Some Bifurcations and Wave Patterns Arising in Excitable and Oscillatory Models of Neuroscience Context

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MS 60

The FitzHugh-Nagumo (FHN) Reaction-Diffusion (RD) system

$$\begin{cases} \epsilon u_t &= f(u) - v + du_{xx},\ (x,t) \in \Omega \times [0,+\infty[\\ v_t &= u - c(x) \end{cases} \tag{1}$$

whith $f(u) = -u^3 + 3u$, ϵ small, and Neumann Boundary conditions (NBC). In the Neuroscience context, u represents a potential, v a recovery variable and Ω is a bounded domain in $\mathbb{R}^2$ or $\mathbb{R}$.

Outline

1. Background and history

2. Patterns, Wave Propagations, Synchronization

3. Qualitative analysis-Bifurcations

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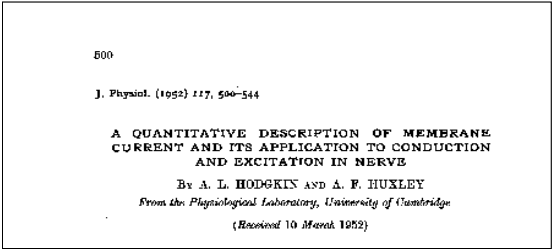
1. Background and history
2. Patterns, Wave Propagations, Synchronization
3. Qualitative analysis-Bifurcations

Hodgkin and Huxley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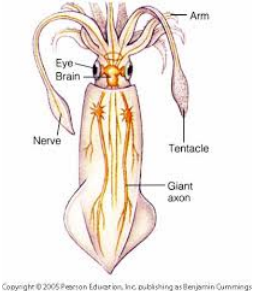
J. Physiol. (1952) 117, 500–544

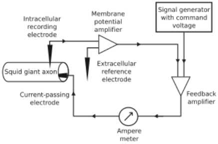
A QUANTITATIVE DESCRIPTION OF MEMBRANE CURRENT AND ITS APPLICATION TO CONDUCTION AND EXCITATION IN NERVE

By A. L. HODGKIN and A. F. HUXLEY

From the Physiological Laboratory, University of Cambridge

(*Received* 10 *March* 1952)





Hodgkin and Huxley

From eqn. (6) this may be transformed into a form suitable for comparison with the experimental results, i.e.

$$g_{\mathrm{K}} = \{(g_{\mathrm{K}\infty})^{\frac{1}{4}} - [(g_{\mathrm{K}\infty})^{\frac{1}{4}} - (g_{\mathrm{K}0})^{\frac{1}{4}}] \exp(-t/\tau_n)\}^4, \tag{11}$$

where $g_{\mathrm{K}\infty}$ is the value which the conductance finally attains and $g_{\mathrm{K}0}$ is the conductance at $t=0$. The smooth curves in Fig. 3 were calculated from

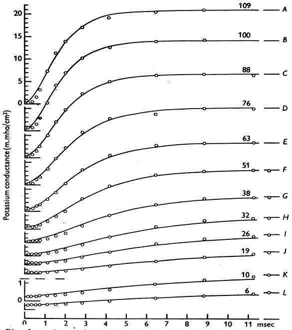
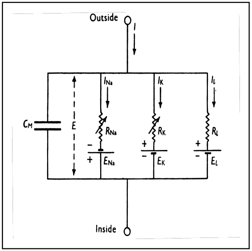
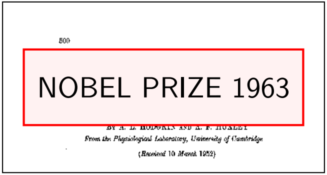


Fig. 3. Rise of potassium conductance associated with different depolarizations. The circles are experimental points obtained on axon 17, temperature 6–7° C, using observations in sea water and choline sea water (see Hodgkin & Huxley, 1952*a*). The smooth curves were drawn from eqn. (11) with $g_{\mathrm{K}0}=0{\cdot}24$ m.mho/cm² and other parameters as shown in Table 1. The time scale applies to all records. The ordinate scale is the same in the upper ten curves (*A* to *J*) and is increased fourfold in the lower two curves (*K* and *L*). The number on each curve gives the depolarization in mV.

