SIAM Conference on
Applications of Dynamical Systems

Figure courtesy J. Meiss and D. Simpson, DSWeb media gallery.

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Snowbird, Utah, USA

Interpreting Huybers' model as a nonsmooth dynamical system

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Joint work with Richard McGehee

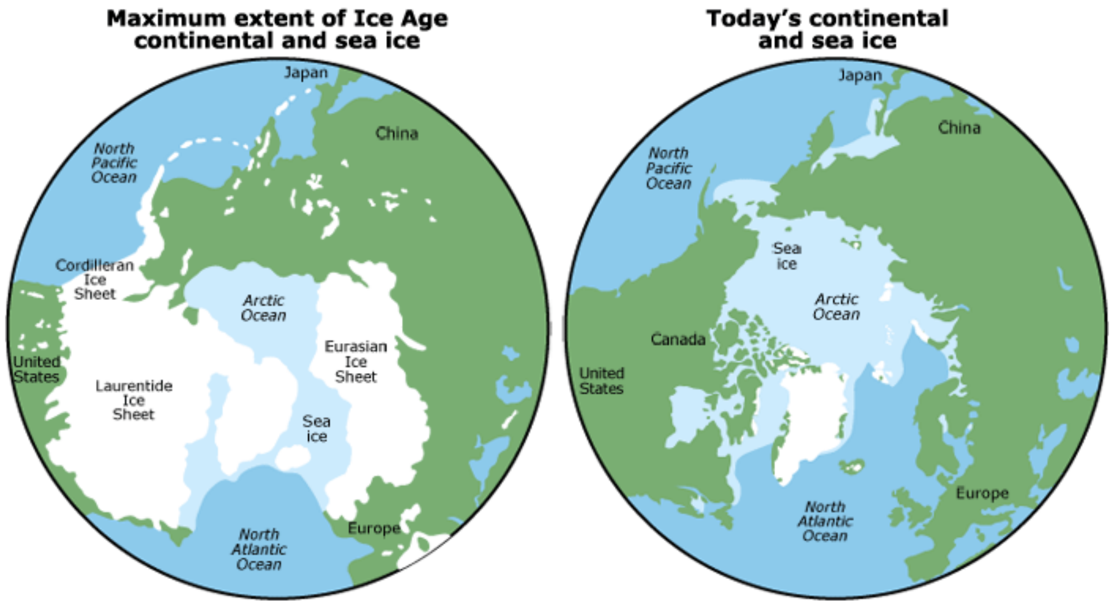
05/24/2017

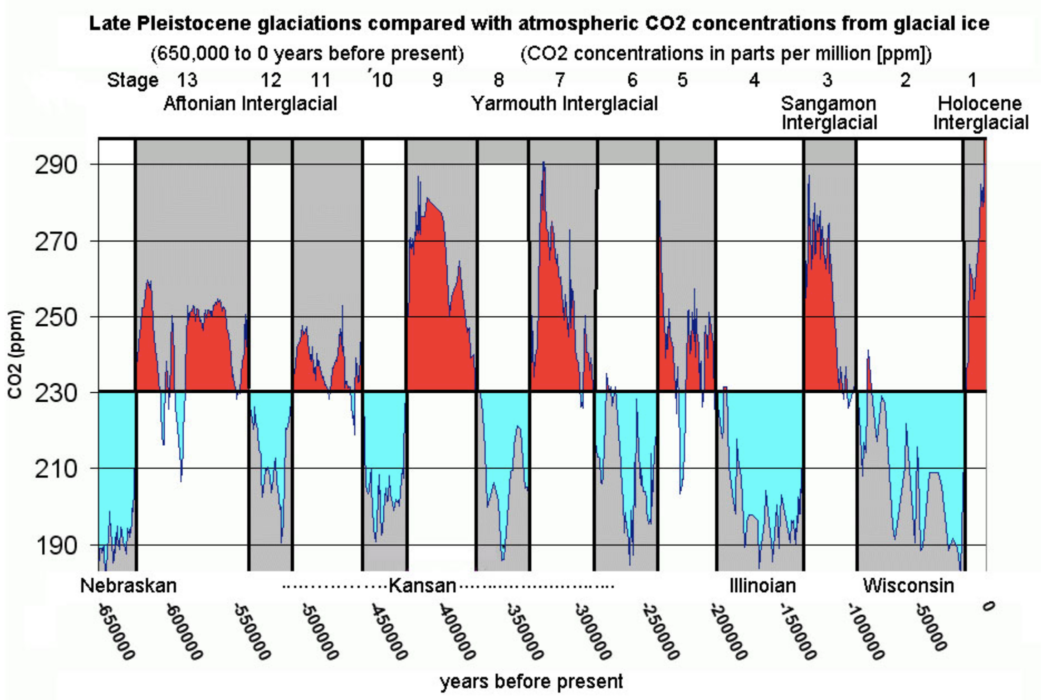
University of Minnesota

Outline of the talk

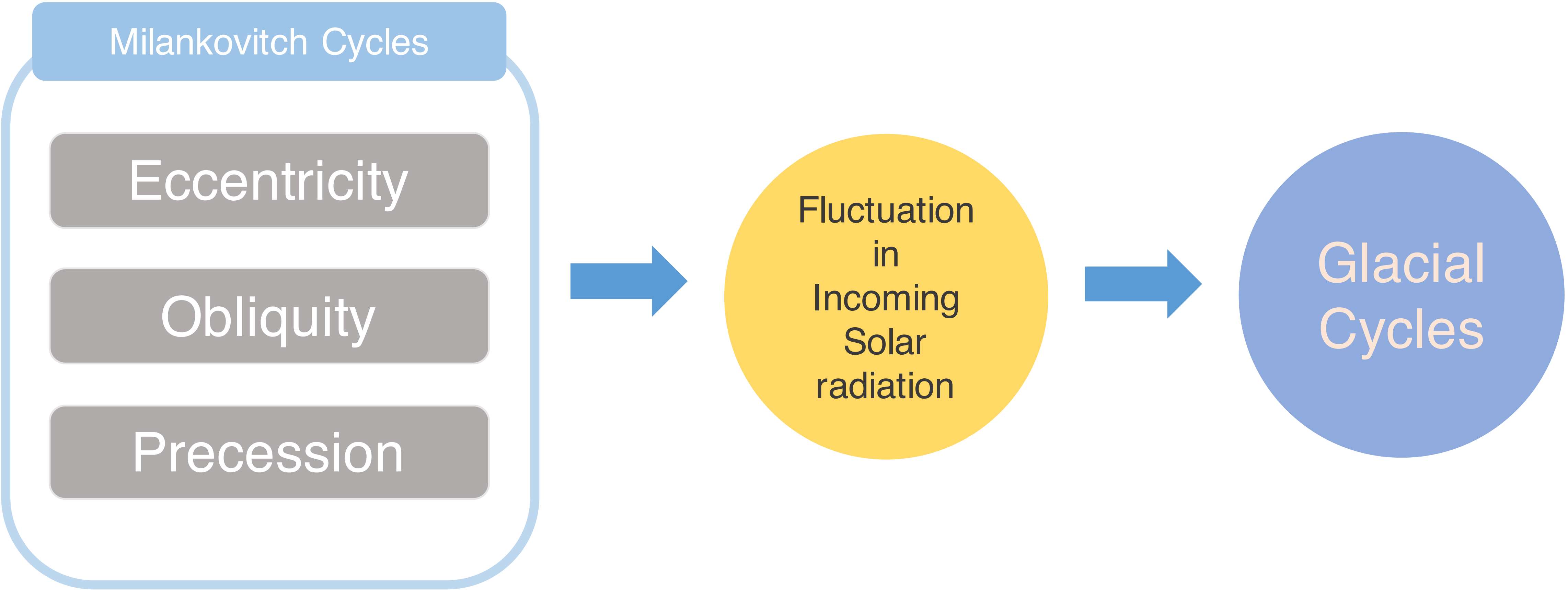
- Background and motivation for Huybers' model
- Huybers' glacial cycles model
- Construction of a cylinder from Huybers' model
- Interpretation of cylinder as a nonsmooth system
- Dynamics on the Filippov system

What are glacial cycles?





Milankovitch cycles drive glacial cycles

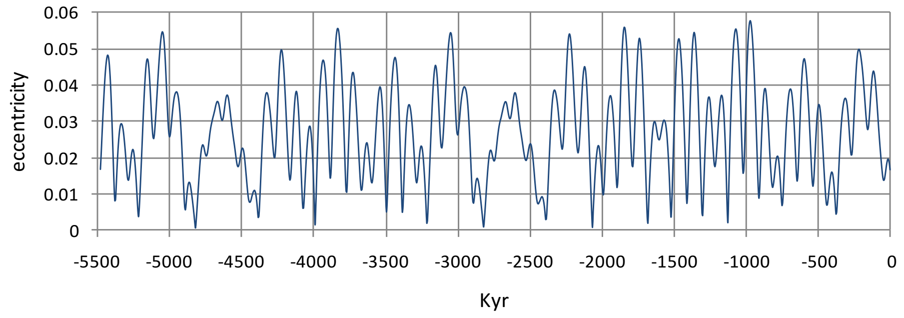


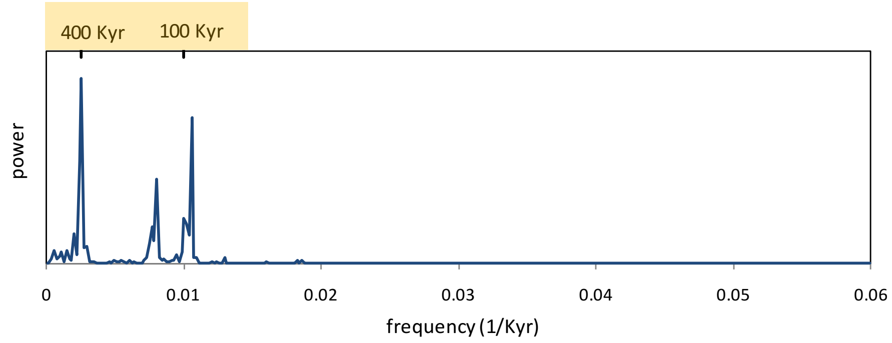
Analysis of Milankovitch cycles' periodicity

