Global Existence for the Derivative Nonlinear Schrödinger Equation by the Method of Inverse Scattering

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Introduction

The Derivative Nonlinear Schrödinger equation (DNLS)

For $(x, t) \in \mathbb{R} \times \mathbb{R}^{+}$, we study the following Cauchy problem

$$
\begin{aligned}
& iu_t + u_{xx} + i(|u|^2 u)_x = 0 \\
& u(x, 0) = q(x)
\end{aligned}
$$

Physical Background:
The DNLS is a canonical dispersive equation derived from the Magneto-Hydrodynamic equations in the presence of the Hall effect. It equation models the dynamics of Alfvén waves propagating along an ambient magnetic field in a long-wave, weakly nonlinear scaling regime.

Some History:

It was proved by Hayashi & Ozawa [3] that solutions exist locally in time in the Sobolev space $H^1(\mathbb{R})$ and they can be extended for all time if the L^2-norm of the initial condition is small enough, namely if $\|u_0\|_2 < \sqrt{2\pi}$. Recently, this upper bound has been improved to $\sqrt{4\pi}$ by Wu [1].

The question of global wellposedness of solutions of DNLS with large data is an important open problem. We apply the inverse scattering method to establish the global existence of the solution at the expense of more stringent regularity and decay assumptions on the initial data.

A central property of DNLS discovered by Kaup and Newell [2] is that it is solvable through the inverse scattering transform method. The IST method for an integrable nonlinear PDE can be explained with the help of the diagram

$$
\begin{array}{ccc}
u(x,0) & \xrightarrow{\ \mathcal{R}\ } & \rho(\lambda,0) \\
\text{Cauchy problem}\ \big\downarrow & & \big\downarrow\ \text{time evolution} \\
u(x,t) & \xleftarrow[\ \mathcal{I}\ ]{} & \rho(\lambda,t)
\end{array}
$$

In Lee's paper [4], it is shown that for Schwartz class initial condition q_0 satisfying certain spectral conditions, the formula

$$q(x,t) = \mathcal{I}\left[e^{4it(\diamond)^2}\left(\mathcal{R}q_0\right)(\diamond)\right](x) \tag{1}$$

gives a classical solution to the DNLS equation.

Main result:

Theorem

There is an open subset of $H^{2,2}(\mathbb{R})$ *containing a neighborhood of* 0 *so that the solution map* (1)

$$\begin{gathered} H^{2,2}(\mathbb{R}) \times \mathbb{R} \longrightarrow H^{2,2}(\mathbb{R}) \\ (q_0, t) \mapsto q(\cdot, t) \end{gathered}$$

is continuous, and Lipschitz continuous in q_0 *for each* t.

where

$$H^{2,2}(\mathbb{R}) = \{f(x) : \|(1+x^2)^{1/2} f\|_{L^2} < \infty,\ \|f''\|_{L^2} < \infty\}$$

The Direct Scattering Problem

It is common to rewrite DNLS in the form

$$iu_t + u_{xx} = i(|u|^2 u)_x \tag{2}$$

A gauge transformation plays an important role in the analysis. It has the form

$$q(x,t) = u(x,t)\exp\left(-i\int_{-\infty}^{x} |u(y,t)|^2 dy\right) \tag{3}$$

and maps solutions of (2) into solutions of

$$iq_t + q_{xx} + iq^2\overline{q}_x + \frac{1}{2}|q|^4 q = 0. \tag{4}$$

We will actually solve (4) by inverse scattering and use the inverse of the gauge transformation (3) to obtain the solution of (2).

The integrable equations (2) and (4) each admit a Lax representation

$$L_t - A_x + [L, A] = 0$$

for suitable operators L and A. Equivalently, (2) and (4) are the compatibility conditions for the system of equations

$$\psi_x = L\psi, \quad \psi_t = A\psi. \tag{5}$$

The equivalence of operators A and L for (2) and (4) can be shown through the gauge transformation.