Hodgkin-Huxley Reaction-Diffusion system

$$
\begin{cases}
C\dfrac{dV}{dt} &= V_{xx} + I - \overline{g}_{Na} m^3 h (V - E_{Na}) - \overline{g}_K n^4 (V - E_K), \\
 & \quad -\overline{g}_L (V - E_L) \\
\dfrac{dn}{dt} &= \alpha_n(V)(1-n) - \beta_n(V) n \\
\dfrac{dm}{dt} &= \alpha_m(V)(1-m) - \beta_m(V) m \\
\dfrac{dh}{dt} &= \alpha_h(V)(1-h) - \beta_h(V) h
\end{cases}
$$

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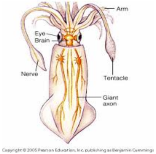
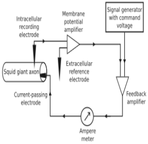
NOBEL PRIZE 1963

From the Physiological Laboratory, University of Cambridge

(Received 10 March 1952)

" For their discoveries concerning the ionic mechanisms involved in excitation and inhibition in the peripheral and central portions of the nerve cell membrane." See www.nobelprize.org.



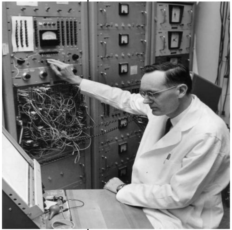


"To this day their work stands as one of the best examples of how scientists can use mathematics to provide insights into complicated biological systems." Nature Education-2010

The Bonhoeffer-Van der Pol model (FitzHugh-Nagumo equations)

IMPULSES AND PHYSIOLOGICAL STATES IN THEORETICAL MODELS OF NERVE MEMBRANE

RICHARD FITZHUGH
From the National Institutes of Health, Bethesda



$$\begin{cases} x_t = c(y + x - \frac{x^3}{3} + z) \\ y_t = -(x - a + by)/c \end{cases} \qquad (2)$$

with $1 - 2b/3 < a$, $0 < b < 1$, $b < c^2$; z stimulus intensity, an arbitrary function of t which can be a Dirac
(Original FitzHugh paper, p 447)

FitzHugh (1961)

"The one to be described in the present paper considers the HH as one member of a large class of non-linear systems showing excitable and oscillatory behavior."

This approach is, however, not so informative in explaining how trains of impulses occur in the HH equations, where interactions between all four variables are essential. Two other approaches to this problem, also based on phase space methods, are more useful. The one to be described in the present paper considers the HH model as one member of a large class of non-linear systems showing excitable and oscillatory behavior. The phase plane model used by Bonhoeffer (1941, 1948, 1953) and Bonhoeffer and Langhammer (1948) to explain the behavior of passi-

445

Van der Pol (1926)

(After Liénard's Transformation)

$$\begin{cases} x_t &= c(y + x - \frac{x^3}{3} + z) \\ y_t &= -x/c \end{cases} \tag{3}$$

Bonhoeffer (1948)

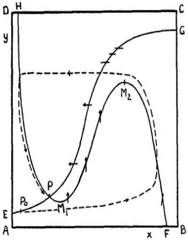


FIG. 8. xy diagram for current densities barely above rheobase. Single activation. Fig. 8 is obtained from Fig. 6 by raising the curves HF_1 and LM_2F_2. The notations and the meaning of the lines are the same as in Fig. 6.

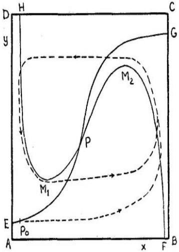
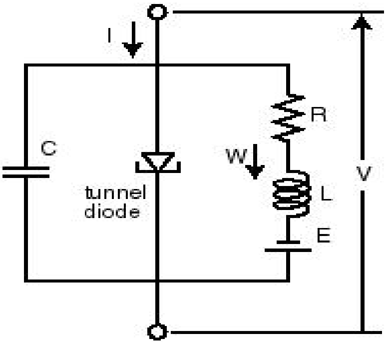


FIG. 9. xy diagram for higher current densities. Rhythmic activation. The notation and the meaning of the lines are the same as in Fig. 6.

FitzHugh-Nagumo equations

In 1962, Nagumo *et al.* provided the analog equivalent circuit.



See:

Nagumo J., Arimoto S., and Yoshizawa S. (1962) An active pulse transmission line simulating nerve axon. Proc. IRE. 50:2061–2070.

Now, the BVP model is called the FHN system.

The FHN ODE system

$$\left\{ \begin{array}{rcl} \epsilon u_t & = & f(u) - v \\ v_t & = & u - c \end{array} \right. \qquad (4)$$

whith $f(u) = -u^3 + 3u$, ϵ small.

The FHN ODE system : excitable and oscillatory behavior

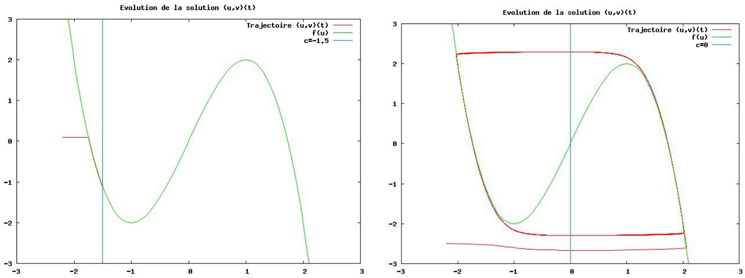


Figure: Solutions of system (4), for typical values of c.

