Sonia Kovalevsky Lecture: Adiabatic Theorem

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A Noisy Adiabatic Theorem: Wilkinson Meets Schrödinger's Cat

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The Plan

- Sonia Kovalevsky
- The problem: perturbations in quantum computing
- Four slides on quantum computing
- The adiabatic theorem and variants
- Applications
- Conclusions

Sonia Kovalevsky (1850-1891)

- 19th century Russian mathematician.
- First woman in Europe to earn a doctorate in mathematics (1874, University of Göttengen, under direction of Weierstrass).
- "S. Kovalevsky" awarded the Prix Bordin of the French Academy of Sciences in 1888 "On the Problem of the Rotation of a Solid Body about a Fixed Point."

"... in this work not only the power of an expansive and profound mind, but also a great spirit of invention."

- *Summa cum laude* dissertation in three parts:

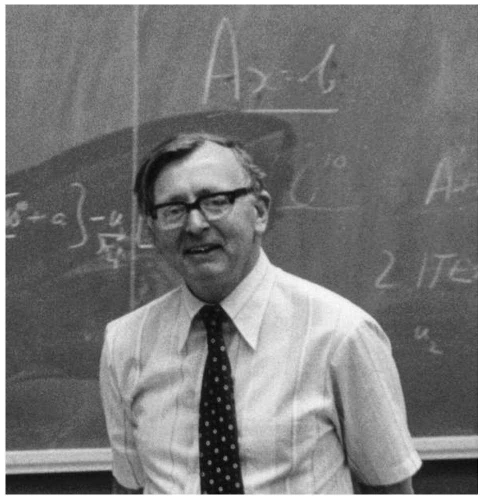
  - "Towards a Theory of Partial Differential Equations". Cauchy-Kovalevsky Theorem is at the foundation of most graduate courses in partial differential equations. First existence theorem for second order linear PDE's in one-dependent and n independent variables.
  - "Supplements and Remarks to Laplace's Investigation of the Form of Saturn's Rings".
  - "On the Reduction of a Class of Abelian Integrals of the Third Rank to Elliptic Integrals".
- Before doctorate: largely self-taught or tutored, since she was excluded from most (all-male) classes. Needed a "marriage of convenience" in order to travel.
- After doctorate: could not find employment in mathematics, so she turned to writing.

- Doctorate: 1874
- During periods of mathematical unemployment, wrote newspaper articles, poetry, criticisms, and a novel, mostly centered on the theme of women's rights.
- Finally received an academic position at Stockholm in 1883, lecturing on differential equations.
- Prix Bordin of the French Academy of Sciences: 1888
- First woman Corresponding Member of the Russian Academy of Sciences (1889), but still could not get a job in Russia.
- Died of flu/pneumonia in Sweden in 1891 at age 41.
- The Kovalevsakya crater on the Moon is named in her honour.
- Various confusing spellings of her name.

"Sofia Kovalevskaya," Wikipedia
"Sonya Kovalevsky," http://scidiv.bcc.ctc.edu/Math/Kovalevsky.html

Perturbation analysis of adiabatic quantum computing

Wilkinson and Schrödinger's cat

- Jim Wilkinson made fundamental contributions to understanding the effects of perturbations on floating-point computation.
- Erwin Schrödinger laid the foundations for understanding quantum phenomena, and I use his (imaginary) cat to symbolize that work.

So what is the effect of perturbations on quantum computing?

Four slides on quantum computing

The main idea

Suppose we want to perform a computation that has 2^p possible answers.

Example: For a (scrambled) list of the numbers 0 to $2^p - 1$, find the index of the item whose value is a given number.

Two models have been proposed for computing using quantum systems:

- Gated systems,
- Adiabatic systems.

The key: In both models, we need only p quantum entities (ions, photons, etc.), not the 2^p entities needed for an exhaustive search in conventional computing.