Laskar's computations

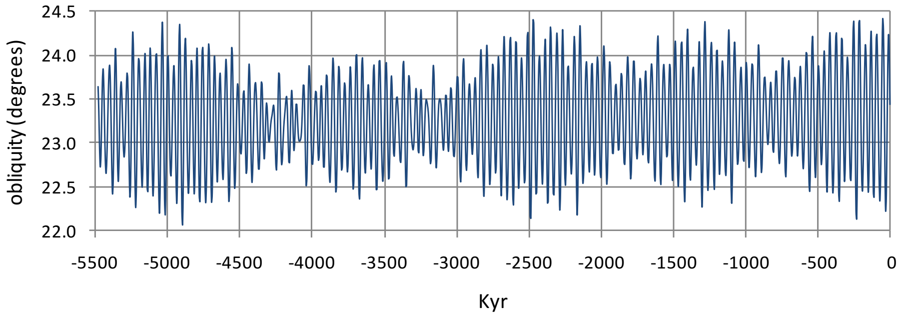
Spectra

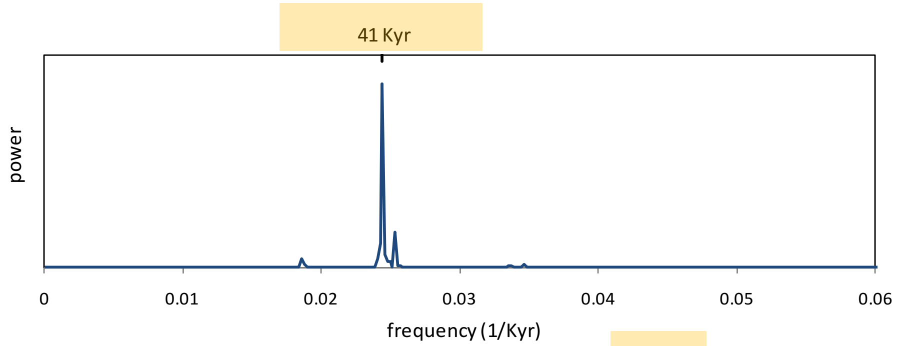
Eccentricity



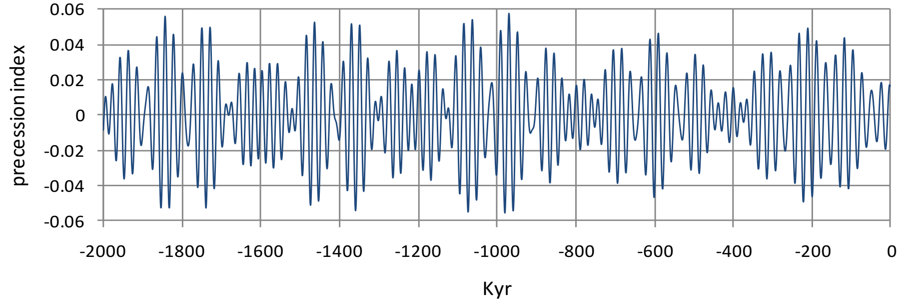


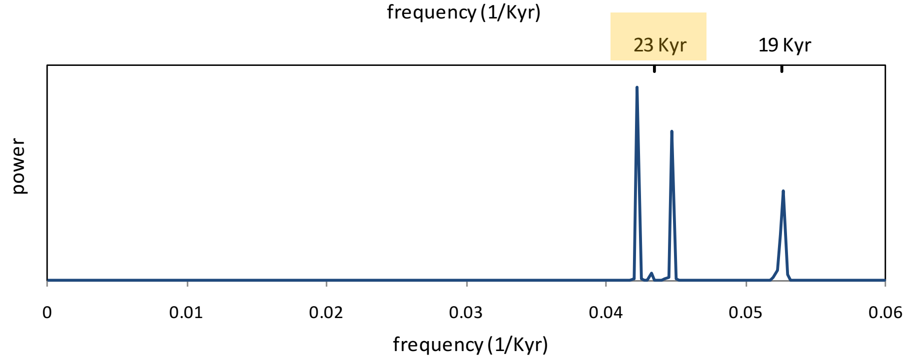
Obliquity



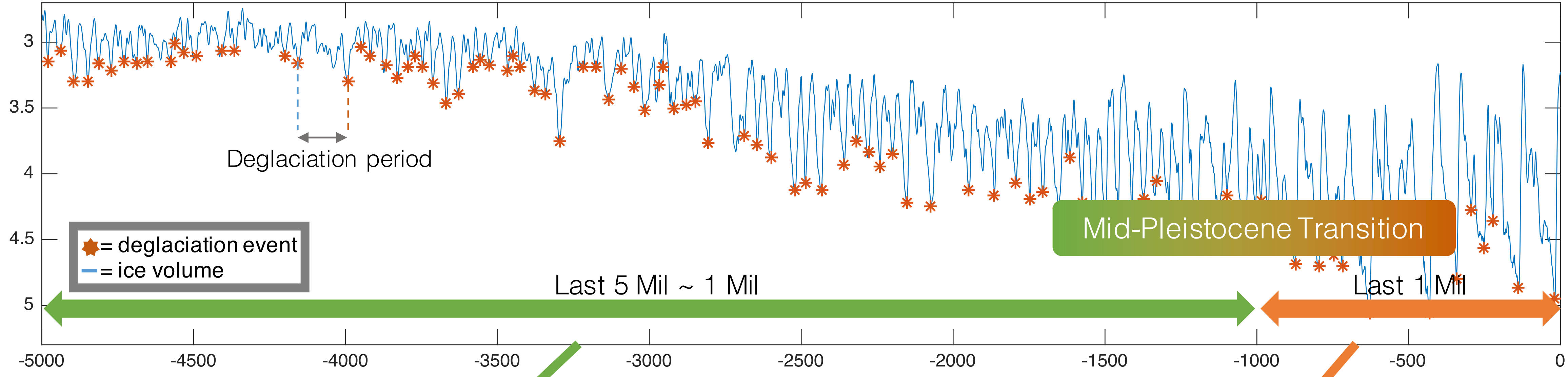


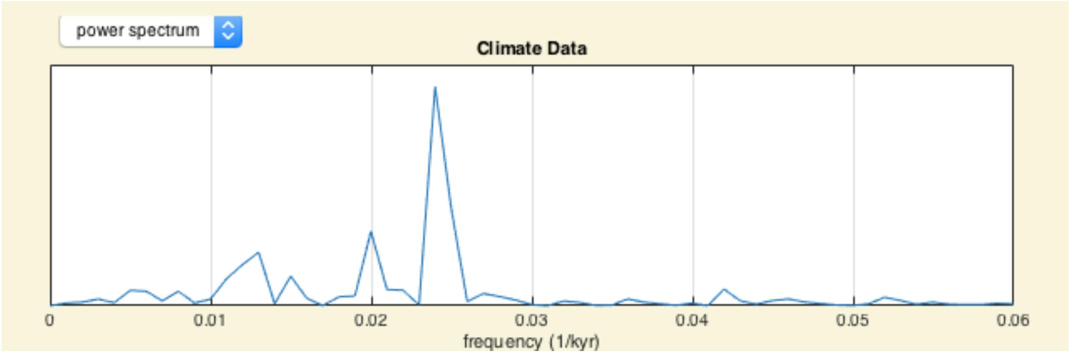
Precession



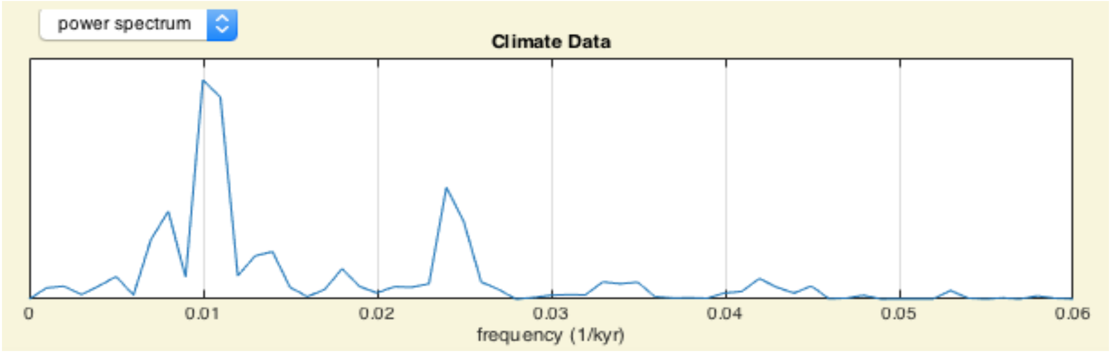


Power spectrum of glacial cycles data

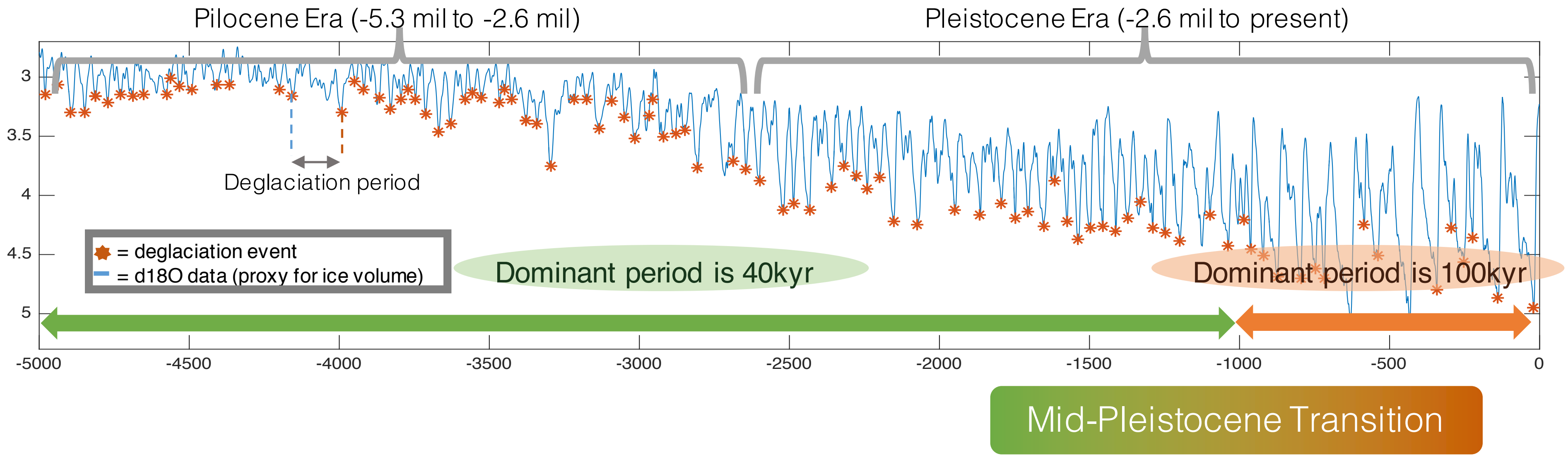




Dominant peak at ~ 0.25 = 40kyr period



Dominant peak at ~ 0.1 = 100kyr period



"Did the main forcing for glacial cycles change from obliquity to eccentricity?"