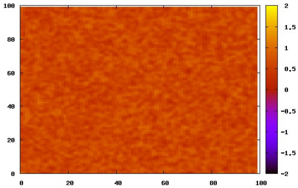
The FHN ODE system

Theorem 1

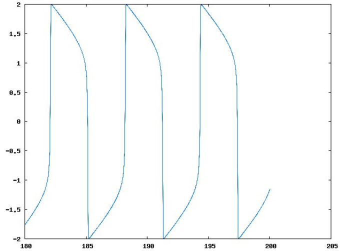
There exists a unique stationary point. If $|c| \geq 1$ the stationary point is globally asymptotically stable, whereas if $|c| < 1$, it is unstable and there exists a unique limit-cycle which attracts all the non constant trajectories. Furthermore, at $|c| = 1$, there is a supercritical Hopf bifurcation.

Outline

$\mathbf{c=0}$;IC:Uniform law on [0,1]



Asymptotic homogeneous space behavior for (1) ($u(x_1, x_2, 0)$). Initial conditions: uniform law on $[0, 1]$.

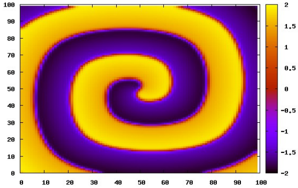


Asymptotic evolution of a solution of (1) at some space points. Red line: $u(x_1, x_2, t)$ for $(x_1, x_2) = (50, 50, t)$, for time $t \in [180, 200]$. Red line: $(x_1, x_2) = (50, 100, t)$. Blue line: $\int_\Omega u(x, t)dx$.

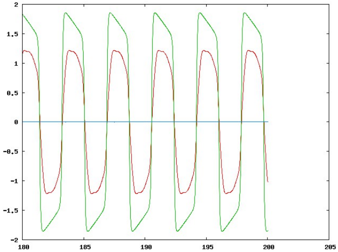
General case

Is there other solutions?

$\mathbf{c=0}$; IC:Specific

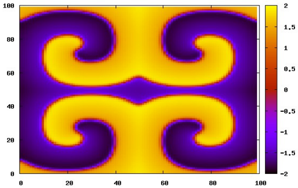


Asymptotic non-homogeneous space behavior of spiral type for (1) ($u(x_1, x_2, 190)$).
Initial conditions:
$(u_0(x), v_0(x)) = (1, 0)$ on Left Top (LT) square, $(0, 1)$ on RT, $(0, -1)$ on LB , $(-1, 0)$ on RB.

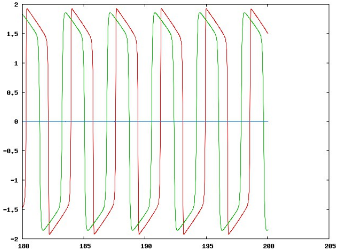


Asymptotic evolution of a solution of (1) at some space points. Red line: $u(x_1, x_2, t)$ for $(x_1, x_2) = (50, 50, t)$, for time $t \in [180, 200]$. Green line: $(x_1, x_2) = (50, 100, t)$. Blue line: $\int_\Omega u(x, t)dx$.

Numerical simulations



Asymptotic non-homogeneous space behavior of four spiral type for (1) ($u(x_1, x_2, 190)$). Initial conditions: we reproduce four times the previous one with symmetry.



Asymptotic evolution of a solution of (1) at some space points. Green line: $u(x_1, x_2, t)$ for $(x_1, x_2) = (50, 50, t)$, for time $t \in [180, 200]$. Red line: $(x_1, x_2) = (50, 100, t)$. Blue line: $\int_\Omega u(x,t)dx$.

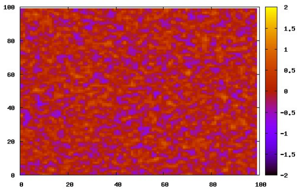
$\mathbf{c=0}$;Invariant subspace

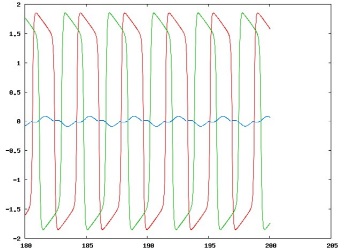
Theorem 2

Suppose that we can divide the domain into a partition $\Omega = (\cup_{i\in\{1,\dots,l\}} U_i) \cup (\cup_{i\in\{1,\dots,l\}} V_i)$ such that there exists a diffeomorphism ϕ that maps each U_i to V_i, $i \in \{1,\dots,l\}$, with $|\det J_\phi| = 1$, where J_ϕ is the jacobian of ϕ and initial conditions such that for all $x \in \cup_{i\in\{1,\dots,l\}} U_i$ and for all $t \in \mathbb{R}^+$, $(u(\phi(x),t), v(\phi(x),t)) = -(u(x,t), v(x,t))$ then the solution of (1) cannot evolve asymptotically around $(\bar{u}, \bar{v})$.