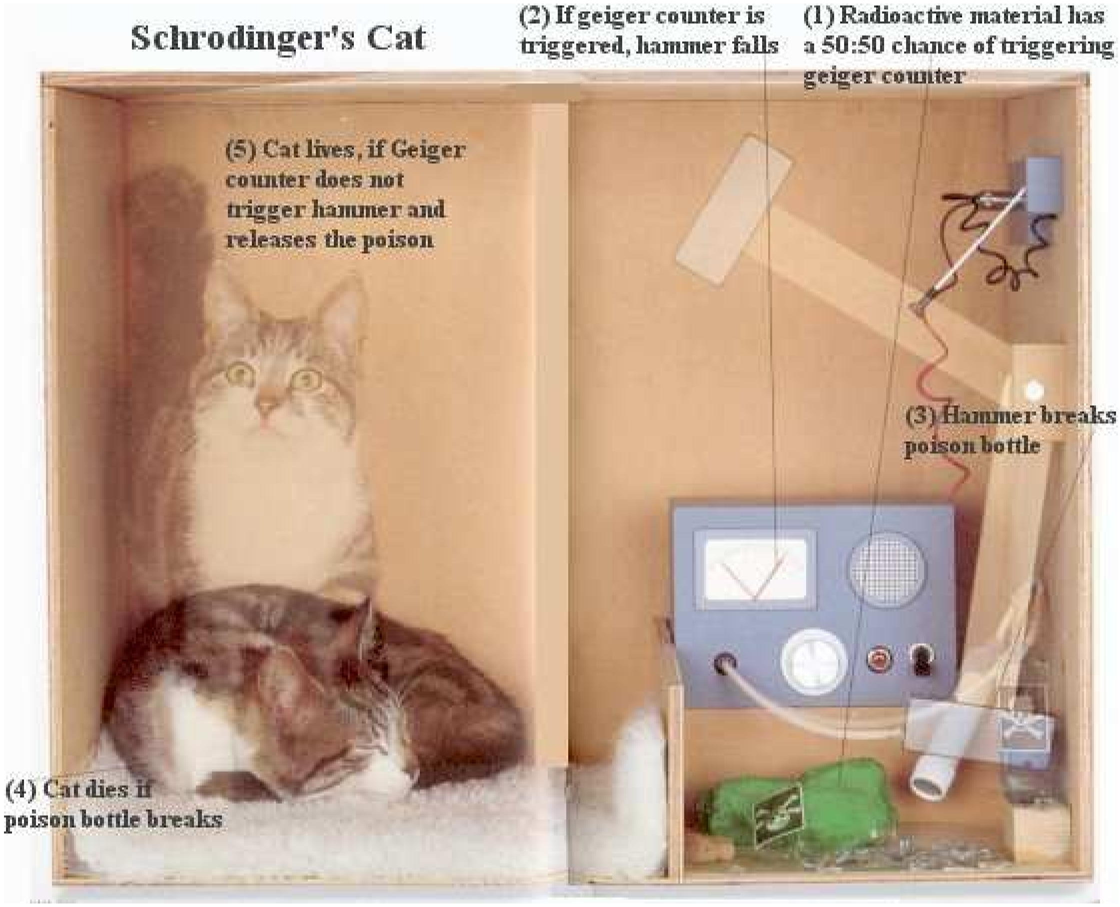
We only consider adiabatic quantum computing in this talk.

Adiabatic systems:

The idea resembles that behind continuation (homotopy) methods.

- Start with an "easy" Hermitian operator $\mathcal{H}(0)$ with a known smallest eigenvalue and corresponding ground-state eigenvector.
- "Evolve" the system very slowly to end with the "desired" operator $\mathcal{H}(1)$, with ground-state eigenvector $\boldsymbol{\psi}(1)$, so that the absolute values of the components of $\boldsymbol{\psi}(1)$ are large if the index corresponds to a correct answer and small otherwise.
- Sample the indices, with these probabilities, to determine the correct answer.
- Key: Until sampling, the system can be regarded as being in a superposition containing all 2^p possible outcomes, with the assigned probabilities.