(40kyr phase) (100kyr phase)

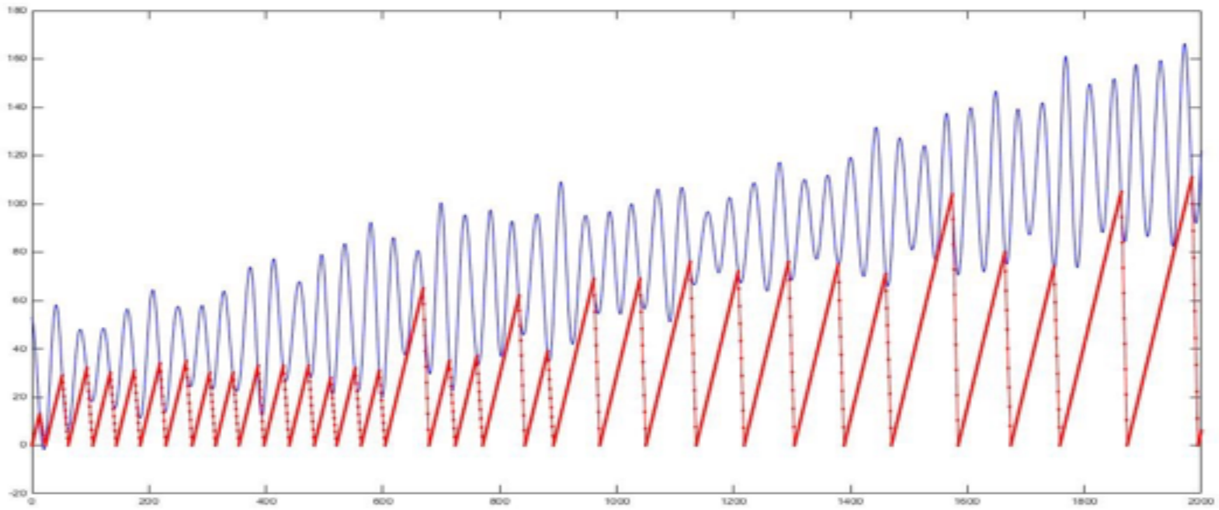
Huybers' Model

$$V_t = V_{t-1} + k_t$$ ---------- Discrete Ice Volume Growth

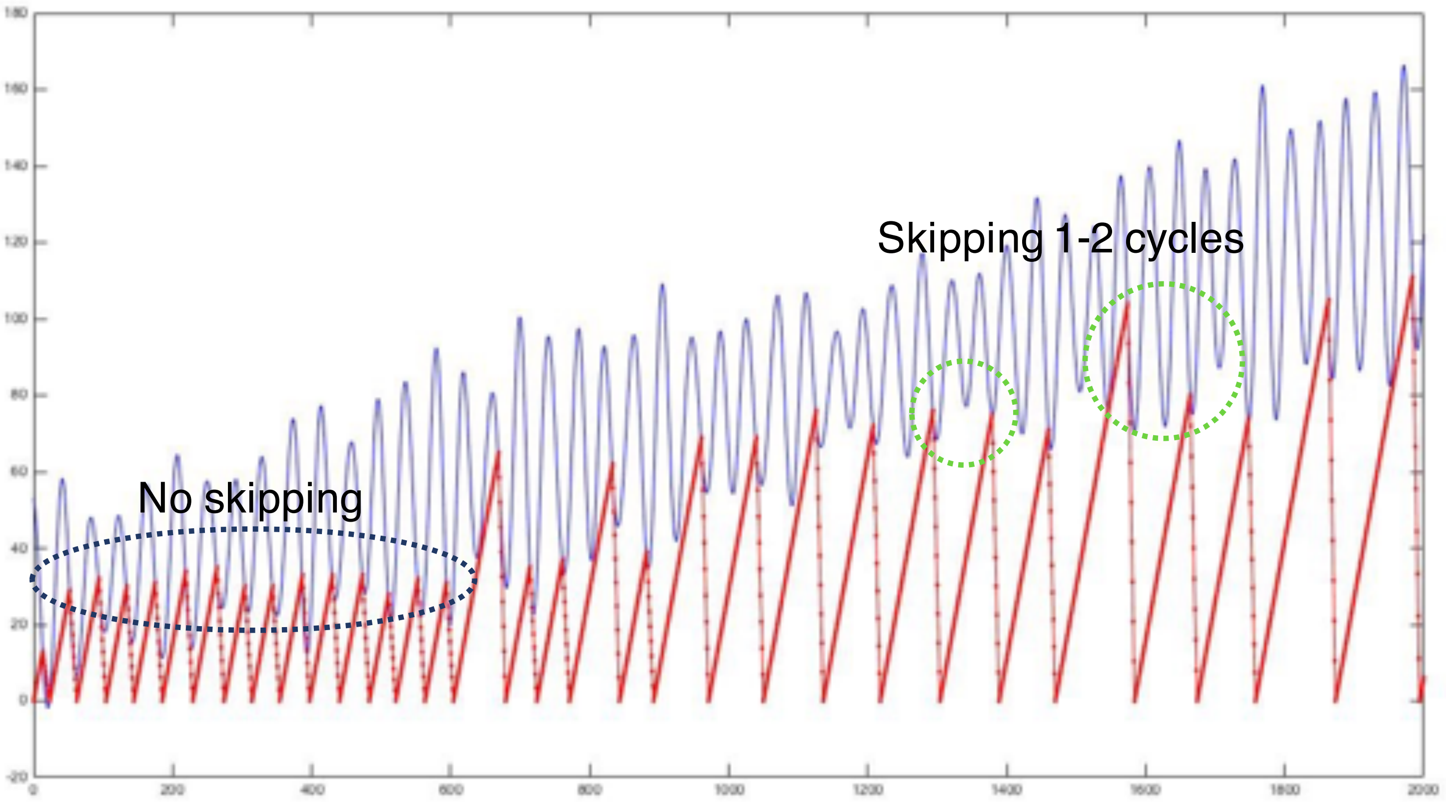
$$T_t = at + b + c\theta_*(t)$$ ---------- Threshold (θ_*= scaled obliquity)

$$\text{If} \quad V_t \geq T_t, \text{ then reset over 10kyr to } V_t = 0$$ ---------- Growth Terminating criterion

Figure:
Model simulation for last 2 Mil years
with a=0.05, b=126, c=20
and k(t) = 1



How did obliquity give rise to the shift to 100kyr period?



"…An explanation for the 100 Ka glacial cycles only requires a change in the likelihood of skipping an obliquity cycle, rather than new sources of long-period variability. "

- Peter Huybers, 2007

Impose a vector field on Huybers' model

$$V_t = V_{t-1} + k_t$$
$$T_t = at + b + c\theta_*$$
$$\text{If} \quad V_t \geq T_t, \text{ then reset over 10kyr to } V_t = 0$$

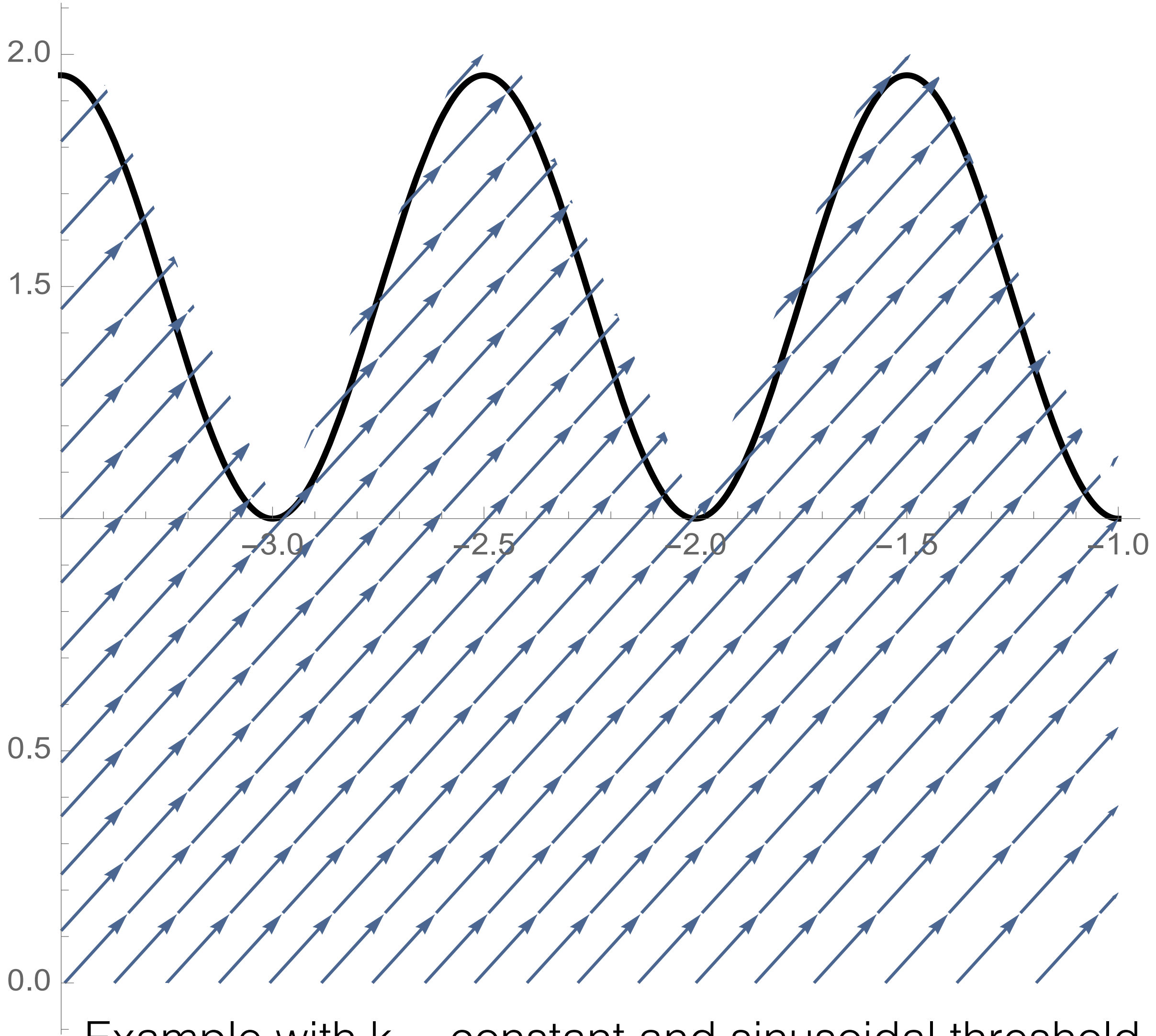
↓

$$V_t = V_{t-\Delta t} + (\Delta t)k_t$$
$$T_t = at + b + c\theta_*$$
$$\text{If } V_t \geq T_t, \text{ then reset over 10kyr to } V_t = 0$$

