A Game Theory Course for Mathematics Students

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Noncooperative game theory

Noncooperative game theory is study of a certain situation.

- Several actors.
- Each has a choice of several actions.
- Once all have chosen their actions, payoffs to all players are determined.

Each player wants to maximize her own payoff, but

- she only controls her own actions;
- her payoff is affected by other players' actions.

Game theory in the undergraduate curriculum

Usually taught at the junior-senior level in economics or political science departments, sometimes mathematics departments.

Sometimes taught at the freshman-sophomore level as an introduction to the social sciences. Text: A. Dixit and S. Skeath, *Games of Strategy*, Norton. "We believe that there is a strong case for reversing the usual order whereby general introductory courses in each subject are followed by advanced subject-specific courses in game theory. In the more natural progression, all students interested in the social and biological sciences would complete a freshman course in elementary game theory before going on to more detailed study of one of the specialized fields." Such a course would also satisfy a social-science distribution requirement for other students.

Game theory in the undergraduate curriculum

Could be taught at the freshman-sophomore level in mathematics departments as a general education class that transmits the idea of mathematical modeling.

Motivation

In 2005 I started a game theory class in at the advanced undergraduate level in the NC State Mathematics Department.

Motivation

- Personal: sense that game theory was a way of looking at the world that gave insight.
- Mathematical: evolutionary game theory, in which the prevalence of strategies in a population evolves over time, uses ODEs.
- Educational: lack of interesting electives for undergraduate mathematics majors.

Sadly, I didn't know much about the subject.

History

- 2005: special topics course.
- 2007: became a regular course, taught each fall.
- ODE prerequisite.
- Originally limited to 20–25 students, raised to 35 in 2010 with homework required to be done in groups.
- The course fills each fall.
- Now in practice limited to mathematics majors and minors.
- The course is not popular because I am an exceptional teacher. It is popular because the subject interests students.

At NC State there are no game theory courses in other departments.

Who would want to teach game theory?

Some problems and examples

- White House tapes
- Israelis and Palestinians
- Global warming negotiations
- Second-price auctions
- Keynes's theory of economic depressions
- The Troubled Asset Relief Program
- Braess's paradox
- Oracle at Delphi
- Cuban Missile Crisis
- Crime control
- Value of college
- Samaritan's dilemma
- Rotten Kid Theorem
- Kitty Genovese case

Game Theory Evolving by Herbert Gintis (Princeton University Press 2000)

I wanted a text that included evolutionary game theory using differential equations.

Gintis's description of *Game Theory Evolving*:

"For most topics, I provide just enough in the way of definitions, concepts, theorems, and examples begin solving problems. Learning and insight come from grappling with and solving problems. The expositional material and many problems are appropriate for an undergraduate course, but some problems, especially those situated toward the end of a chapter, are better suited for a graduate course."

Game Theory Evolving

"Game theory is adventure and fantasy."

"Game theory is a universal language for the unification of the behavioral sciences."

"Game theory is about the emergence, transformation, diffusion, and stabilization of forms of behavior."

Lots of opinions about questions of economic and social science theory.

Organization

Introduce ideas, then use them in variety of situations.

Original order (Gintis's order disaggregated)

1. Games in extensive form with complete information
2. Normal form games solved by elimination of dominated strategies
3. Pure-strategy Nash equilibrium
4. Games in extensive form with incomplete information
5. Mixed-strategy Nash equilibria
6. More about games in extensive form (subgame perfect equilibria, repeated games . . .)
7. Evolutionary stability
8. Differential equations
9. Evolutionary dynamics

A rather natural progression . . .

Comments

Tried to be true to spirit of text.

- Learn an idea, then use it in a variety of problems.
- Problems almost all involve some sort of story.

Devoted a lot of effort to solving problems, understanding how the material fits together, adding problems from other sources when I learned of a striking application of game theory.

Due to personal inclination, avoided Bayesian games, stochastic games, software.

Class consists of lectures, problem sets to be graded, two tests, final exam.

Optional paper “should be on an application of game theory in an area of interest to you, and should include a description of the situation, appropriate game theory models with justification, analysis of the models, and conclusions about the original situation.”