Schrödinger's thought experiment



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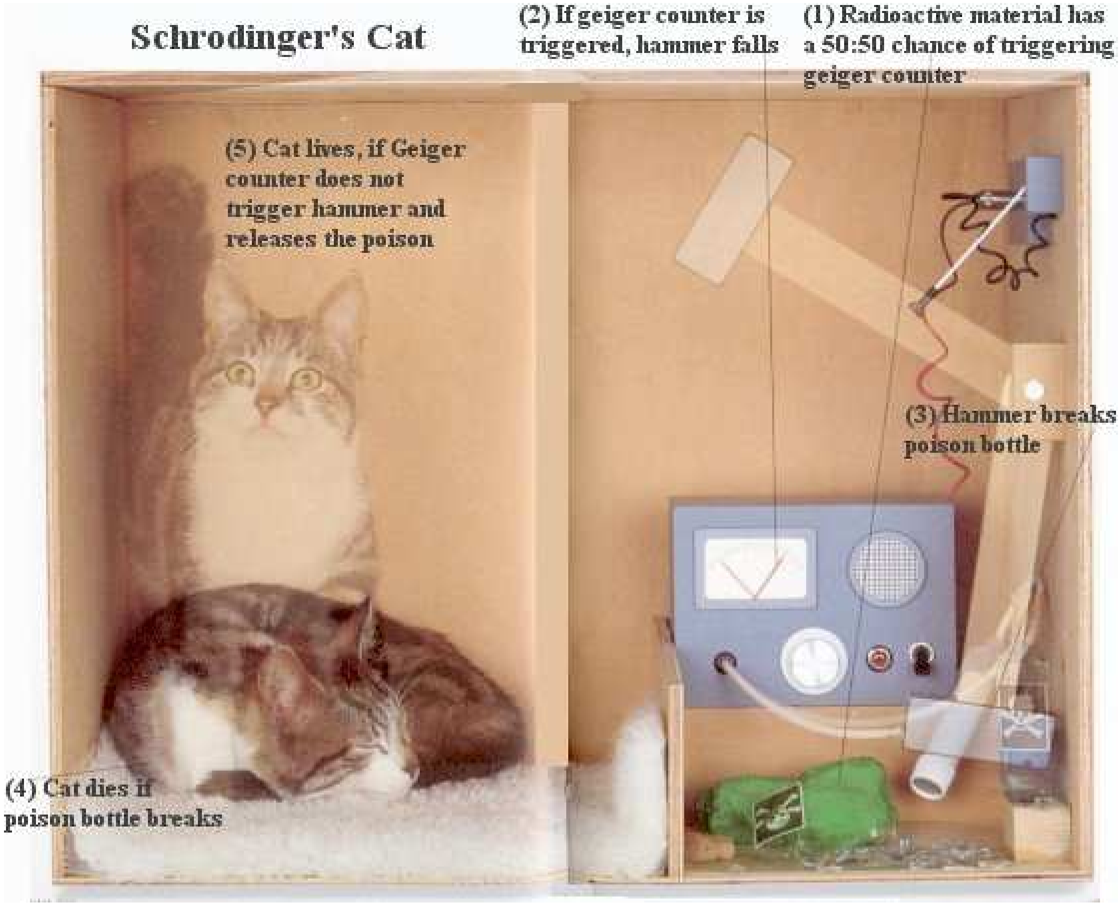
Schrödinger's thought experiment

What happens if the atom is in a superposition of states?



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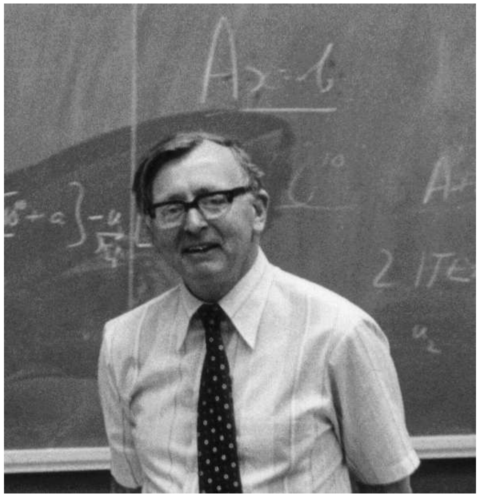
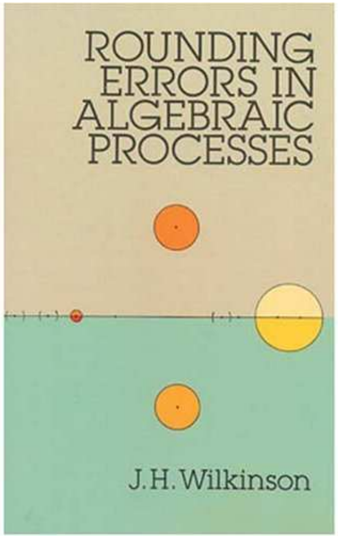
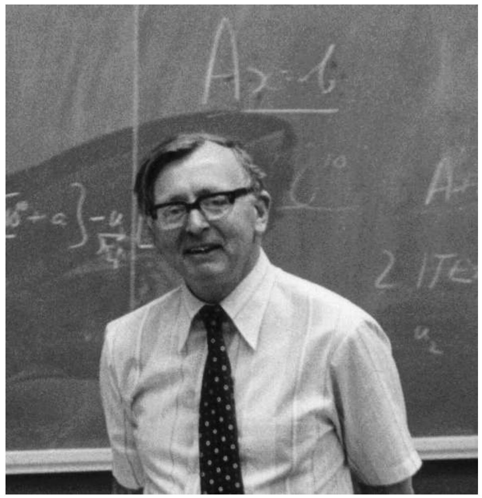
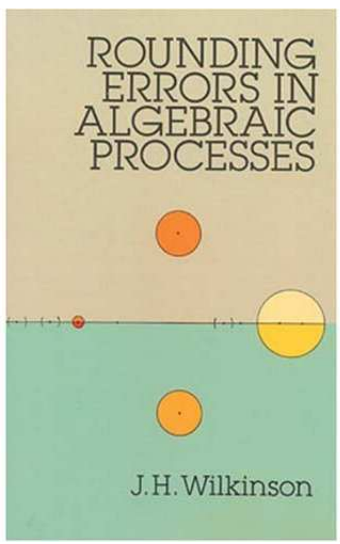
Wilkinson-style perturbation results