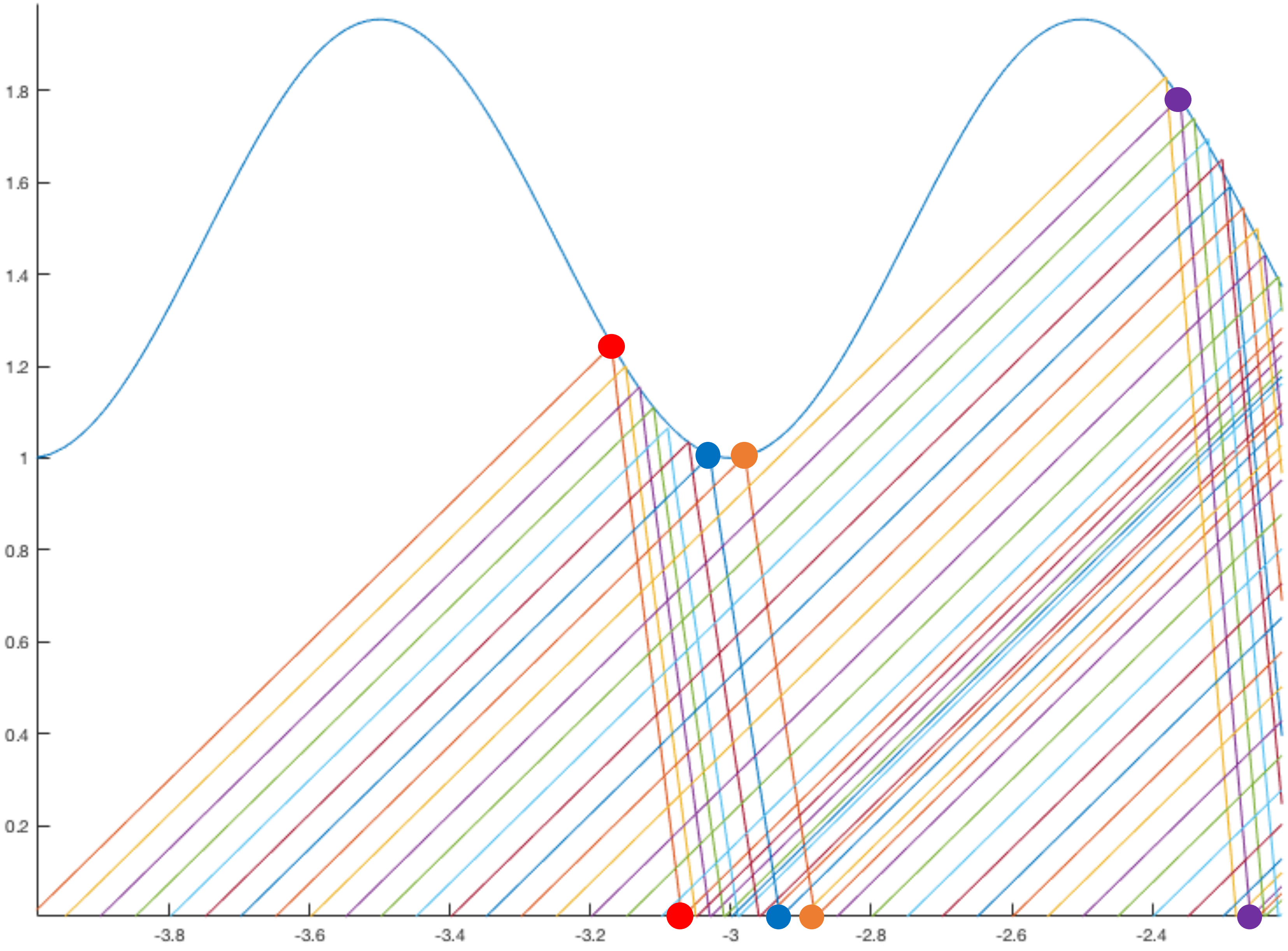
↓ $\Delta t \to 0$

$$\frac{dV}{dt} = k(t)$$
$$T(t) = at + b + c\theta_*$$
$$\text{If } V(t) \geq T(t), \text{ then reset over 10kyr to } V(t) = 0$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k(t) \end{bmatrix}$$



Example with k = constant and sinusoidal threshold


1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
-3.8
-3.6
-3.4
-3.2
-3
-2.8
-2.6
-2.4

Modify the growth terminating condition

$$V_t = V_{t-1} + k_t$$

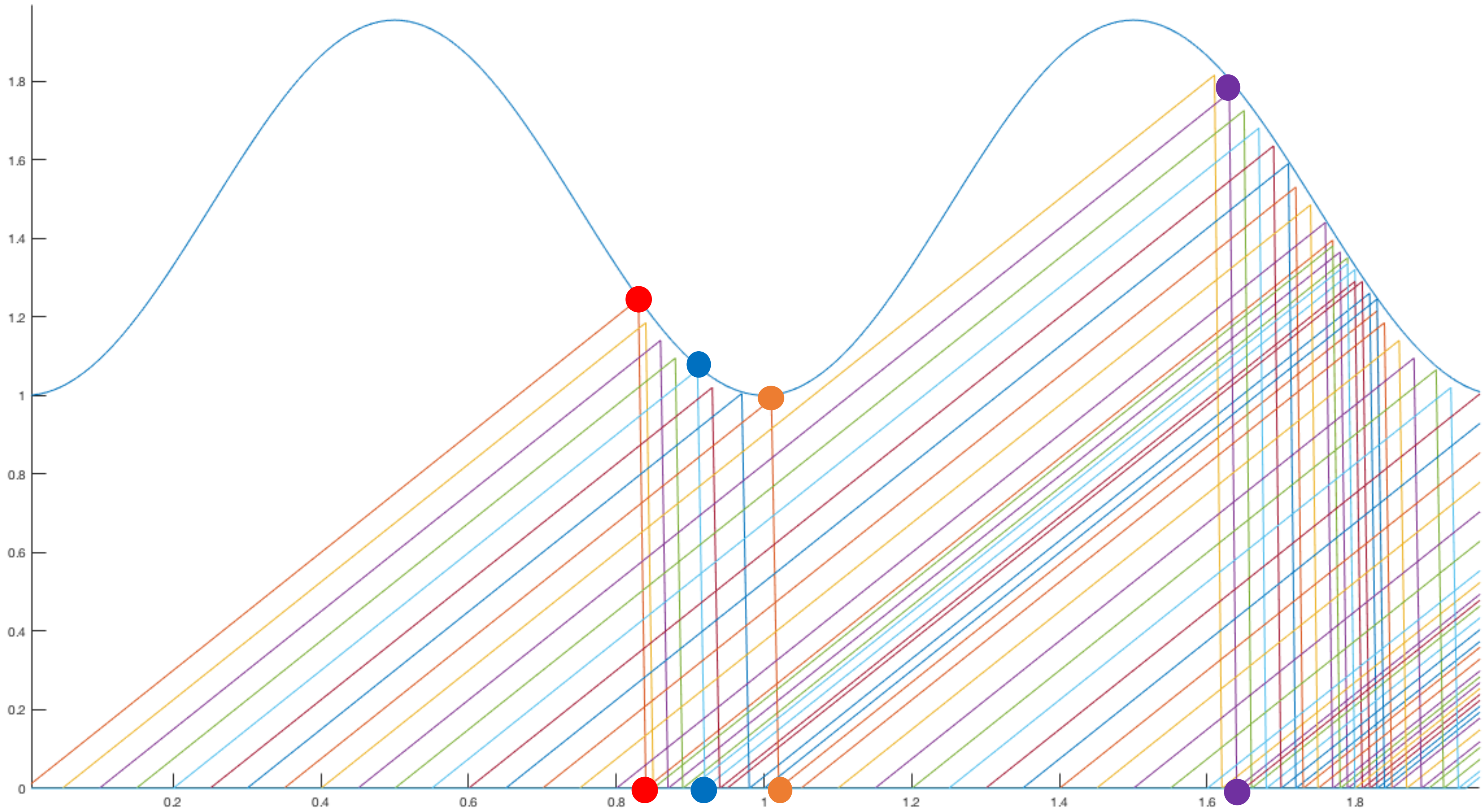
$$T_t = at + b + c\theta_*$$

$$\text{If} \quad V_t \geq T_t, \text{ then reset over 10kyr to } V_t = 0$$

$$V_t = V_{t-1} + k_t$$

$$T_t = at + b + c\theta_*$$

$$\text{If} \quad V_t \geq T_t, \text{ then reset instantaneously to } V_t = 0$$


1.8
1.6
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0.8
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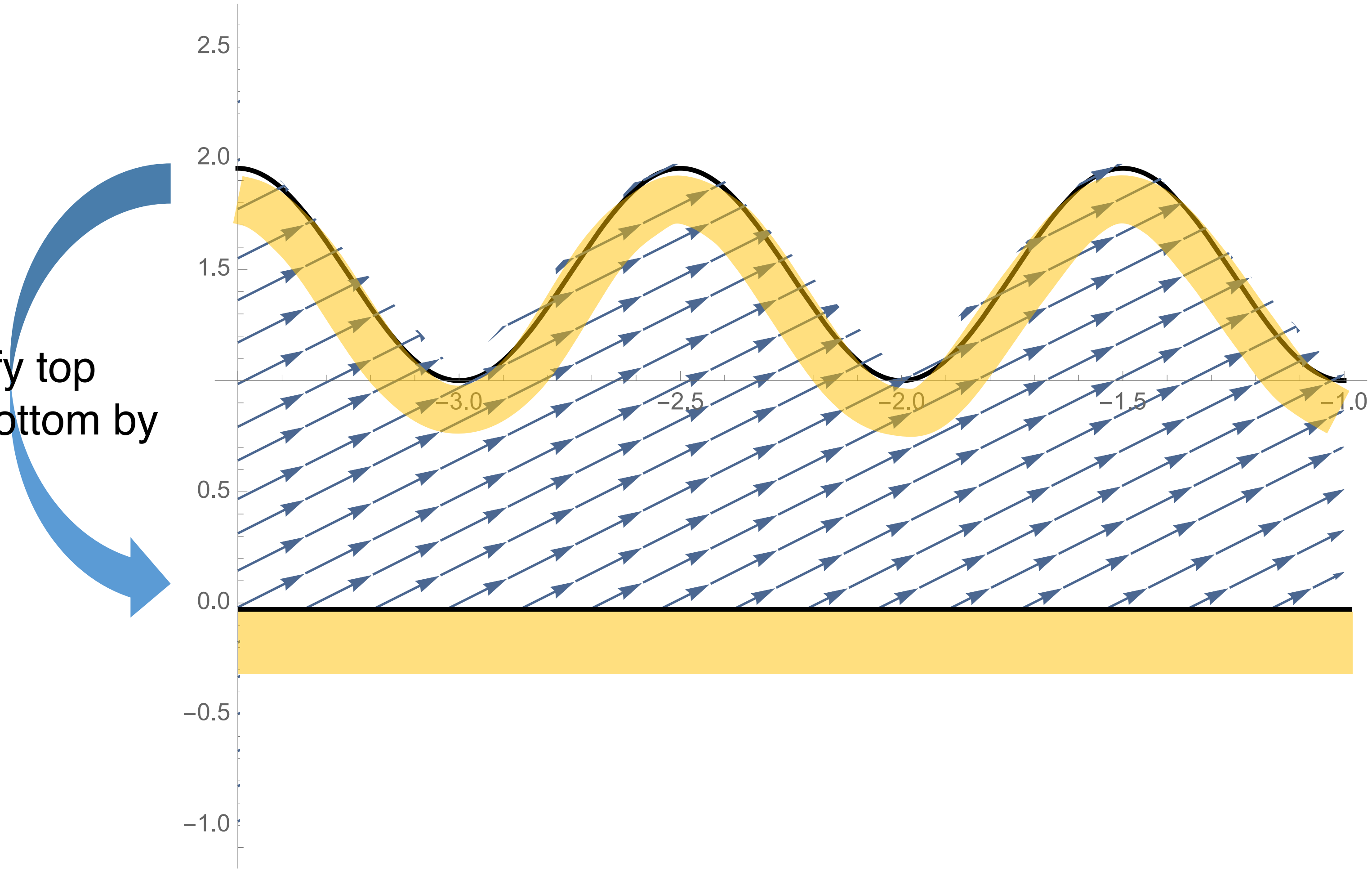
0.8
1
1.2
1.4
1.6
1.8