See: B. A., M.A. Aziz-Alaoui, Basin of Attraction of Solutions with Pattern Formation in Slow–Fast Reaction–Diffusion Systems Acta Biotheoretica 64 (4), (2016), 311-325.

$\mathbf{c=0}$; IC:Uniform law on [-1,1]





Asymptotic non-homogeneous space behavior of multiple spiral type for (1) ($u(x_1, x_2, 0)$). Initial conditions: uniform law on $[-1, 1]$.

Asymptotic evolution of a solution of (1) at some space points. Red line: $u(x_1, x_2, t)$ for $(x_1, x_2) = (50, 50, t)$, for time $t \in [180, 200]$. Green line: $(x_1, x_2) = (50, 100, t)$. Blue line: $\int_\Omega u(x, t)dx$.

c **x-dependant**. Propagation of oscillatory signals

Is the system (1) able to generate oscillatory signals and to propagate it?

Is the system (1) able to generate oscillatory signals and to propagate it?
Yes, the idea being:
" Oscillatory signal initiates at some point and propagates along excitatory tissue thanks to diffusion."

c **x-dependant**. Propagation of oscillatory signals

We will consider functions such that:

- $c(x) < -1$ for x close to the border, (Excitatory dynamics for the ODE system)
- $c(x) = 0$ for x close to the center, (Oscillatory dynamics for the ODE system)

c x-dependant. Propagation of oscillatory signals (2D)

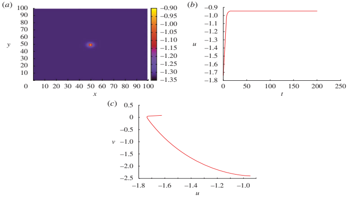


Figure 1. Solutions for $\delta = 0.01$, $c_0 = -1.3$ and (a) $u(x, y, 50)$, (b) $u(50, 50, t)$ (solid line) and (c) $(u, v)(50, 50, t)$ (solid line). (a) Evolution to stationary solution for $c_0 = -1.3$ and $t = 50$; (b) evolution of the variable u for a central cell and $c_0 = -1.3$; and (c) evolution of (u, v) for a central cell and $c_0 = -1.3$.

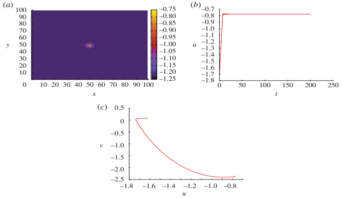


Figure 2. Solutions for $\delta = 0.01$, $c_0 = -1.195$. (a) $u(x, y, 50)$, (b) $u(50, 50, t)$ (solid line) and (c) $(u, v)(50, 50, t)$ (solid line). (a) Evolution to stationary solution for $c_0 = -1.195$ and $t = 50$; (b) evolution of the variable u for a central cell and $c_0 = -1.195$; and (c) evolution of (u, v) for a central cell and $c_0 = -1.195$.

c x-dependant. Propagation of oscillatory signals (2D)

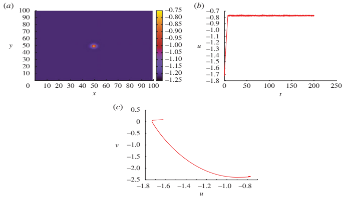


Figure 3. Solutions for $\delta = 0.01$, $c_0 = -1.19302$ ($T = 200$). (a) $u(x, y, 50)$, (b) $u(50, 50, t)$ (solid line) and (c) $(u, v)(50, 50, t)$ (solid line). (a) Solution for $c_0 = -1.19302$ and $t = 50$; (b) evolution of the variable u for a central cell and $c_0 = -1.19302$; and (c) evolution of (u, v) for a central cell and $c_0 = -1.19302$.