Wilkinson-style perturbation results for quantum computing?





???

Even if you don't "believe" in quantum computing ...

.. the Adiabatic Theorem also gives insight into the behavior of quantum mechanical systems such as

- superconductors,
- superfluids,
- radiation of black bodies,
- electron orbits.

So the Adiabatic Theorem is at least as important today as it was when first formulated by Born and Fock (1928).

The adiabatic theorem and variants

The setup

- Schrödinger's equation: $\boldsymbol{U}(s)$ is a $2^p \times 2^p$ unitary operator satisfying

$$\dot{\boldsymbol{U}}(s) = -i\tau\mathcal{H}(s)\,\boldsymbol{U}(s)\ .$$

- A twice-continuously differentiable Hamiltonian evolution $\mathcal{H}(s)$ parameterized by $s \in [0,1]$.
- An evolution time τ, so that the Hamiltonian at time t is $\mathcal{H}(t/\tau)$.
- $\mathcal{H}(s)$ with countable eigenstates $\{\boldsymbol{\psi}_j(s)\}$ and eigenvalues $\lambda_0(s) \leq \lambda_1(s)$..., and a subset of distinguished eigenstates

$$\Psi(s) = \mathrm{span}\,\{\boldsymbol{\psi}_m(s), ..., \boldsymbol{\psi}_n(s)\}\ \ ,$$

- The projector $\boldsymbol{P}(s)$ onto $\Psi(s)$ and $\boldsymbol{Q}(s) = \boldsymbol{I} - \boldsymbol{P}(s)$ onto the complement of $\Psi(s)$.

The error in the adiabatic approximation

The adiabatic approximation says that if you start a system in its groundstate and evolve it slowly enough, the system remains in its groundstate. It is an informal statement of the adiabatic theorem.

We apply $\boldsymbol{P}(0)$ to obtain the component of the initial state contained in $\Psi(0)$, evolve it forward in time by applying $\boldsymbol{U}(s)$, and then apply $\boldsymbol{Q}(s) = \boldsymbol{I} - \boldsymbol{P}(s)$ to compute the component of the state outside $\Psi(s)$. Therefore, the error operator is

$$\boldsymbol{Q}(s)\,\boldsymbol{U}(s)\,\boldsymbol{P}(0).$$

Formulating the error operator allows a formal statement of the theorem.