The flow defined by (4) is linearized by the scattering transform associated with the linear problem

$$\frac{d}{dx}\Psi = -i\zeta^2\sigma\Psi + \zeta Q(x)\Psi + P(x)\Psi. \tag{6}$$

where $\mathrm{Im}(\zeta^2) = 0$ and the Jost solution Ψ is a 2×2 matrix and

$$\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad Q(x) = \begin{pmatrix} 0 & q(x) \\ \overline{q(x)} & 0 \end{pmatrix},$$

$$P(x) = \begin{pmatrix} p_1(x) & 0 \\ 0 & p_2(x) \end{pmatrix}$$

with

$$p_1(x) = -(i/2)|q(x)|^2, \quad p_2(x) = (i/2)|q(x)|^2. \tag{7}$$

The Jost solutions $\Psi^{\pm}$ obey the relation

$$\Psi^{+}(x,\zeta) = \Psi^{-}(x,\zeta)T(\zeta) \tag{8}$$

where

$$T(\zeta) = \begin{pmatrix} a(\zeta) & \breve{b}(\zeta) \\ b(\zeta) & \breve{a}(\zeta) \end{pmatrix} \tag{9}$$

with $a(\zeta)\breve{a}(\zeta) - b(\zeta)\breve{b}(\zeta) = 1$ called the *transition matrix* and the symmetry reduction

$$\breve{a}(\zeta) = \overline{a(\overline{\zeta})}, \quad \breve{b}(\zeta) = \overline{b(\overline{\zeta})}$$

Analytically, it is more convenient to work with the *normalized Jost solutions* $m^{\pm}(x,\zeta) = \Psi^{\pm}(x,\zeta)e^{ix\zeta^2\sigma}$. These functions solve the equation

$$\frac{d}{dx}m = -i\zeta^2 ad(\sigma)m + \zeta Q(x)m + P(x)m \tag{10}$$

where

$$ad(\sigma)(A) = \sigma A - A\sigma.$$

and the Jost solutions of (10) are related by

$$m^{+} = m^{-}e^{-ix\zeta^2 ad(\sigma)}T(\zeta)$$

Letting $\lambda = \zeta^2$, we get a new linear problem from (10)

$$\frac{dn^{\pm}}{dx} = -i\lambda ad(\sigma)n^{\pm} + \begin{pmatrix} 0 & q \\ \lambda\overline{q} & 0 \end{pmatrix} n^{\pm} + Pn^{\pm} \quad (11a)$$

$$\lim_{x\to\pm\infty} n^{\pm}(x,\lambda) = \mathbf{1} \quad (11b)$$

and are related by

$$n^{+}(x,\lambda) = n^{-}(x,\lambda)e^{-i\lambda x ad(\sigma)} \begin{pmatrix} \alpha(\lambda) & \beta(\lambda) \\ \lambda\overline{\beta(\lambda)} & \overline{\alpha(\lambda)} \end{pmatrix} \quad (12)$$

where

$$\rho(\lambda) = \beta(\lambda)/\alpha(\lambda)$$

Setting

$$e_\lambda(x) = e^{2i\lambda x},$$

we have from (11a)-(11b) that $n_{11}^{\pm}$ and $n_{21}^{\pm}$ obey the integral equations

$$n_{11}^{\pm}(x,\lambda) = 1 - \int_0^{\pm\infty} \left(q(y)n_{21}^{\pm}(y,\lambda) + p_1(y)n_{11}^{\pm}(y,\lambda)\right)\, dy \quad (13a)$$

$$n_{21}^{\pm}(x,\lambda) = -\int_0^{\pm\infty} e_\lambda(x-y)\left(\lambda\overline{q(y)}n_{11}^{\pm}(y,\lambda) + p_2(y)n_{21}^{\pm}(y,\lambda)\right)\, dy. \quad (13b)$$