Identify top and bottom by gluing

Define an equivalence relation

Given

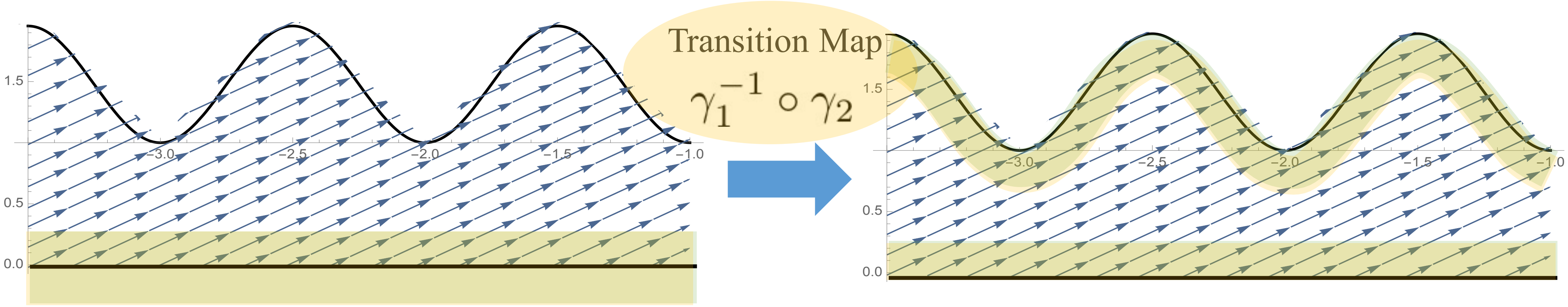
$$X = \{(t, V) : 0 \leq V \leq T(t)\}$$

Define the equivalence relation and the quotient space to be the following:

$$M = X/\sim$$

$$(t, V(t)) \sim (t, 0) \text{ if } V(t) = T(t)$$

Make a cylinder using the equivalence relation

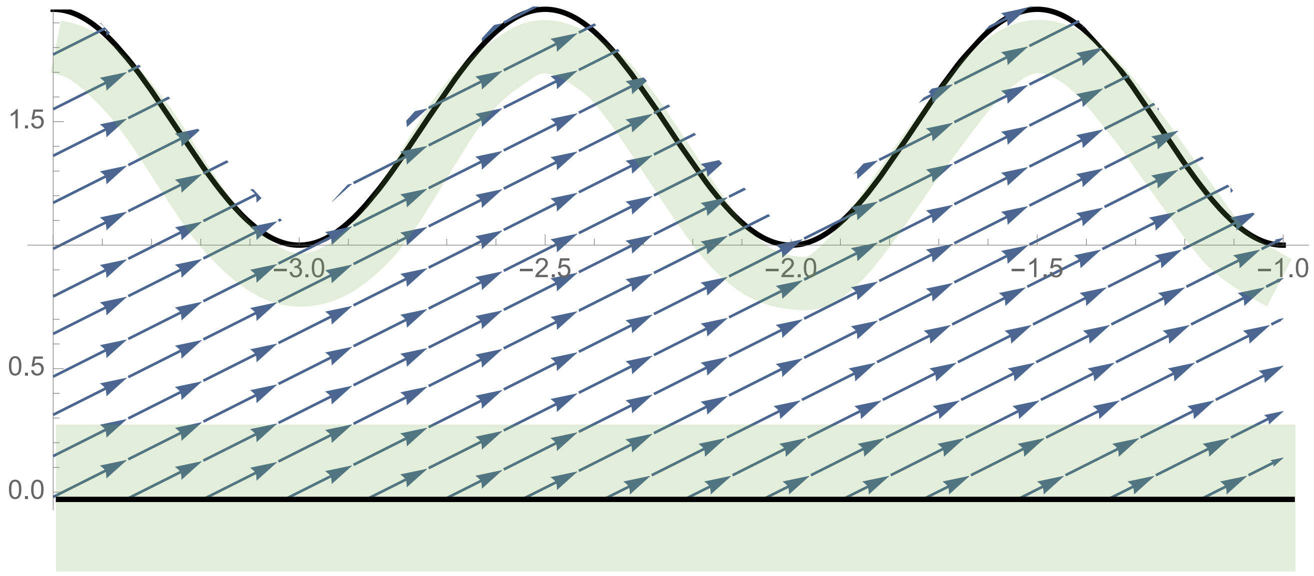


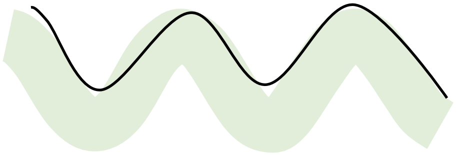
$$\gamma_1 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{pmatrix} \nu \\ \eta \end{pmatrix}$$

$$\gamma_2 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{cases} (\nu, \eta) & 0 \leq \eta < \epsilon \\ (\nu, \eta + T(\nu)) & -\epsilon < \nu \leq 0 \end{cases}$$

$u_1 = X^o =$ ^{o}

$u_2 = \mathbb{R} \times (-\epsilon, \epsilon) =$



Note, and  are the same strip after gluing.

Question: The first strip has the constant vector field. How does this vector field get deformed by the gluing?

$$u_1 = X^{\mathrm{o}} \qquad \gamma_1 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{pmatrix} \nu \\ \eta \end{pmatrix}$$

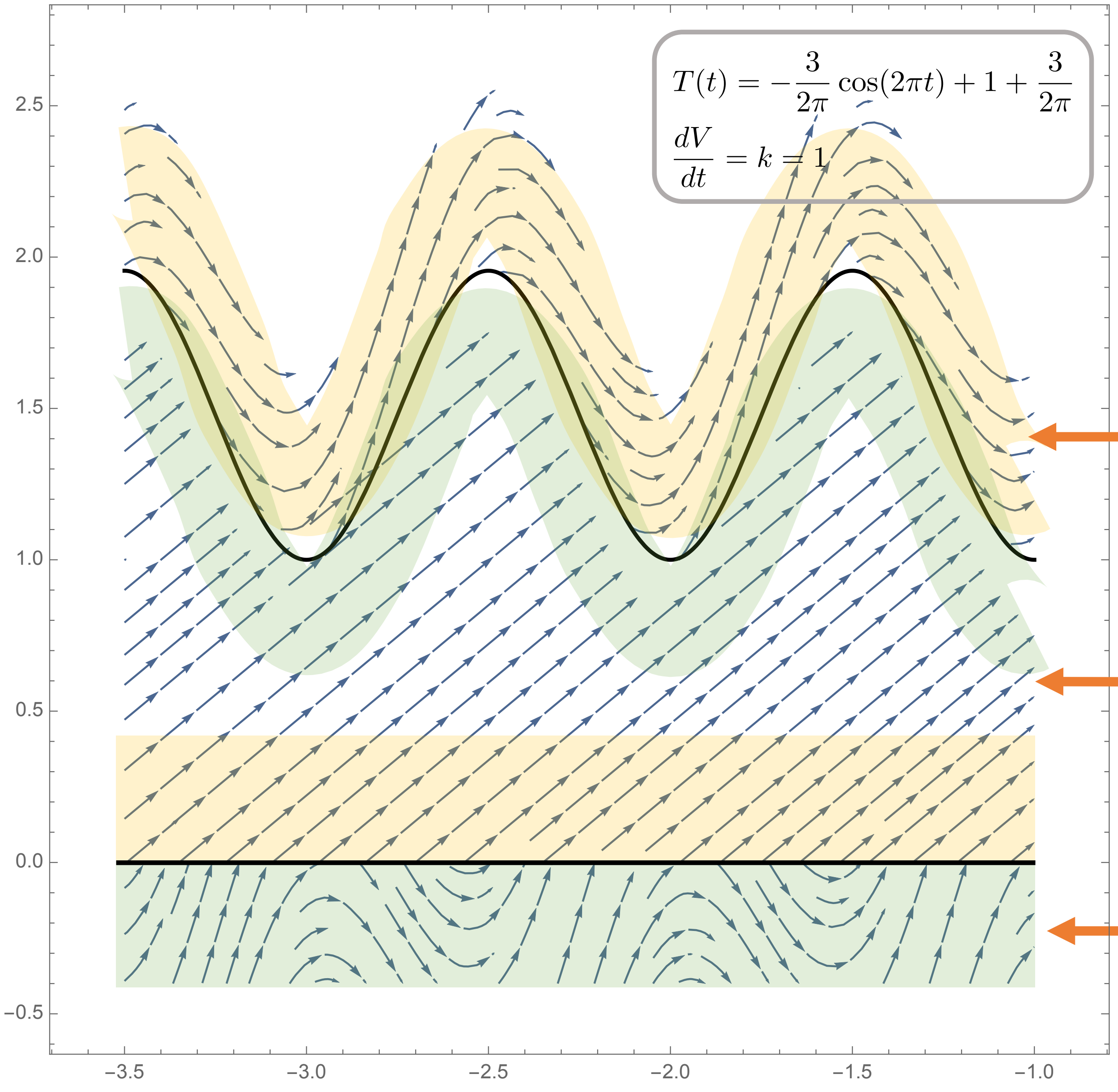
$$u_2 = \mathbb{R} \times (-\epsilon, \epsilon) \qquad \gamma_2 \begin{pmatrix} \nu \\ \eta \end{pmatrix} = \begin{cases} (\nu, \eta) & 0 \leq \eta < \epsilon \\ (\nu, \eta + T(\nu)) & -\epsilon < \nu \leq 0 \end{cases}$$

Observe that in $\mathbb{R} \times (-\epsilon, 0)$ we have:

$$\gamma_1^{-1} \circ \gamma_2 \ (\nu, \eta) = (\nu, \eta + T(\nu))$$

Jacobian of this transformation gives the transformation of vector field:

$$D^{-1}(\gamma_1^{-1} \circ \gamma_2(\nu, \eta)) \cdot \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -T'(\nu) & 1 \end{bmatrix} \begin{bmatrix} 1 \\ k \end{bmatrix} = \begin{bmatrix} 1 \\ k - T'(\nu) \end{bmatrix}$$



$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k + T'(t) \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k \end{bmatrix}$$

$$\frac{d}{dt}\begin{bmatrix} t \\ V \end{bmatrix} = \begin{bmatrix} 1 \\ k - T'(t) \end{bmatrix}$$

Filippov systems

- It is a piecewise-smooth system of the form:

$$\dot{x} = \begin{cases} F_1(x) & H(x) > 0 \\ F_2(x) & H(x) < 0 \end{cases}$$

  where $H(x)$ is smooth.