The Adiabatic Theorem

$$\dot{\boldsymbol{U}}(s) = -i\tau\mathcal{H}(s)\,\boldsymbol{U}(s),$$

$$\begin{aligned}
||\dot{\mathcal{H}}(s)|| &\le b_1(s), \\
||\ddot{\mathcal{H}}(s)|| &\le b_2(s), \\
\gamma(s) &= \min\{\lambda_{n+1}(s) - \lambda_n(s),\ \lambda_m(s) - \lambda_{m-1}(s)\} > 0, \\
w(s) &= \lambda_n(s) - \lambda_m(s), \\
D(s) &= 1 + \frac{2w(s)}{\pi\gamma(s)}\,.
\end{aligned}$$

Then

$$\begin{aligned}
||\boldsymbol{Q}(s)\,\boldsymbol{U}(s)\boldsymbol{P}(0)|| \le\ & \frac{8D^2(0)b_1(0)}{\tau\gamma^2(0)} + \frac{8D^2(s)b_1(s)}{\tau\gamma^2(s)} \\
& + \int_{r=0}^{s} \frac{8D^2(r)}{\tau\gamma^2(r)} \left(\frac{8(1+D(r))b_1^2(r)}{\gamma(r)} + b_2(r) \right) dr\,.
\end{aligned}$$

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\end{aligned}$$

Background

Original result: Born and Fock (1928).

Our version: closely related to that of Reichardt (2004), which is based on that by Avron, Seiler, and Yaffe (1987).

Our contribution:

- Explicit definitions of constants, so it can make quantitative predictions for specific physical systems.
- An integral formulation, for tighter bounds when the energy gap is widely varying.
- Formulation for subspaces rather than just a nondegenerate groundstate. (But for simplicity, the remaining slides consider a single nondegenerate groundstate: $(m = n = 0)$ and $s = 1$.)
- Four variants that give insight into various physical systems.

Four variants of the theorem

- What happens if the initial state is perturbed from its ideal value?

$$\boldsymbol{\psi}_0(0) \longrightarrow \boldsymbol{\phi}(0) = \eta\left(\boldsymbol{\psi}_0(0) + \delta\boldsymbol{\psi}_\perp\right)$$

- What happens if there is a perturbation to the Hamiltonian on a different time-scale?

$$\mathcal{H}(s) \longrightarrow \mathcal{H}_\tau(s) = \mathcal{H}(s) + \mathcal{H}_{noise}(s\tau)$$

- What happens if there is a smooth perturbation to the Hamiltonian?

$$\mathcal{H}(s) \longrightarrow \mathcal{H}_\epsilon(s)$$

- What happens if there is coupling between the Hamiltonian and the external world?

$$\mathcal{H}(s) \longrightarrow \mathcal{H}_\epsilon(s) = \mathcal{H}(s) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \mathcal{H}_{env} + \epsilon\Delta(s)$$

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Theorem: Perturbation in the initial state

Let the initial state be

$$\boldsymbol{\phi}(0) = \eta\left(\boldsymbol{\psi}_0(0) + \delta\boldsymbol{\psi}_\perp\right).$$

Let

$$\begin{aligned} \left|\left|\dot{\mathcal{H}}(s)\right|\right| &\leq b_1(s) \leq \bar{b}_1, \\ \left|\left|\ddot{\mathcal{H}}(s)\right|\right| &\leq b_2(s) \leq \bar{b}_2 \\ \bar{\gamma} &\leq \gamma(s). \end{aligned}$$

Then

$$||\,\boldsymbol{Q}(1)\,\boldsymbol{U}(1)\boldsymbol{\phi}(0)|| \leq |\eta| \left(|\delta| + \frac{8}{\tau\bar{\gamma}^2}\left(2\bar{b}_1 + \bar{b}_2 + \frac{16\bar{b}_1^2}{\bar{\gamma}}\right)\right) \; .$$

Theorem: Hamiltonian evolutions on two time scales

$$\mathcal{H}_\tau(s) = \mathcal{H}(s) + \mathcal{H}_{noise}(s\tau)$$

Assume

$$\left|\left|\dot{\mathcal{H}}(s)\right|\right| \le b_1 \qquad \left|\left|\dot{\mathcal{H}}_{noise}(t)\right|\right| \le d_1$$

$$\left|\left|\ddot{\mathcal{H}}(s)\right|\right| \le b_2 \qquad \left|\left|\ddot{\mathcal{H}}_{noise}(t)\right|\right| \le d_2$$

$$\sqrt{1 - |\langle \boldsymbol{\psi}_0(0), \boldsymbol{\phi}_0(0)\rangle|^2} = \delta_0 \qquad \sqrt{1 - |\langle \boldsymbol{\psi}_0(1), \boldsymbol{\phi}_0(1)\rangle|^2} = \delta_1 \ ,$$

where $\boldsymbol{\phi}_0(s)$ is the ground state of $\mathcal{H}_\tau(s)$ and $\boldsymbol{\psi}_0(s)$ is the ground state of $\mathcal{H}(s)$. Further assume that $0 < \bar{\gamma} \le \gamma_\tau(s)$ for all s and τ.