In order to compute α and β from $n_{11}^{\pm}$ and $n_{21}^{\pm}$, we evaluate (12) at $x = 0$ and use the following symmetry relation

$$n_{22}(x,\lambda) = \overline{n_{11}(x,\lambda)}, \quad n_{12}(x,\lambda) = \lambda^{-1}\overline{n_{21}(x,\lambda)}$$

to conclude that

$$\alpha(\lambda) = n_{11}^{+}(0,\lambda)\overline{n_{11}^{-}(0,\lambda)} - \lambda^{-1} n_{21}^{-}(0,\lambda)\overline{n_{21}^{+}(0,\lambda)} \tag{14}$$

$$\beta(\lambda) = \frac{1}{\lambda}\left(n_{11}^{-}(0,\lambda)\overline{n_{21}^{+}(0,\lambda)} - n_{11}^{+}(0,\lambda)\overline{n_{21}^{-}(0,\lambda)}\right) \tag{15}$$

Note that both α and β are regular at $\lambda = 0$.

Theorem 2.1

There is an open subset U of $H^{2,2}(\mathbb{R})$ containing a neighborhood of 0 so that the direct scattering map $\mathcal{R}$, initially defined on $\mathcal{S}(\mathbb{R}) \cap U$, extends to a Lipschitz continuous map from U into $H^{2,2}(\mathbb{R})$. Moreover, $\mathcal{R}(U)$ is invariant under the map $r \mapsto e^{4it(\diamond)^2} r(\diamond)$, and also contains an open neighborhood of 0 in $H^{2,2}(\mathbb{R})$.

For $i, j = 1, 2$ we formulate integral equations of the form

$$n_{i\,j}(x, \lambda) = h(x, \lambda) + (Tn_{i\,j})(x, \lambda)$$

where $h(x, \lambda)$ lies in some space with mixed norm. And T is an integral operator whose associated resolvent $(I - T)^{-1}$ is a bounded operator on that space.

The Inverse Scattering problem

We now turn to the inverse map. We begins by constructing from $m^{\pm}$, the Beals-Coifman solutions $M(x,\zeta)$ for $\zeta \in \mathbb{C} \setminus \Sigma$, with the properties that

(1) $M(x,\zeta)$ solves the linear spectral problem and $\lim_{x\to+\infty} M(x,\zeta) = \mathbf{1}$,

(2) $\lim_{\zeta\to\infty} M(x,\zeta) = \mathbf{1}$,

(3) $M(x,\zeta)$ is bounded as $x \to -\infty$, and

(4) The boundary values $M^{\pm}(x,\zeta) = \lim_{z\to\zeta,\, \pm Im(z^2)>0} M(x,z)$ exist on Σ and obey the jump relation

$$M_+(x,\zeta) = M_-(x,\zeta)e^{-ix\zeta^2 ad(\sigma)}J(\zeta)$$

where

$$J(\zeta) = \begin{pmatrix} 1 - b\breve{b}/a\breve{a} & \breve{b}/a \\ -b/\breve{a} & 1 \end{pmatrix}$$

Properties (2)–(4) define a Riemann-Hilbert problem which can be solved to construct the solutions $M(x, \zeta)$ and recover Q (and hence q) from the large-ζ asymptotics of $M(x, \zeta)$. Explicitly, writing

$$M(x, \zeta) = \mathbf{1} + \zeta^{-1} M_{-1}(x) + o(1/\zeta)$$

we have

$$Q(x) = i\, ad(\sigma) M_{-1}(x)$$

As with the direct map, one can reduce this problem to a RHP with contour $\mathbb{R}$ using symmetry. It can be shown that the diagonal of $M(x,\zeta)$ is even under the reflection $\zeta \mapsto -\zeta$, while the off-diagonal is odd. Hence, by the change of variables

$$M^{\sharp}(x,\zeta^2) = \begin{pmatrix} M_{11}(x,\zeta) & \zeta^{-1}M_{12}(x,\zeta) \\ \zeta M_{21}(x,\zeta) & M_{22}(x,\zeta) \end{pmatrix}$$

one arrives at the Riemann-Hilbert problem

$$\begin{aligned} M^{\sharp}_{+}(x,\lambda) &= M^{\sharp}_{-}(x,\lambda)e^{-i\lambda x ad(\sigma)}J(\lambda) \\ J(\lambda) &= \begin{pmatrix} 1-\lambda|\rho(\lambda)|^2 & \rho(\lambda) \\ -\lambda\overline{\rho(\lambda)} & 1 \end{pmatrix} \end{aligned} \tag{16}$$

