D_{V_2} = 25.4080$, $Q_{zz}Q_{zw} > 0$, $Q_{ww}Q_{wz} > 0$.

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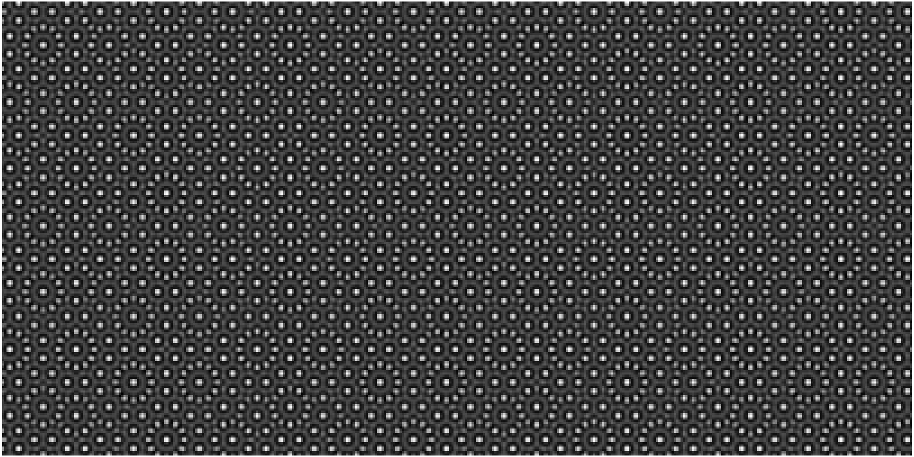
Coupled Turing VI



112×112.

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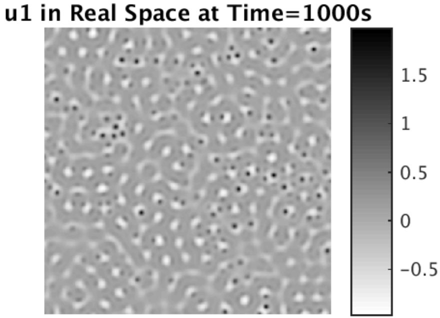
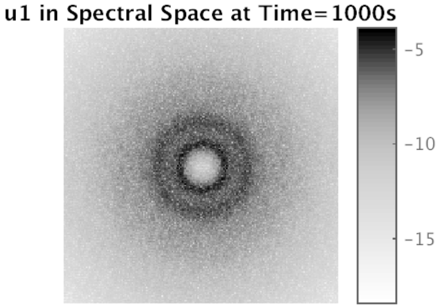
Coupled Turing VII





$\alpha = 5.0$, $\beta = 1.0$, $\mu = -0.095$, $\nu = 0.029$, 30×30, $Q_{zz}Q_{zw} < 0$, $Q_{ww}Q_{wz} < 0$.

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Conclusions

- If the ratio of wavenumbers q is between $\frac{1}{2}$ and 2, mode interactions in both directions must be taken in to account.
- Most values of q in this range lead to the possibility of generating an infinite number of interacting waves.
- The outcome of the mode interactions will be influenced by the signs of the quadratic coefficients, with time-dependence (and spatiotemporal chaos) most likely in the case of (both pairs of) quadratic coefficients with opposite sign.
- These ideas can help find quasipatterns and spatiotemporal chaos in coupled reaction–diffusion problems (work ongoing).