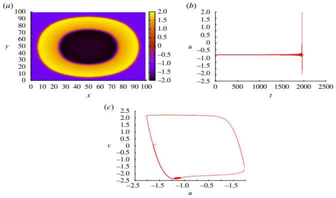


Figure 4. Solutions for $\delta = 0.01$, $c_0 = -1.19302$ ($T = 2000$). (a) $u(x, y, 1950)$, (b) $u(50, 50, t)$ (solid line) and (c) $(u, v)(50, 50, t)$ (solid line). (a) Solution for $c_0 = -1.19302$ and $t = 50$; (b) evolution of the variable u for a central cell and $c_0 = -1.19302$; and (c) evolution of (u, v) for a central cell and $c_0 = -1.19302$.

c x-dependant. Propagation of oscillatory signals (2D)

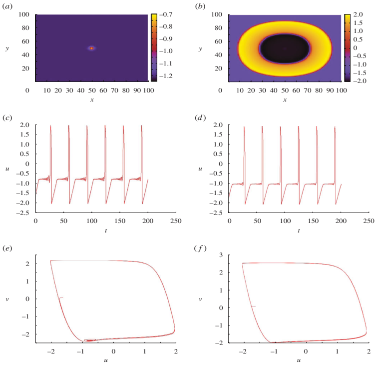


Figure 5. Solutions for $\delta = 0.01$, $c_0 = -1.19$. (a) $u(x, y, 50)$, (b) $u(x, y, 62)$, (c) $u(50, 50, t)$ (solid line), (d) $u(51, 50, t)$ (solid line), (e) $(u, v)(50, 50, t)$ (solid line) and (f) $(u, v)(51, 50, t)$ (solid line). (a) Solution for $c_0 = -1.19$ and $t = 50$; (b) solution for $c_0 = -1.19$ and $t = 62$; (c) evolution of the variable u for a central cell and $c_0 = -1.19$; (d) evolution of the variable u for a non-central cell and $c_0 = -1.19$; (e) evolution of (u, v) for a central cell and $c_0 = -1.19$; and (f) evolution of (u, v) for a non-central cell and $c_0 = -1.19$.

(b) *Numerical simulations of system (4.1) and propagation of the bursting oscillations*

The same result is observed numerically: there is a threshold such that, if the excitability c_0 is below the threshold, the solution evolves to a stationary solution. If it is above, there is a propagation of the bursting oscillations. The number of

c x-dependant. Propagation of oscillatory signals (2D)

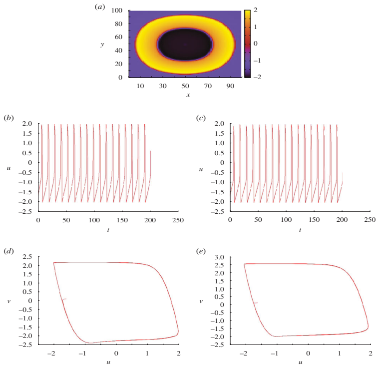


Figure 6. Solutions for $\delta = 0.01$, $c_0 = -1.15$. (*a*) $u(x, y, 55)$, (*b*) $u(50, 50, t)$ (solid line), (*c*) $u(51, 50, t)$ (solid line), (*d*) $(u, v)(50, 50, t)$ (solid line) and (*e*) $(u, v)(51, 50, t)$ (solid line). (*a*) Solution for $c_0 = -1.15$ and $t = 55$; (*b*) evolution of the variable u for a central cell and $c_0 = -1.15$; (*c*) evolution of the variable u for a non-central cell and $c_0 = -1.15$; (*d*) evolution of (u, v) for a central cell and $c_0 = -1.15$; and (*e*) evolution of (u, v) for a central cell and $c_0 = -1.15$.

spikes increases as $\gamma \to 0$ and/or as $c_0 \to -1$. Initial conditions are around the values $u(0, x) = -1$, $v(0, x) = 0.1$, $\delta = 0.01$; propagation of bursting oscillations is seen in figure 7.

c x and t-dependant. Propagation of bursting signals (2D)

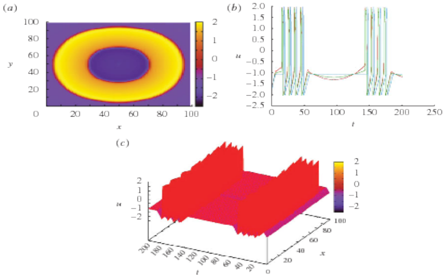
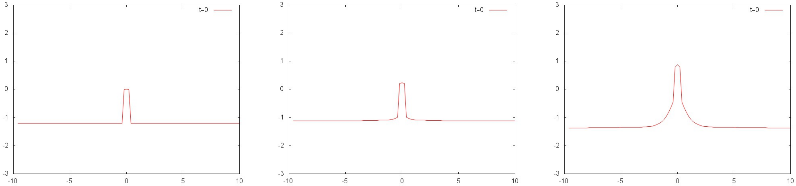


Figure 7. Solutions of the system (4.1), for $\gamma = 0.05$ and $c = -1, 05$. (*a*) $u(x, y, 28)$, (*b*) $u(50, 50, t)$ (red line), $u(51, 50, t)$ (green line) and $u(99, 50, t)$ (blue line) and (*c*) $u(x, 49, t)$ (red line).