However this RHP is not properly normalized (a careful computation shows that $M^{\sharp}(x,\lambda) \not\to \mathbf{1}$ as $\lambda \to \infty$) and it is more effective to consider the row-wise RHP

$$N_+(x,\lambda) = N_-(x,\lambda) e^{-i\lambda x \, ad(\sigma)} J(\lambda) \tag{17a}$$

$$\lim_{\lambda\to\infty} N(x,\lambda) = (1,0). \tag{17b}$$

where

$$N(x,\lambda) = \left(N_{11}(x,\lambda), N_{12}(x,\lambda) \right).$$

One recovers q from the relation

$$q(x) = 2i \lim_{z \to \infty} z\, N_{12}(x, z) \tag{18}$$

where the limit is taken non-tangentially in $\mathbb{C} \setminus \mathbb{R}$. One can also compute the limit in (18) from the integral formula

$$q(x) = -\frac{1}{\pi} \int e^{-2i\lambda x} \rho(\lambda) \nu_{11}(x, \lambda)\, d\lambda. \tag{19}$$

The mapping $\rho \mapsto q$ determined by the RHP (17) and the asymptotic formula (18) is called the *inverse scattering map* and denoted $\mathcal{I}$. The last step is the reconstruction of the potential at time $t > 0$ which follows from the time evolution of the reflection coefficient $\rho(\lambda, t)$.

Now we formulate the Beals-Coifman equation. The jump matrix (16) is factorized as

$$J(\lambda) = (I - w^-(\lambda))^{-1}(I + w^+(\lambda)) \tag{20}$$

where

$$w^-(\lambda) = \begin{pmatrix} 0 & \rho(\lambda) \\ 0 & 0 \end{pmatrix}, \quad w^+(\lambda) = \begin{pmatrix} 0 & 0 \\ -\lambda\overline{\rho(\lambda)} & 0 \end{pmatrix}. \tag{21}$$

The row vector $\nu = (\nu_{11}, \nu_{12})$ defined as

$$\nu(x,\lambda) = N^+(x,\lambda)(1 + w_x^+)^{-1} = N^-(1 - w_x^-)^{-1},$$

with

$$w_x^\pm(\lambda) = e^{i\lambda x ad(\sigma)} w^\pm(\lambda)$$

obeys the Beals-Coifman integral equation

$$\nu = \mathbf{e}_1 + \mathcal{C}_w(\nu) \tag{22}$$

where the operator C_w is defined by its action on a row vector-valued function f as

$$\mathcal{C}_w(f)(\lambda) = C_+(f w_x^-)(\lambda) + C_-(f w_x^+)(\lambda). \tag{23}$$

Equivalently, in component form,

$$\nu_{11}(x,\lambda) = 1 - C_-\left(\nu_{12}(x,\,\cdot\,)(\,\cdot\,)\overline{\rho_x(\,\cdot\,)}\right)(\lambda) \tag{24a}$$

$$\nu_{12}(x,\lambda) = C_+\left(\nu_{11}(x,\,\cdot\,)\rho_x(\,\cdot\,)\right)(\lambda) \tag{24b}$$

where

$$\rho_x(\lambda) = e^{-2i\lambda x}\rho(\lambda).$$

We can iterate these equations to obtain a single integral equation for ν_{11}:

$$\nu_{11}(x,\lambda) = 1 + S\left[\nu_{11}(x,\,\cdot\,)\right](\lambda) \tag{25}$$

where

$$S[f](\lambda) = -C_-\left[C_+\left(f(\,\cdot\,)\rho_x(\,\cdot\,)\right)(\diamond)\overline{\rho_x}(\diamond)\right](\lambda). \tag{26}$$