Then $||\,\boldsymbol{Q}(1)\,\boldsymbol{U}_\tau(1)\boldsymbol{P}(0)||$ is bounded by

$$\frac{8}{\bar{\gamma}^2}\left[\left(d_2 + \frac{16d_1^2}{\bar{\gamma}}\right)\tau + 2d_1\left(1 + \frac{16b_1}{\bar{\gamma}}\right) + \left(2b_1 + b_2 + \frac{16b_1^2}{\bar{\gamma}}\right)\frac{1}{\tau}\right] + \delta_0 + \delta_1 + \delta_0\delta_1 \ .$$

Some practicalities

$$\mathcal{H}_\tau(s) = \mathcal{H}(s) + \mathcal{H}_{noise}(s\tau), \quad \Delta(t) = \|\mathcal{H}_{noise}(s\tau)\|.$$

When it is inconvenient to compute $\delta_0 = \sqrt{1 - |\langle \boldsymbol{\psi}_0(0), \boldsymbol{\phi}_0(0)\rangle|^2}$, it can be bounded using the "$\sin(\Theta)$ theorem" and the Bauer-Fike theorem:

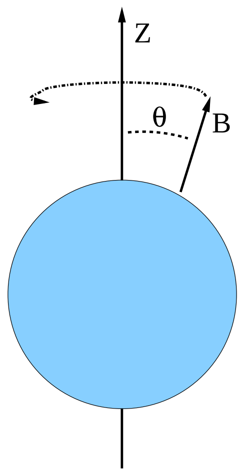
$$\delta_0 \leq \frac{\Delta(0)}{\gamma(0) - \Delta(0)},$$

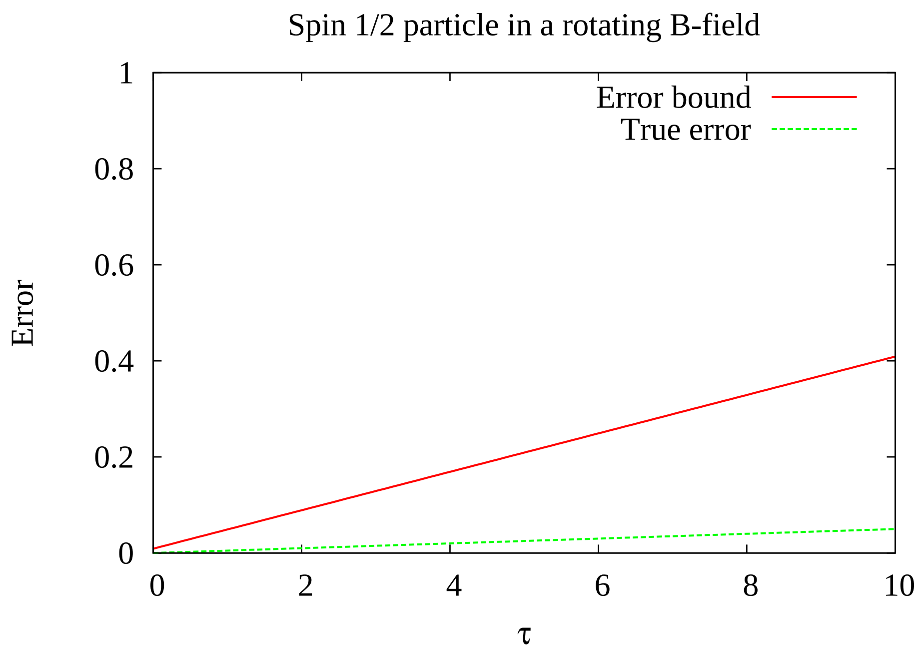
where $\gamma(0)$ is the energy gap of the unperturbed Hamiltonian $\mathcal{H}(0)$.

Similar results hold for δ_1.

