An FBSDE approach to pricing in carbon markets

Hinesh Chotai
(work in progress with D. Crisan, JF Chassagneux, M Muûls)

Imperial College London
Mathematics of Planet Earth Centre for Doctoral Training (`mpecdt.org`, `@MPECDT`)

SIAM Conference on Mathematics of Planet Earth
September 30 - October 2, 2016
DoubleTree by Hilton Hotel Philadelphia Center City

Motivation

- Climate change represents an urgent challenge for humanity.
- Carbon markets are currently being implemented worldwide (e.g. EU ETS).
- More mathematical studies of such markets are needed.

Carbon markets: overview

The EU ETS is described in the following way.

- Cap and trade scheme. European Commission sets the cap and market participants trade allowances.
- At the end of each trading year, firms must submit a report of their emissions, and subsequently surrender one EUA for each ton of CO2 emitted.
- For any other CO2 emissions, they are charged the penalty (currently 100 Euros) per ton of CO2.

Carbon markets: overview

More on the EU ETS:

- EUAs that are not used for compliance can be retained and used in the next year, though this was not the case during phase one.
- There was a huge over-supply of allowances during phase 2 (2008-2012).
- A market stability reserve will be in place from 2018.

FBSDEs: an overview

- $T > 0$ terminal time.
- W an n-dimensional Brownian motion over $[0, T]$
- $b : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$,
  $\sigma : [0, T] \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^{n \times n}$,
  $g : \mathbb{R}^n \to \mathbb{R}$,
  $f : [0, T] \times \mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$
  measurable functions.

Find a triple of adapted processes (X, Y, Z) satisfying the forward-backward stochastic differential equation (FBSDE):

$$\begin{aligned} X_t &= x + \int_0^t b(s, X_s, Y_s)\,\mathrm{d}s + \int_0^t \sigma(s, X_s, Y_s)\,\mathrm{d}W_s, \\ Y_t &= g(X_T) + \int_t^T f(s, X_s, Y_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s\,\mathrm{d}W_s. \end{aligned} \tag{1}$$

FBSDEs: an overview

- Applications: stochastic control, mathematical finance, numerical methods for PDE, and others.
- If the system is *decoupled* with $b(s,x,y) \equiv b(s,x)$, $\sigma(s,x,y) \equiv \sigma(s,x)$, $f(s,x,y,z) \equiv f(s,y,z)$, then '$b$, σ, f, g all Lipschitz continuous with linear growth' $\Longrightarrow$ 'FBSDE has a unique adapted solution'.
- For a fully coupled system, these conditions are not sufficient for the existence of a unique adapted solution.
- Fully coupled FBSDE have been studied with the following conditions (this list is not exhaustive):
  -Non-degenerate diffusion coefficient, σ (Delarue, 2002),
  -Small terminal time T (Antonelli, 1993),
  -A monotonicity condition on f and g (Pardoux & Tang, 1999).

Setup: carbon markets

- Primary example is the EU ETS (in operation since 2005).
- **Single-period setting**: firms submit allowances at time T for their emissions during the period $[0, T]$. Excess emissions (over the cap Λ) are penalized at a penalty ($\pi = 1$ here). After that, emissions regulation is no longer in effect.
- **Two-period setting:** two compliance periods $[0, T_1]$ and $[T_1, T_2]$ with separate caps Λ_1 and Λ_2 at T_1 and T_2, respectively. Excess allowances at T_1 continue into $[T_1, T_2]$.

Setup: carbon markets

We study FBSDE of the following form (called 'singular FBSDE')

$$\begin{aligned} \mathrm{d}P_t &= b(t, P_t)\,\mathrm{d}t + \sigma(P_t)\cdot \mathrm{d}W_t, & P_0 &= p \in \mathbb{R}^N, \\ \mathrm{d}E_t &= \mu_e(P_t, Y_t)\,\mathrm{d}t, & E_0 &= e \in \mathbb{R}, \\ \mathrm{d}Y_t &= Z_t \cdot \mathrm{d}W_t, & Y_T &= \phi(E_T). \end{aligned} \tag{2}$$

where ϕ is a monotonic increasing and *discontinuous* function. Note that $X = (P, E)$ is the forward component. In applications, we often choose

$$\phi(x) = \mathbf{1}_{[\Lambda, +\infty)}(x). \tag{3}$$

FBSDE of this kind were studied in (Carmona et al, 2013) and (Carmona & Delarue, 2013).