It is useful to define $\nu^\sharp = \nu_{11} - 1$ satisfying

$$\nu^\sharp = S[1] + S\left[\nu^\sharp(x,\,\cdot\,)\right](\lambda) \tag{27}$$

where

$$S[1](\lambda) = C_-\left[C_+\left((\,\cdot\,)\rho_x(\,\cdot\,)\right)(\diamond)\overline{\rho_x}(\diamond)\right](\lambda)$$

is well-defined as an element of $L^2(\mathbb{R})$ for each fixed x since $\rho \in H^{2,2}(\mathbb{R})$ and $C_\pm$ are isometries of L^2.

We show that if ρ belongs to $H^{2,2}(\mathbb{R})$, then q belongs to $H^{2,2}(\mathbb{R}^+)$. It suffices to estimate the L^2-norms of q, $x^2 q$, and q''. From the reconstruction formula, we get

$$q(x) = -\frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho(\lambda)\,d\lambda - \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho(\lambda)\nu^\sharp\,d\lambda$$

$$\begin{aligned} q''(x) &= \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}4\lambda^2\rho(\lambda)\,d\lambda + \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}4\lambda^2\rho(\lambda)\nu^\sharp\,d\lambda \\ &\quad + \frac{1}{\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\left[2i\lambda\rho(\lambda)\frac{\partial\nu^\sharp}{\partial x}(x,\lambda) - \rho(\lambda)\frac{\partial^2\nu^\sharp}{\partial x^2}(x,\lambda)\right]\,d\lambda \end{aligned}$$

and

$$x^2 q(x) = \frac{1}{4\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho''(\lambda)\,d\lambda + \frac{1}{4\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\rho''(\lambda)(\nu^\sharp)\,d\lambda$$
$$+ \frac{1}{4\pi}\int_{\mathbb{R}} e^{-2i\lambda x}\left[2\rho'(\lambda)\frac{\partial \nu^\sharp}{\partial\lambda}(x,\lambda) + \rho(\lambda)\frac{\partial^2\nu^\sharp}{\partial\lambda^2}(x,\lambda)\right]\,d\lambda.$$

In each of these formulas, the first (Fourier) terms clearly have the correct mapping properties, and it remains to estimate the remaining terms involving ν_{11} and its derivatives. We denote by $L^2(\mathbb{R}^+ \times \mathbb{R})$ the space $L^2((\mathbb{R}^+ \times \mathbb{R}), dx\,d\lambda)$.

Proposition

3.1 The map $\rho \mapsto q$ is a Lipschitz continuous map from B a bounded subset of $H^{2,2}(\mathbb{R}^+)$ to $H^{2,2}(\mathbb{R}^+)$, provided that the following maps are Lipschitz maps from B into a bounded subset $L^2(\mathbb{R}^+ \times \mathbb{R})$:

(i) $\rho \mapsto \nu^\sharp$,

(ii) $\rho \mapsto (\partial \nu^\sharp / \partial x)$,

(iii) $\rho \mapsto (\partial^2 \nu^\sharp / \partial x^2)$,

(iv) $\rho \mapsto (\partial \nu^\sharp / \partial \lambda)$, and

(v) $\rho \mapsto \langle \lambda \rangle^{-1} (\partial^2 \nu^\sharp / \partial \lambda^2)$