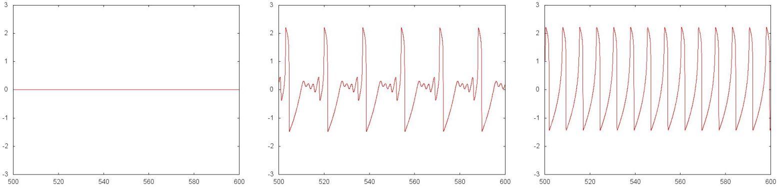
c x-dependant. Propagation of oscillatory signals (1D)



(a) Solution stationnaire (b) Bifurcation (c) Propagation d'ondes

Figure: Bifurcation from stationnary solution to wave propagation.

c x-dependant. Propagation of oscillatory signals (1D)



(a) Solution stationnaire (b) Bifurcation (c) Propagation d'ondes

Figure: Bifurcation from stationnary solution to wave propagation.

Networks: Synchronization of Patterns

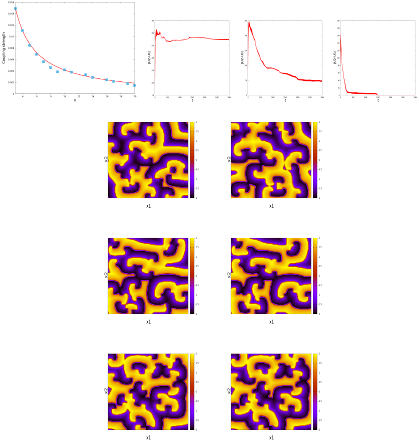


Figure: **Fully connected network. Case $c(x) = 0$.**

See B. A., M A Aziz-Alaoui. V L E Phan, "Large time behaviour and synchronization of complex networks of reaction–diffusion systems of FitzHugh–Nagumo type" IMA JAM,84(2), (2019), 416-443

Outline

Notations

We set

$$\mathcal{H} = L^2(0,1) \times L^2(0,1)$$

$\mathcal{V} = H^1(0,1) \times H^1(0,1)$ where $H^1(0,1)$ is the classical Sobolev space.

$||\cdot||$ will denote the norm on $\mathcal{H}$.

A toy model

We consider

$$\begin{cases} u_t &= \alpha u - u^3 - v + u_{xx} \\ v_t &= u \end{cases} \tag{5}$$

on the domain $(0, 1)$ with Neumann Boundary conditions.

Linearization around $(0,0)$

Note first that $(0,0)$ is a constant solution of (5). The linearized system around this point is given by:

$$\left\{\begin{array}{lcl} u_t & = & \alpha u - v + u_{xx} \\ v_t & = & u \end{array}\right. \qquad (6)$$

on the domain $(0,1)$ with Neumann Boundary conditions.

Linearization around $(0,0)$

Using the spectral decomposition, we can give a detailed and comprehensive analysis of the qualitative behavior of (6). Classically, we set:

$$\varphi_0(x) = 1, \text{ and } \forall k \in \mathbb{N}^* \, \varphi_k(x) = \sqrt{2}\cos(k\pi x).$$

We recall that the familly $(\varphi_k)_{k\in\mathbb{N}}$ is an othornormal basis of L^2, and that the funcions φ_k satisfy:

$$-(\varphi_k)_{xx} = \lambda_k \varphi_k$$

and

$$(\varphi_k)_x(0) = (\varphi_k)_x(1) = 0,$$

with

$$\lambda_k = k^2\pi^2.$$

Linearization around $(0, 0)$

Looking for solutions of the form,

$$u(t) = \sum_{k=0}^{\infty} u_k(t)\varphi_k, \; v(t) = \sum_{k=0}^{\infty} v_k(t)\varphi_k$$

leads by projection on the eigenspace generated by (φ_k, φ_k) to the resolution of the two dimensional ODE systems indexed by k, and denoted by E_k:

$$(E_k) \left\{ \begin{array}{lcl} u_{kt} & = & (\alpha_k - \lambda_k)u_k - v_k \\ v_{kt} & = & u_k \end{array} \right. \tag{7}$$

Linearization around $(0,0)$

The eigenvalues of matrix

$$A_k = \begin{pmatrix} \alpha - \lambda_k & -1 \\ 1 & 0 \end{pmatrix}$$

are given by

$$\sigma_k^1 = \frac{1}{2}\left(\alpha - \lambda_k - \sqrt{(\alpha - \lambda_k)^2 - 4}\right), \; \sigma_k^2 = \frac{1}{2}\left(\alpha - \lambda_k + \sqrt{(\alpha - \lambda_k)^2 - 4}\right).$$

We summarize the remarkable properties of σ_k^1 and σ_k^2 in the following proposition.