Working assumptions

Assumption 1

There exist three constants $L \geq 1$, $l_1, l_2 > 0$, $1/L \leq l_1 \leq l_2 \leq L$ satisfying

1. *b, σ have L-linear growth and are L-Lipschitz continuous (this holds uniformly in time for b):*
2. *μ_e also has L-linear growth, and is Lipschitz continuous, satisfying*

$$|\mu_e(p,y)| \leq L(1+|p|+|y|),$$
$$|\mu_e(p,y)-\mu_e(p',y')| \leq L\left(|p-p'|+|y-y'|\right),$$
$$p, p' \in \mathbb{R}^N, y, y' \in \mathbb{R}.$$

3. *Finally, for any $p \in \mathbb{R}^N$, the real function $y \mapsto \mu_e(p,y)$ is strictly decreasing and $-\mu_e$ satisfies the following monotonicity condition*

$$l_1|y-y'|^2 \leq (y-y')\left(\mu_e(p,y')-\mu_e(p,y)\right) \leq l_2|y-y'|^2,$$
$$p \in \mathbb{R}^N, y, y' \in \mathbb{R}$$

Existing results

Theorem 1 (Carmona & Delarue 2013)

Under the assumptions set out above, and given $(p, e) \in \mathbb{R}^2$, (2) *admits a unique progressively measurable 4-tuple* (P, E, Y, Z) *with*

$$\mathbb{E}\left[\sup_{t\in[0,T]}\left(|P_t|^2 + |E_t|^2 + |Y_t|^2\right) + \int_0^T |Z_s|^2 \, \mathrm{d}s\right] < \infty,$$

such that $P_0 = p$, $E_0 = e$, *and* (P, E, Y, Z) *satisfies the dynamics in* (2) *over* $[0, T)$, *but* Y_T *only satisfies*

$$\mathbb{P}[\phi_-(E_T) \leq Y_T \leq \phi_+(E_T)] = 1$$

where ϕ_- *and* ϕ_+ *are the left and right continuous versions, respectively, of* ϕ. *Finally,* $|Z_t| \leq C \quad$ *for every* $t \in [0, T]$, *where* C *is a constant depending only on* T *and the Lipschitz constants in Assumption 1.*

Existing results

Theorem 2 (Carmona & Delarue 2013)

Consider the case $\phi(x) = \mathbf{1}_{[\Lambda,+\infty)}(x)$. In addition to the earlier assumptions, assume also that σ is uniformly elliptic in the sense that

$$(\sigma(p))^{\mathrm{T}}\sigma(p) \geq L^{-1} > 0, \qquad p \in \mathbb{R}^N.$$

Also, assume that the function $p \mapsto \mu_e(p,0)$ is uniformly continuous over $\mathbb{R}^n$ and satisfies

$$|\partial_p \mu_e(p,0)| \geq L^{-1}, \qquad p \in \mathbb{R}^N.$$

Then, for any starting point $(P_0, E_0) = (p, e)$, the solution (P, E, Y, Z) to (2) satisfies

$$\mathbb{P}[E_T = \Lambda] > 0.$$

A model for a two-period carbon market

Consider two periods $[0, T_1]$, $[T_1, T_2]$ and two caps Λ_1, Λ_2 and a penalty $\pi > 0$ for each period. Let

$$\mathrm{d}P_t = b(P_t)\,\mathrm{d}t + \sigma(P_t)\,\mathrm{d}W_t, \qquad P_0 = p \in \mathbb{R}^n, t \in [0, T_2]$$

and consider a pair of FBSDE in the following form.