"Boundedness of B" means that $\forall \rho \in B$, $\exists$ some constant C_B such that

$$\|\rho\|_{H^{2,2}(\mathbb{R})} \leq C_B.$$

Note that

$$\nu^{\sharp} = S\left[1\right] + S\left[\nu^{\sharp}(x,\,\cdot\,)\right](\lambda)$$

is a singular integral equation. We use the Fourier transform form of Cauchy projection

$$\left(C^{+}f\right)(\lambda) = \frac{1}{\pi}\int_{0}^{\infty} e^{2i\lambda\zeta}\widehat{f}(\zeta)\,d\zeta \tag{28}$$

$$\left(C^{-}f\right)(\lambda) = \frac{1}{\pi}\int_{0}^{-\infty} e^{2i\lambda\zeta}\widehat{f}(\zeta)\,d\zeta \tag{29}$$

to write down explicitly

$$S\left[1\right](\lambda) = \frac{-1}{2i\pi^2}\int_{0}^{-\infty} e^{2i\lambda\xi}\int_{x}^{\infty}\widehat{\rho}(\xi')\widehat{\overline{\rho}}\,'(\xi-\xi')d\xi' d\xi.$$

Proposition 3.2:
Suppose $\rho \in H^{2,2}(\mathbb{R})$. Then

$$\|S[1]\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\|\rho\|_{H^{2,2}}\|\overline{\rho}\|_{H^{2,2}} \tag{30}$$

$$\left\|\frac{\partial S[1]}{\partial x}\right\|_{L^2_x L^2_\lambda} \lesssim \|\rho\|_{H^{2,2}}\|\overline{\rho}\|_{H^{2,2}}, \quad \left\|\frac{\partial^2 S[1]}{\partial x^2}\right\|_{L^2_x L^2_\lambda} \lesssim \|\rho\|_{H^{2,2}}\|\overline{\rho}\|_{H^{2,2}} \tag{31}$$

$$\left\|\frac{\partial S[1]}{\partial \lambda}\right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\|\rho\|_{H^{2,2}}\|\overline{\rho}\|_{H^{2,2}} \tag{32}$$

$$\left\|\langle\lambda\rangle^{-1}\frac{\partial^2 S[1]}{\partial \lambda^2}\right\|_{L^2_\lambda} \lesssim (1+|x|)^{-1}\|\rho\|_{H^{2,2}}\|\overline{\rho}\|_{H^{2,2}} \tag{33}$$

Moreover, the functions appearing in the left-hand-side are all Lipschitz continuous in ρ.

and

$$S[h](\lambda) = -\frac{1}{2i\pi^2}\int_0^{-\infty} e^{2i\lambda\xi}\int_x^{\infty}(\widehat{\rho}*\widehat{h})(\xi')\widehat{\overline{\rho}}\,'(\xi-\xi')d\xi' d\xi$$

Lemma

3.3 Suppose that $x \geq 0$ and $\rho \in H^{1,1}_\alpha(\mathbb{R})$ where $1/2 < \alpha < 1$. Then:

(i) *The operator S is bounded on L^2 and Hilbert-Schmidt. Moreover, for any fixed $M > 0$,*

$$\lim_{x\to\infty} \sup\{\|S\| : \|\rho\|_{H^{1,1}_\alpha} \leq M\} = 0.$$

(ii) *The map $(x,\rho) \in \mathbb{R}^+ \times H^{1,1}_\alpha(\mathbb{R}) \longrightarrow S_{x,\rho} \in \mathcal{B}(L^2)$ is continuous.*

We proceed as follows:

- We begin by proving important estimates on the inhomogeneous term $S[1]$ of (27) and on the linear operator S. These are done in the previous proposition and lemma.
- Assertions (i)–(v) of Proposition 30 are proven in the following order:
  (1) We show that the resolvent $(I - S)^{-1}$ exists and is a bounded operator from $L_x^2 L_\lambda^2$ to itself. Then we use this estimate to establish assertion (i) of Proposition 3.1.

(2) Having established (i), we differentiate (27) to get the integral equations for the corresponding derivatives in (ii)–(v) and use what we know from the previous step to conclude that assertions (ii)-(v) hold. Note that the λ-derivatives will be studied using a direct computation, while x-derivatives will be studied by differentiating the integral equation with respect to a parameter.

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Thanks for your attention!