Proposition 1

When α crosses λ_k from left to right, σ_k^1 and σ_k^2 cross the imaginary axis from left to right. Furthermore,

$$\lim_{k \to +\infty} \sigma_k^1 = -\infty \textit{ and } \lim_{k \to +\infty} \sigma_k^2 = 0^-.$$

Linearization around $(0, 0)$

Theorem 3 (Linearized System)

For $\alpha < 0$, for any initial condition $(u(\cdot, 0), v(\cdot, 0))$ in $\mathcal{H}$, we have

$$\lim_{t \to +\infty} ||(u, v)(t)|| = 0.$$

Linearization around $(0,0)$

Theorem 4

Let $k \in \mathbb{N}^*$. For $\alpha = \lambda_k$, $(0,0)$ is a center for system E_k, a source for E_l if $l < k$ and a sink for E_l if $l > k$. Furthermore, if: $u_l(0) = v_l(0) = 0$ for $l \in \{0, ..., k-1\}$ then

$$\lim_{t\to+\infty} ||(u,v)(t) - \varphi_k(u_k(t), v_k(t))|| = 0.$$

Otherwise,

$$\lim_{t\to+\infty} ||(u,v)(t)|| = +\infty.$$

Linearization around $(0, 0)$

Theorem 4 (part 2)

For $\lambda_k < \alpha < \lambda_{k+1}$, $(0, 0)$ is a source for E_l si $l \leq k$ and a sink for E_l if $l > k$. Furthermore, if $u_l(0) = v_l(0) = 0$ for $l \in \{1, ..., k\}$ then

$$\lim_{t \to +\infty} ||(u, v)(t)|| = 0.$$

Otherwise

$$\lim_{t \to +\infty} ||(u, v)(t)|| = +\infty.$$

Toy Model

Theorem 5 (Nonlinear System)

For $\alpha < 0$, for any initial condition $(u(\cdot, 0), v(\cdot, 0))$ in $\mathcal{H}$,

$$\lim_{t \to +\infty} ||(u, v)(t)|| = 0.$$

Proof

- Lyapunov Function
- LaSalle's Principle

Theorem 6

For $0 < \alpha < \lambda_1$*, if* $u(x) = -u(1-x)$ *and* $v(x) = -v(1-x)$ *then for all IC in* $\mathcal{H}$

$$\lim_{t \to +\infty} ||(u,v)(t)|| = 0$$

Proof

- Invariant Subspace
- Lyapunov Function
- LaSalle's Principle

Toy Model

Theorem 7

For $0 < \alpha < \lambda_1$, *there exists a sequence* $(\mu_k)_{k\in\mathbb{N}}$ *such that if*

$$(u_k(0), v_k(0)) \in B(0, \mu_k) \subset \mathbb{R}^2$$

then

$$\lim_{t\to+\infty} ||(u(t) - u_0(t), v(t) - v_0(t))|| = 0,$$

where $B(0, \mu_k)$ *is the ball of center* $(0, 0)$ *and radius* μ_k.

Proof

- Projection onto subspaces
- Nonlinear terms bounded by:

$$C\sum_{i=1}^{\infty}|u_i|\sum_{i=1}^{\infty}u_i^2$$

- Estimation on each subspace
- Lyapunov function

Asumptions on $c(x)$

We assume that the function $c(x)$, depending on a parameter $p > 0$, is regular and satisfies the following conditions:

$$
\begin{aligned}
c(x) \leq 0 & \qquad \forall x \in (-a, a), && (8)\\
c(0) = 0, & && (9)\\
c'(x) > 0 & \quad \forall x \in (-a, 0),\ c'(x) < 0\, \forall x \in (0, a), && (10)\\
c'(-a) = c'(a) = 0, & && (11)\\
\forall x \in (-a, a), x \neq 0, & \quad c(x) \text{ is a decreasing function of } p, && (12)\\
\forall x \in (-a, a), x \neq 0, & \qquad \lim_{p\to 0} c(x) = 0, && (13)\\
\forall x \in (-a, a), x \neq 0, & \qquad \lim_{p\to +\infty} c(x) = -\infty. && (14)
\end{aligned}
$$

Typical example

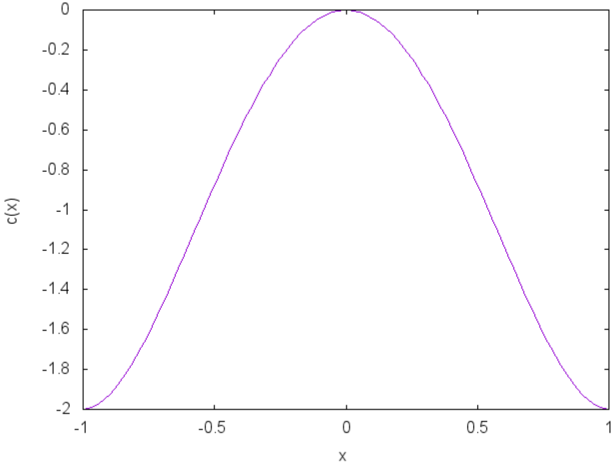


Figure: Graph of $c(x)$ for $p = 5$.