For $t \in [0, T_1]$:

$$\mathrm{d}E_t^1 = \mu_e(P_t, Y_t^1)\,\mathrm{d}t, \quad E_0^1 = e_{T_0},$$
$$\mathrm{d}Y_t^1 = Z_t^1\,\mathrm{d}W_t, \quad Y_{T_1}^1 = \phi_1(E_{T_1}^1) = \begin{cases} Y_{T_1}^2, & \text{if } E_{T_1}^1 < \Lambda_1 + \Lambda_2, \\ \pi, & \text{otherwise.} \end{cases}$$

For $t \in [T_1, T_2]$:

$$\mathrm{d}E_t^2 = \mu_e(P_t, Y_t^2)\,\mathrm{d}t, \quad E_{T_1}^2 = E_{T_1}^1,$$
$$\mathrm{d}Y_t^2 = Z_t^2\,\mathrm{d}W_t, \quad Y_{T_2}^2 = \begin{cases} 0, & \textit{if} \quad E_{T_2}^2 < \hat{\Lambda}_2(E_{T_1}^2) = (\Lambda_1 + \Lambda_2 - E_{T_1}^1)^+, \\ \pi, & \text{otherwise.} \end{cases}$$

A model for a two-period carbon market

- This is a model for a two-period carbon market (with banking, borrowing, and withdrawal of allowances).
- Setting $\pi = 1$, we now need to consider FBSDE of the following form:

For $t \in [0, T_1]$:

$$
\begin{aligned}
dE_t^1 &= \mu_e(P_t, Y_t^1)\,dt, \quad E_0^1 = e_{T_0},\\
dY_t^1 &= Z_t^1\,dW_t,\\
Y_{T_1}^1 &= \phi_1(E_{T_1}^1) = \begin{cases} v^2(T_1, P_{T_1}, E_{T_1}, \hat{\Lambda}_2(E_{T_1}^1)), & \text{if } E_{T_1}^1 < \Lambda_1 + \Lambda_2,\\ 1, & \text{otherwise.}\end{cases}
\end{aligned}
$$

where $v^2(\cdot, \cdot, \cdot, \Lambda)$ is the value function constructed in Theorem 1 with parameter Λ ($Y_t = v^2(t, P_t, E_t, \Lambda)$ in Theorem 1).

Current research

- Consider the terminal condition ϕ_1 and find conditions under which it is continuous or monotonic increasing.
- Show that the two-period pricing problem is well-posed under appropriate conditions (e.g. find conditions under which ϕ_1 is monotonic increasing.)
- Further numerical investigation and calibration (of a model in which P consists of electricity market factors) to real data.

Some numerical results (single period model)

We first approximate the discontinuous terminal condition ϕ by a Lipschitz continuous function. Then we apply the Markovian scheme of Bender and Zhang (2008).

- We set $P_t = (S_t^c, S_t^g, D_t)$ where S^c is the coal price, S^g the gas price, D the demand for electricity.
- The emissions rate $\bar{\mu}_e$ followed the bid-stack approach.

The choice of processes, functions and parameters was as in (Carmona, Coulon and Schwarz, 2012). Please see the figures below for two typical paths (scenarios) of the process (S_t^g, S_t^c, E_t, Y_t).

Some numerical results (single period model)

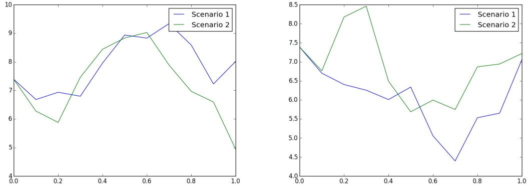

Figure: Realized paths of the gas price , S_t^g(left) and coal price S_t^c (right)

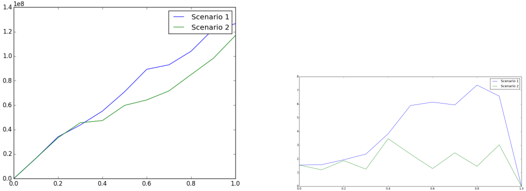

Figure: Realized paths of the emissions E_t (left) and allowance price Y_t (right)

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