Applications

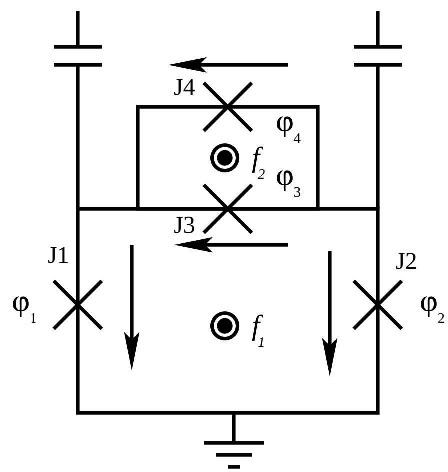
Spin-1/2 particle in a rotating magnetic field

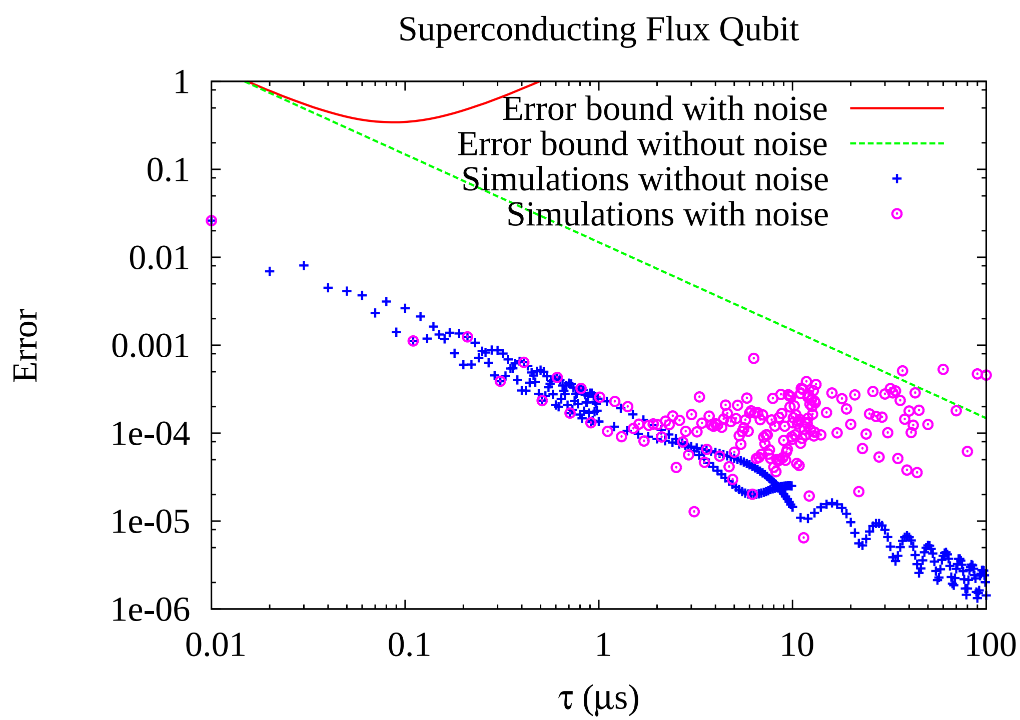




Tong, Singh, Kwek, and Oh (2005) presented an example of a Hamiltonian evolution for which the adiabatic approximation performs poorly. The Hamiltonian is for a spin-1/2 particle in a rotating magnetic field. Our theorem correctly predicts and bounds this failure.

A superconducting flux qubit



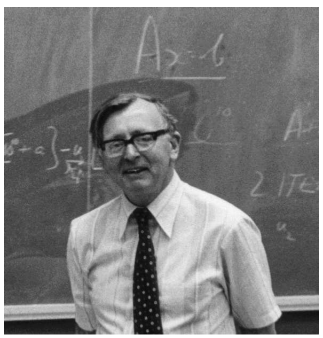


Orlando, Mooij, Tian, van der Wall, Levitov, Lloyd, and Mazo (1999) proposed using flux in a superconductor for quantum computing. Our theorem provides guidance as to how much noise can be tolerated and how slowly the evolution can occur.

Conclusions

- We provide rigorous bounds for the adiabatic approximation under four sources of experimental error:
  - perturbations in the initial state,
  - perturbations on a different time-scale,
  - smooth perturbations to the Hamiltonian,
  - coupling to the external world.
- The results give a perturbation theory for quantum computing analogous in some sense to that of Wilkinson for floating-point computing.
- We applied the results to a spin-$1/2$ particle in a rotating magnetic field and the superconducting flux qubit proposed by Orlando et al.