- $H(x) = 0$ is the discontinuity boundary

What to do on discontinuity boundary?

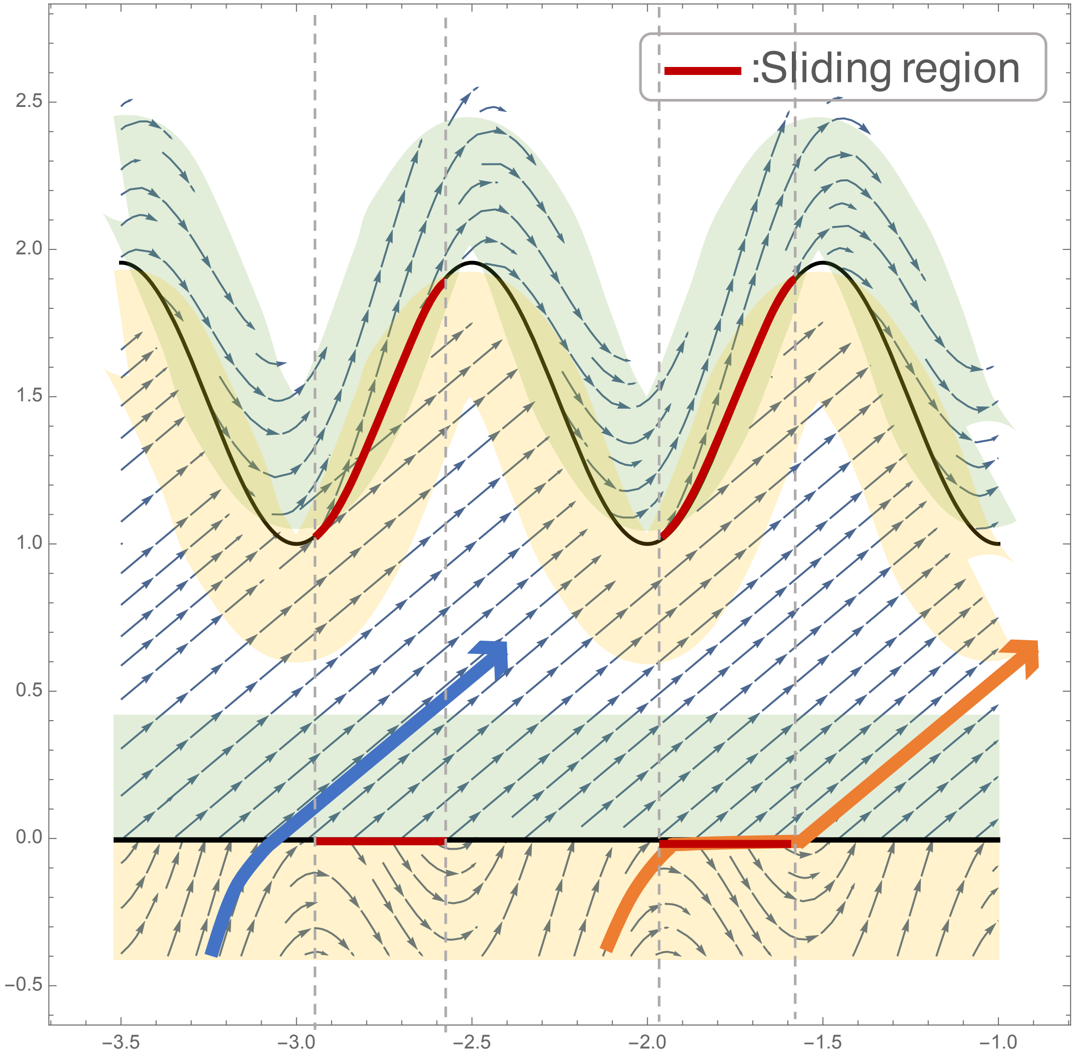
- Transverse Intersection

$$(H_x F_1) * (H_x F_2) > 0$$

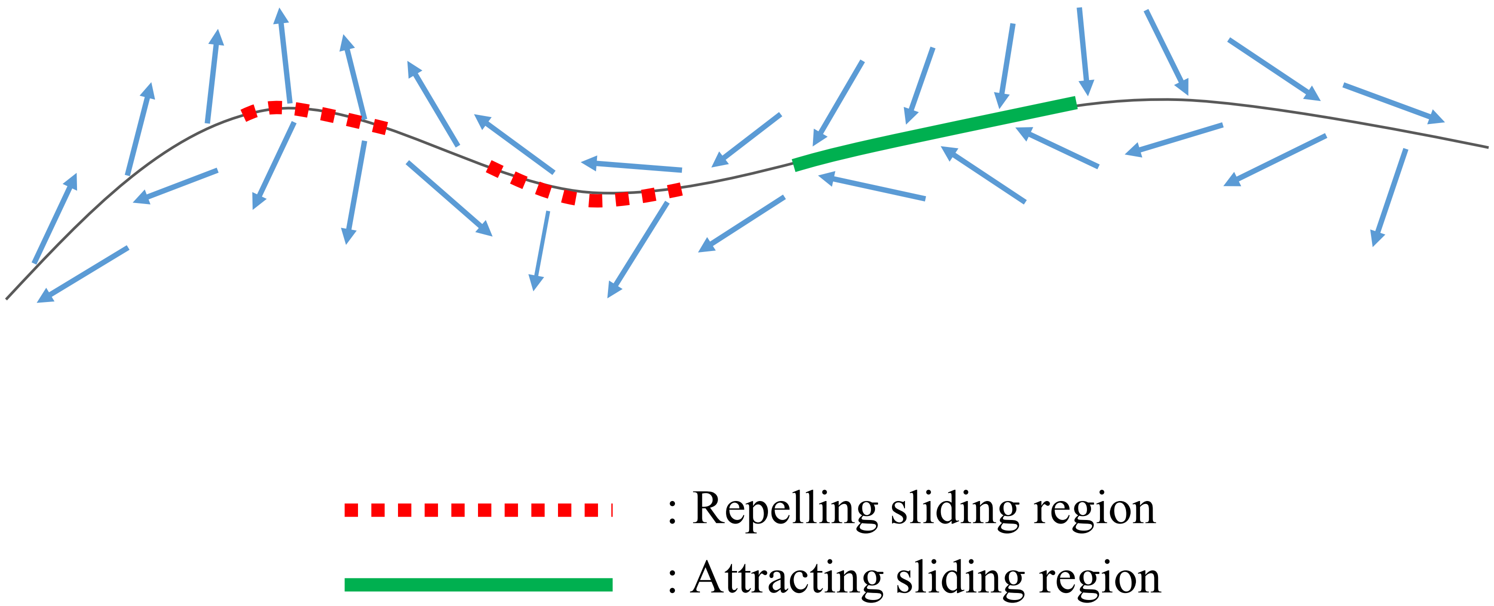
- Sliding mode

$$(H_x F_1) * (H_x F_2) < 0$$

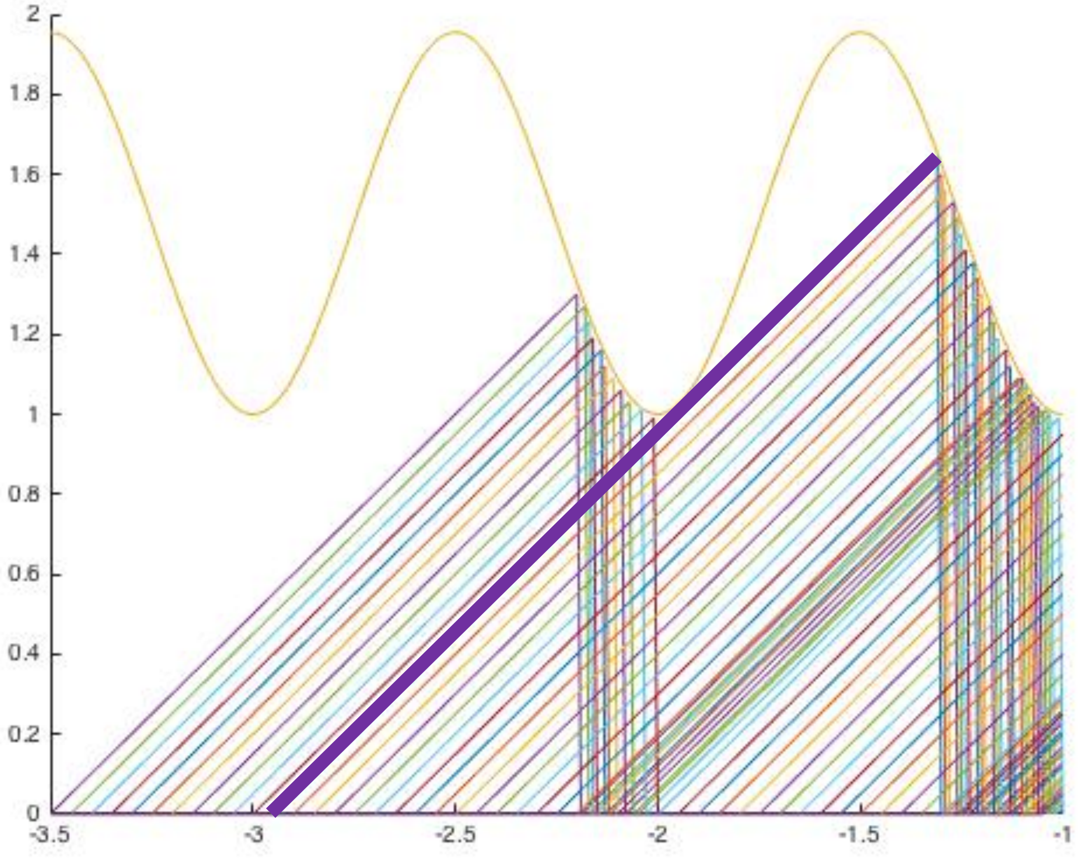
(Note) $H_x F_i$ is component of F_i normal to H.