Stationary solution

The stationary solution is given by

$$\begin{cases} \bar{v} &= f(\bar{u}) + d\bar{u}_{xx} \\ \bar{u} &= c(x) \end{cases} \tag{15}$$

The linearized system around $(\bar{u}, \bar{v})$ writes:

$$\left\{ \begin{array}{rcl} \epsilon u_t & = & f'(\bar{u})u - v + du_{xx} \\ v_t & = & u \end{array} \right. \tag{16}$$

Spectral analysis

We would like to proceed to projection on appropriate subspaces as in previous sections. To that end, we are interested in solutions of the following equation

$$f'(c(x))u + du_{xx} = \lambda u \tag{17}$$

with NBC. Note that equation (17) is a regular Sturm-Liouville problem.

Theorem 8

There exists an increasing sequence of real numbers (λ_n) *and an orthogonal basis* $(\varphi)_{n\in\mathbb{N}}$ *of* $L^2(-a,a)$ *such that:*

$$\begin{array}{rcl} (d\varphi_{nxx} + f'(\bar{u})\varphi_n & = & \lambda_n \varphi_n \\ \varphi_n'(-a) = \varphi_n'(a) & = & 0. \end{array}$$

Furthermore,

$$\lim_{n\to+\infty} \lambda_n = +\infty,$$

$$\lambda_0 = \inf_{u\in H^2(-a,a), |u|_{L^2(-a,a)}=1} d\int_{(-a,a)} |u_x|^2 dx - \int_{(-a,a)} f'(c(x))u^2 dx.$$

and

$$\lambda_n = \frac{\pi^2 n^2}{4a^2} + O(n)$$

Spectral analysis

The projection on the *kth* subspace writes

$$(E_k)\begin{cases} \epsilon u_{kt} & = & -\lambda_k u_k - v_k \\ v_{kt} & = & u_k \end{cases} \tag{18}$$

while the eigenvalues are given by

$$\begin{cases} \sigma_1^k & = & \frac{1}{2\epsilon}\Bigg(- \lambda_k - \sqrt{(\lambda_k^2 - 4\epsilon}\Bigg) \\ \sigma_2^k & = & \frac{1}{2\epsilon}\Bigg(- \lambda_k + \sqrt{\lambda_k^2 - 4\epsilon}\Bigg) \end{cases} \tag{19}$$

Spectral analysis

Theorem 9

For each p, the number of eigenvalues with positive real part is finite. For p small enough, σ_1^0 and σ_2^0 have a positive real part. For p large enough, all the eigenvalues σ_1^k and σ_2^k have negative real part. There is an Hopf Bifurcation: there exists a value p_0 for which as p crosses p_0 from right to left, σ_1^0 and σ_2^0 are complex and their real part increase from negative to positive. The other eigenvalues remaining with negative real parts. Furthermore,

$$\lim_{k\to+\infty} \sigma_k^1 = -\infty \text{ and } \lim_{k\to+\infty} \sigma_k^2 = 0^-.$$

See: B. A.,"Hopf Bifurcation in an Oscillatory-Ecxitable Reaction-Diffusion system with spatial heterogeneity", International Journal of Bifurcation and Chaos, 27(5), (2017)

Theorem 10

For $p > p_0$, there exists a sequence $(\mu_k)_{k\in\mathbb{N}}$ such that if

$$(u_k(0), v_k(0)) \in B(0, \mu_k)$$

then

$$\lim_{t\to+\infty} ||(u(t), v(t))|| = 0,$$

where $B(0, \mu_k)$ is the ball of center $(0, 0)$ and radius μ_k.

Theorem 11

There exists $\delta > 0$ *such that for* $p_0 < p < p_0 + \delta$*, there exists a sequence* $(\mu_k)_{k\in\mathbb{N}}$ *such that if*

$$(u_k(0), v_k(0)) \in B(0, \mu_k)$$

then

$$\lim_{t\to+\infty} ||(u(t) - u_0(t), v(t) - v_0(t))|| = 0,$$

where $B(0, \mu_k)$ *is the ball of center* $(0, 0)$ *and radius* μ_k*.*

Thanks!