Michael J. O'Hara and Dianne P. O'Leary, "The Adiabatic Theorem in the Presence of Noise," *Physical Review A*, 77 (2008) 042319, 20 pages. Chosen for *Virtual Journal of Applications of Superconductivity* and *Virtual Journal of Quantum Information*.



We can continue to look for opportunities for cross-disciplinary results in mathematics and physics.

May Sonia Kovalevsky's struggles inspire us to dismantle artificial barriers to mathematics study and research.

Theorem: Smooth error in the Hamiltonian

$$\mathcal{H}(s) \longrightarrow \mathcal{H}_\epsilon(s)$$

Suppose

$$\begin{aligned}
\left|\left|\dot{\mathcal{H}}_\epsilon(s)\right|\right| &\leq \bar{b}_1, \\
\left|\left|\ddot{\mathcal{H}}_\epsilon(s)\right|\right| &\leq \bar{b}_2, \\
\sqrt{1 - |\langle \boldsymbol{\psi}_0(0), \boldsymbol{\phi}_0(0)|\rangle^2} &= \delta_0, \\
\sqrt{1 - |\langle \boldsymbol{\psi}_0(1), \boldsymbol{\phi}_0(1)|\rangle^2} &= \delta_1 \ ,
\end{aligned}$$

where $\boldsymbol{\psi}_0(s)$ is the ground state of $\mathcal{H}_\epsilon(s)$ and $\boldsymbol{\phi}_0(s)$ is the ground state of $\mathcal{H}(s)$. If $\bar{\gamma}_\epsilon > 0$, then

$$||\, \boldsymbol{Q}(1)\, \boldsymbol{U}_\epsilon(1)\, \boldsymbol{P}(0)|| \leq \frac{8}{\tau\bar{\gamma}_\epsilon^2}\left(2\bar{b}_1 + \bar{b}_2 + \frac{16\bar{b}_1^2}{\bar{\gamma}_\epsilon}\right) + \delta_0 + \delta_1 + \delta_0\delta_1 \ .$$

Theorem: Couplings to the environment

$$\mathcal{H}_\epsilon(s) = \mathcal{H}(s) \otimes \boldsymbol{I} + \boldsymbol{I} \otimes \mathcal{H}_{env} + \epsilon \Delta(s)$$

$$||\mathcal{H}_{env}|| + 2\epsilon\, ||\Delta(s)|| \leq w < \bar{\gamma}\ ,$$

where $\bar{\gamma}$ is the minimum energy gap between the ground state and first excited state of $\mathcal{H}(s)$, and that the ground state of $\mathcal{H}_{env}$ has zero energy. Let

$$\left|\left|\dot{\mathcal{H}}_\epsilon(s)\right|\right| \leq \bar{b}_1\ , \left|\left|\ddot{\mathcal{H}}_\epsilon(s)\right|\right| \leq \bar{b}_2\ ,$$

$$\delta_0 = \frac{\epsilon\, ||\Delta(0)||}{\bar{\gamma} - ||\mathcal{H}_{env}|| - \epsilon\, ||\Delta(0)||}\, , \delta_1 = \frac{\epsilon\, ||\Delta(1)||}{\bar{\gamma} - ||\mathcal{H}_{env}|| - \epsilon\, ||\Delta(1)||}\ ,$$

$$\bar{\gamma}_\epsilon = \begin{cases} \bar{\gamma} - w & : \epsilon > 0 \\ \bar{\gamma} & : \epsilon = 0 \end{cases}\ , \bar{D} = \begin{cases} 1 + \frac{2w}{\pi \bar{\gamma}_\epsilon} & : \epsilon > 0 \\ 1 & : \epsilon = 0 \end{cases}\ .$$

Then

$$||(\boldsymbol{Q}(1) \otimes I)\ \boldsymbol{U}_\epsilon(1)\ (\boldsymbol{P}(0) \otimes I)|| \leq \frac{8\bar{D}^2}{\tau \bar{\gamma}_\epsilon^2} \left(2\bar{b}_1 + \frac{8(1+\bar{D})\bar{b}_1^2}{\bar{\gamma}_\epsilon} + \bar{b}_2 \right) + \delta_0 + \delta_1 + \delta_0\delta_1\ .$$