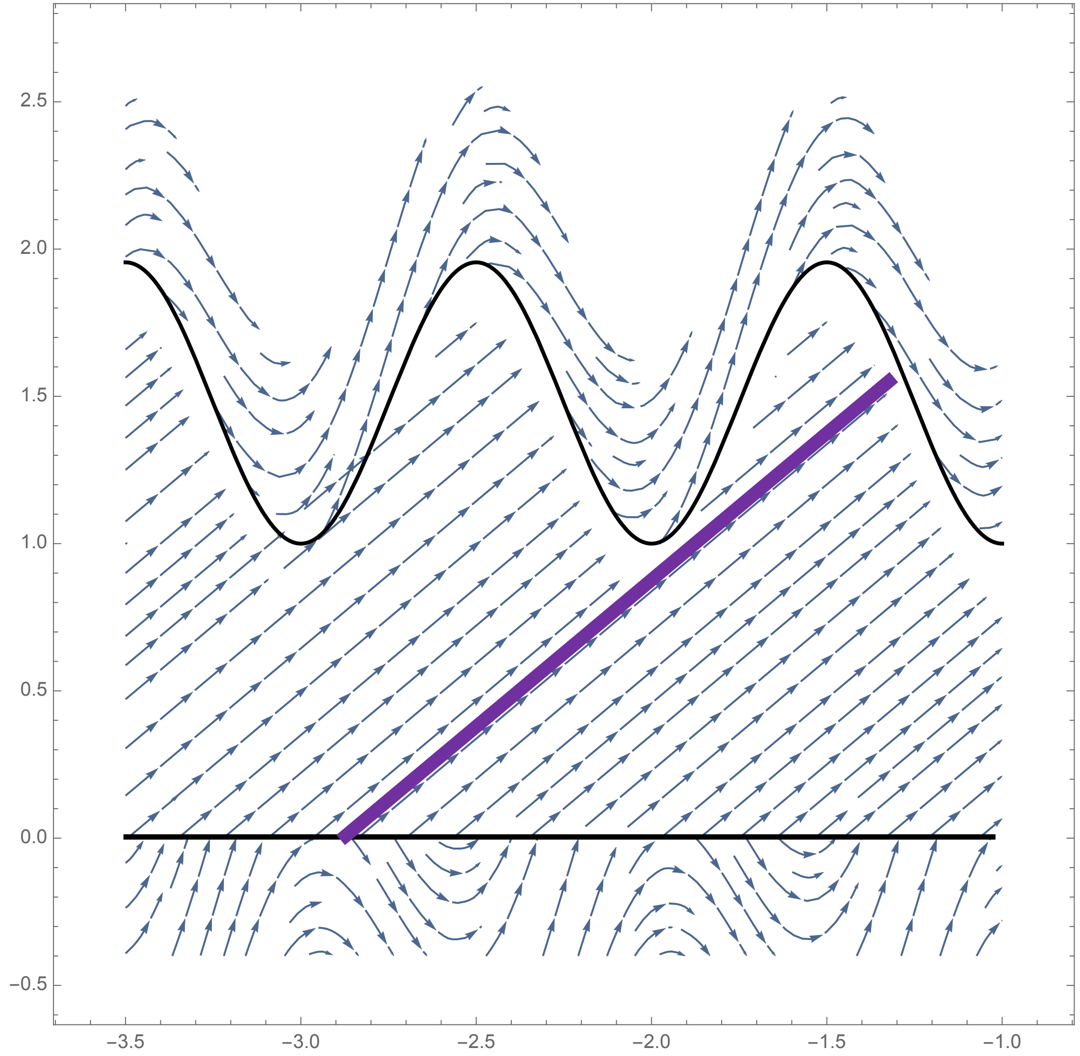


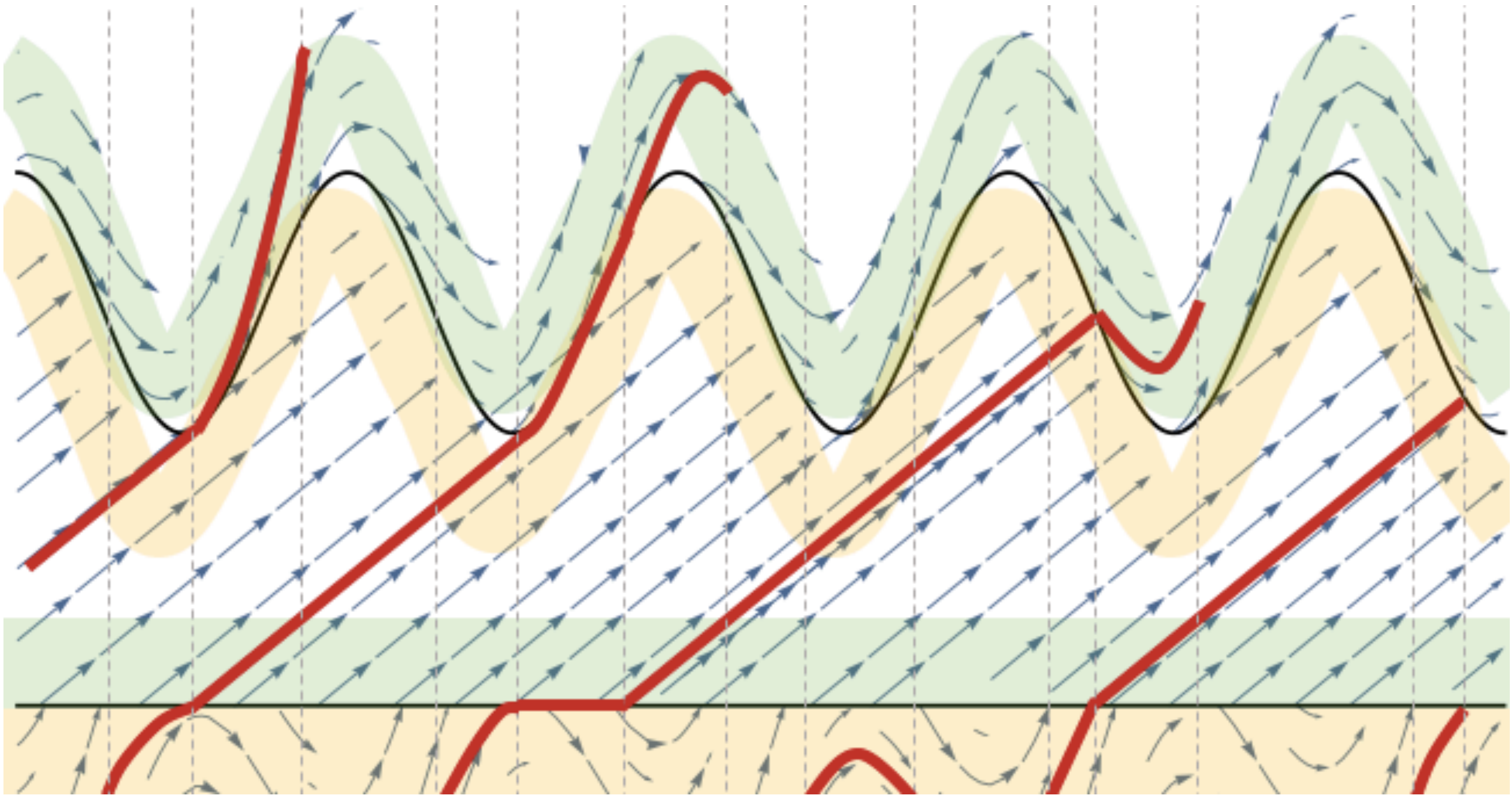
Attracting vs. Repelling Sliding Region

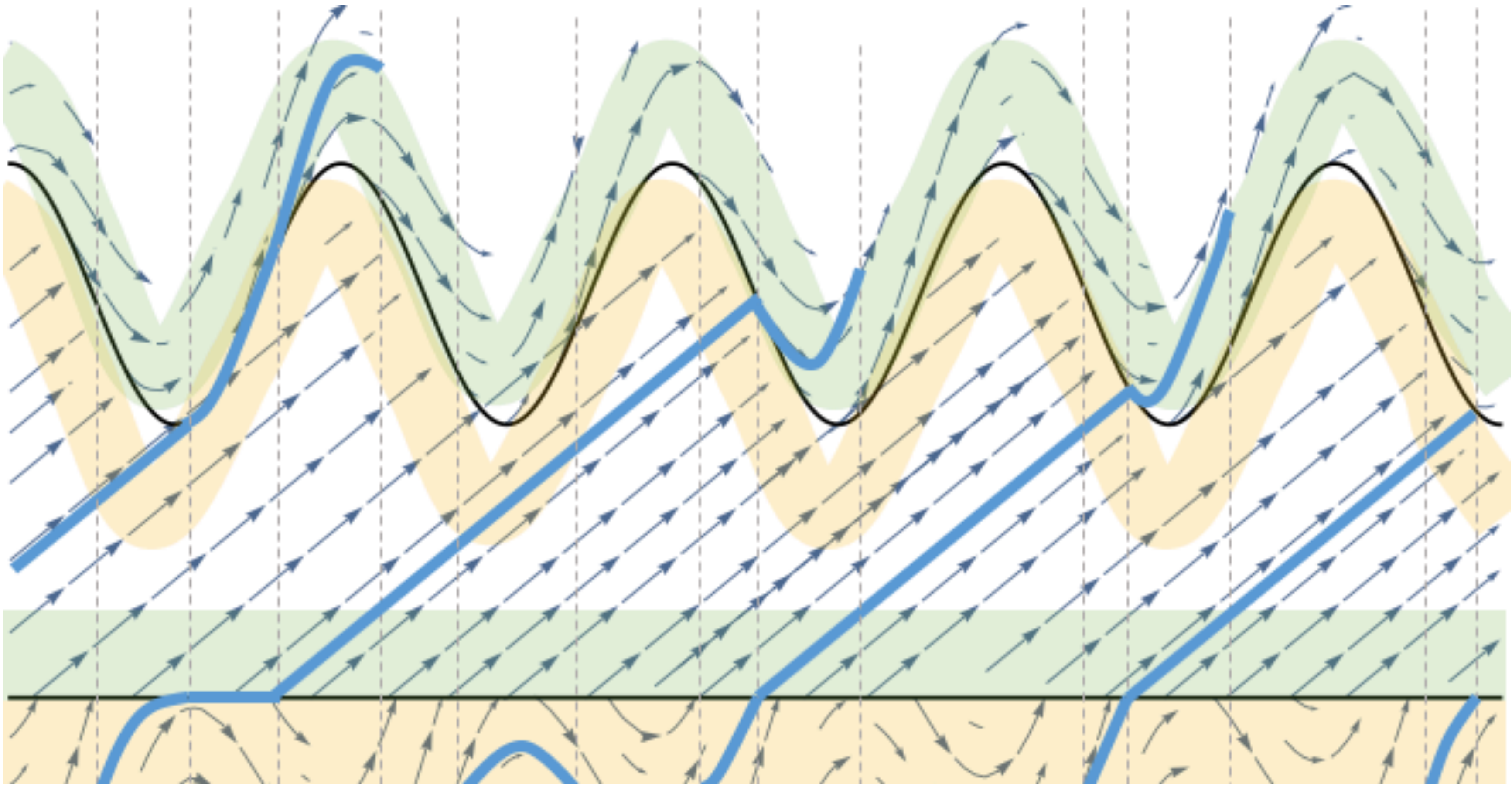


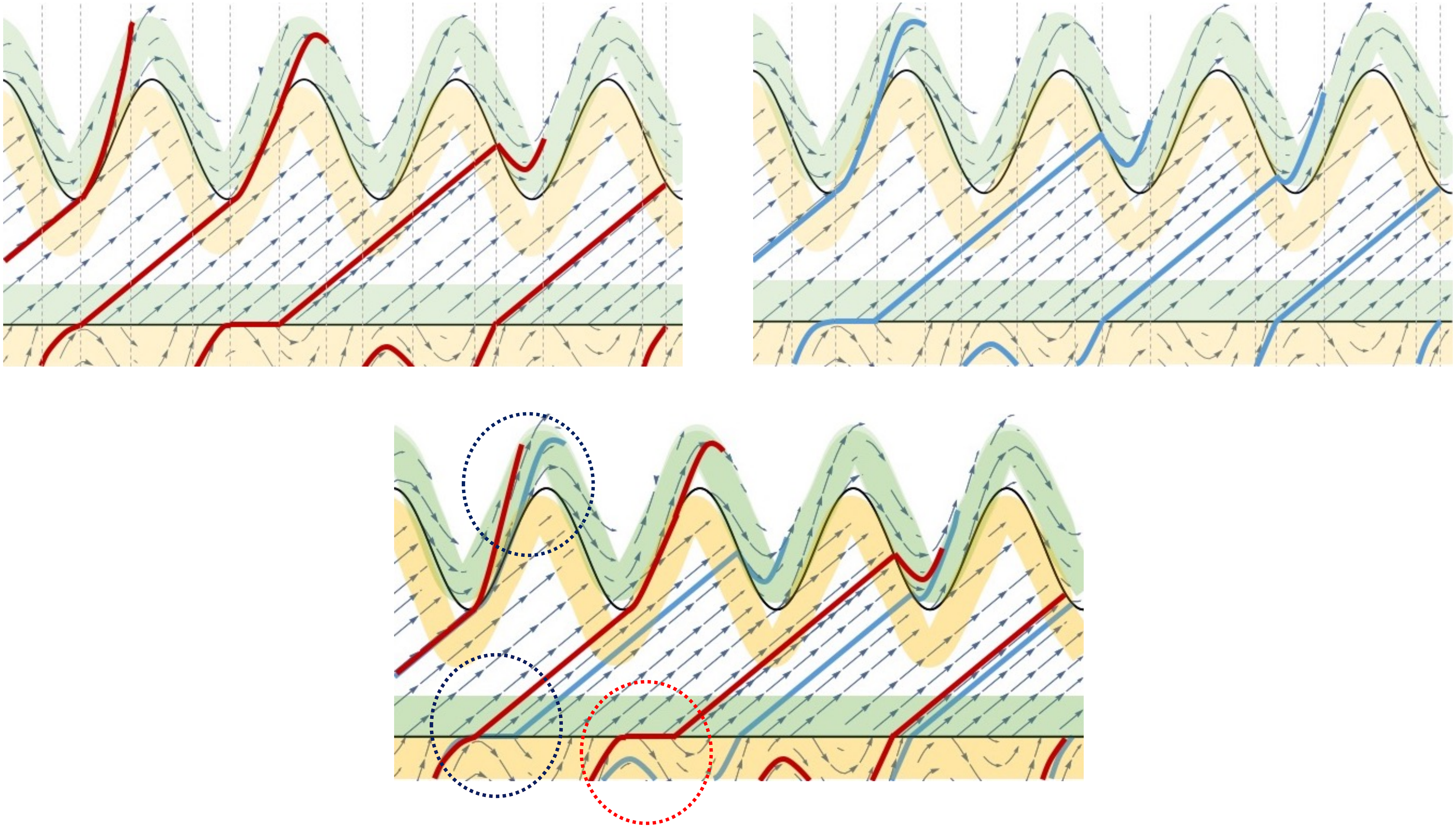
Let’s follow trajectories and see what happens..



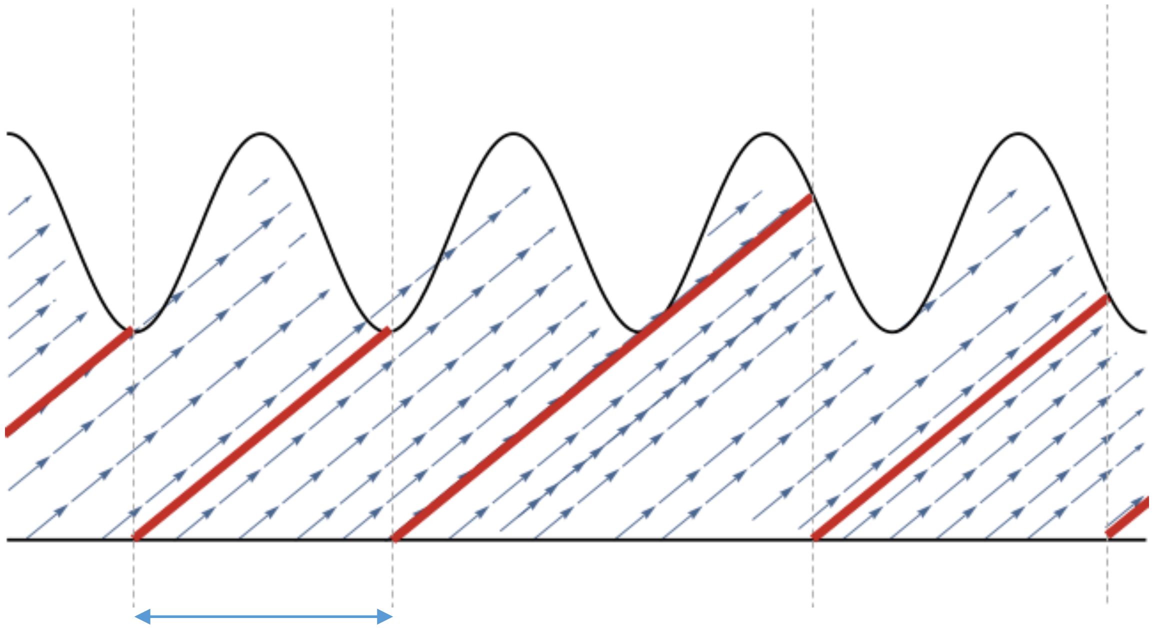






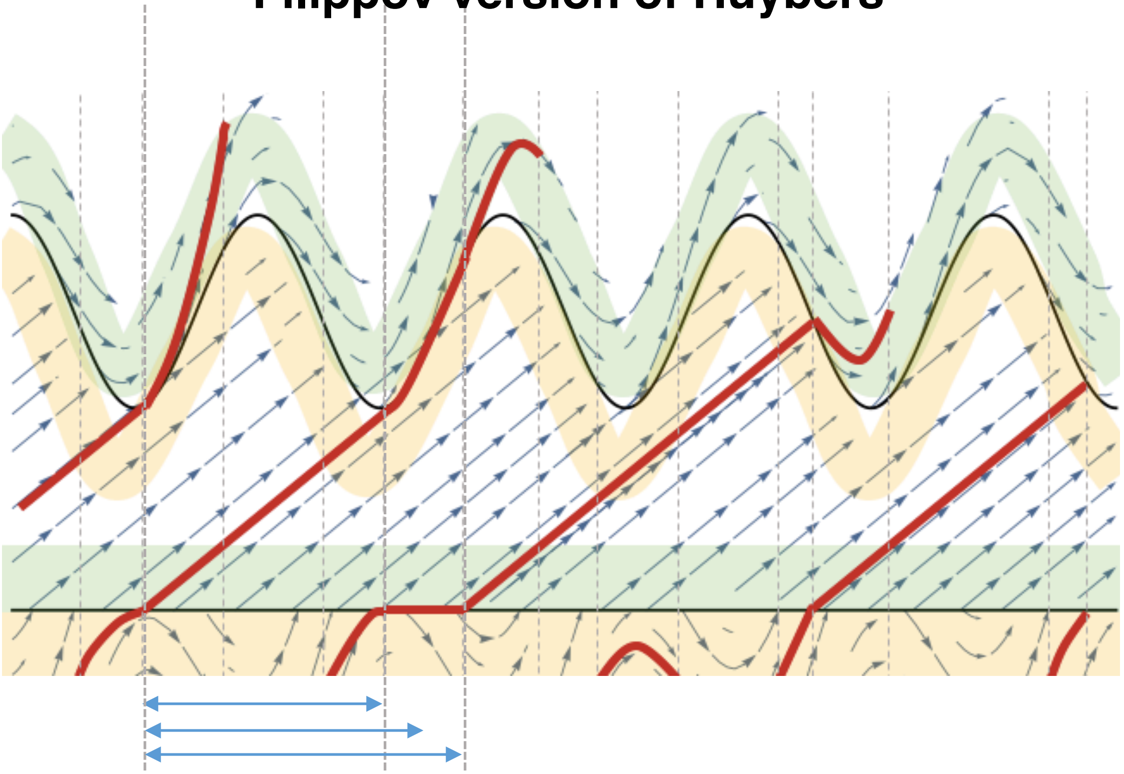


Original Huybers'



Average of all of the widths between glacial growths is the periodicity of glacial cycles

Filippov version of Huybers'



How to define periodicity of glacial cycles here?

Average? Interval?

Summary and Future work

- Constructed a cylinder inspired by the growth terminating condition
- Discovered a Filippov system on the cylinder
  - Possible tool to analyze dynamics of Huybers' model

- Does the Filippov version still reproduce MPT?
  - Simulate using the original threshold and stochastic volume growth