Game Theory in Action by Herbert Gintis and S., Princeton University Press, 2016

After a couple years I started producing supplementary exposition, i.e., typed up and corrected lectures.

Provided worked examples for more types of problems.

After a couple more years I seemed to have the makings of a book.

Gintis agreed to be a coauthor with proviso that book be offered to PUP. Accepted by their economics editor.

Prepared solution manual, having in mind the interested teacher who didnt know much about the subject.

Characteristics

Calculus text approach: universal mathematical ideas with application to wide variety of areas

Mathematician's sensibility rather than social scientist's sensibility or abstract development of subject

Accurate statement of theory, many proofs, emphasis on problems

Shamelessly gathered interesting examples and problems wherever I found them (invented in a few cases)

Plenty of classics (problems with their own Wikipedia page)

Missing:

- proof of existence of a mixed-strategy Nash equilibrium
- conversion of the problem of finding Nash equilibria of a 2-person 0-sum games to a linear programming problem

Games in normal form

Game in normal form:

- Players $1, \ldots, n$.
- Strategy sets $S_1, \ldots S_n$.
- Payoff functions $\pi_i : S_1 \times \cdots \times S_n \to \mathbb{R}$, $i = 1, \ldots, n$.

A *mixed strategy* σ_i for Player i uses each $s \in S_i$ with probability p_s ($p_s \geq 0$, $\sum_{s \in S_i} p_s = 1$).

Define $\pi_i(\sigma_1, \ldots, \sigma_n)$ by assuming the players choose the probabilities independently.

A *mixed strategy Nash equilibrium* of a game in normal form is a mixed strategy profile $(\sigma_1^*, \ldots, \sigma_n^*)$ such that: if any *single* player changes his strategy, his own payoff will not increase:

- For every σ_1, $\pi_1(\sigma_1^*, \sigma_2^*, \ldots, \sigma_n^*) \geq \pi_1(\sigma_1, \sigma_2^*, \ldots, \sigma_n^*)$.
- For every σ_2, $\pi_2(\sigma_1^*, \sigma_2^*, \sigma_3^*, \ldots, \sigma_n^*) \geq \pi_2(\sigma_1^*, \sigma_2, \sigma_3^*, \ldots, \sigma_n^*)$.
- etc.

Nash's Theorem

Theorem

A game in normal form with finite strategy sets has a Nash equilibriium.

Evolutionary stability

A mixed strategy σ can also be viewed as a *population state*: a probability distribution on strategies in use in a population.

Consider a symmetric game with two players (so $S_1 = S_2$) and a finite strategy set. A mixed strategy σ is an *evolutionarily stable state* if for each $\tau \neq \sigma$,

$$\pi_1(\sigma, (1-\epsilon)\sigma + \epsilon\tau) > \pi_1(\tau, (1-\epsilon)\sigma + \epsilon\tau)$$

for small $\epsilon > 0$.

Theorem
If σ is an evolutionarily stable state, then (σ, σ) is a mixed strategy Nash equilibriium.

Symmetric games

Symmetric game:

- $S_1 = \cdots = S_n = S$.
- Suppose $(s_1, \ldots, s_n) \in S \times \cdots \times S$ has payoffs $(\pi_1, \ldots, \pi_n)$, and α is a permutation of $\{1, \ldots, n\}$. Then $\alpha \cdot (s_1, \ldots, s_n)$ has payoffs $\alpha \cdot (\pi_1, \ldots, \pi_n)$.

Theorem

A symmetric game in normal form with finite strategy set has a symmetric mixed strategy Nash equilibrium (all players use the same mixed strategy).

Princeton bar

n men walk into a bar. In the bar is one extremely attractive woman and many attractive women. Each man has two possible pure strategies:

- s: Approach one of the attractive women. (The safe strategy.)
- r: Approach the extremely attractive woman. (The risky strategy.)

The payoffs are:

- $a > 0$ to each man who uses strategy s. (There are many attractive women in the bar; the strategy of approaching one of them will succeed.)
- If there is a unique man who uses strategy r, his payoff is $b > a$. If more than one man uses strategy r, they all have payoff 0. (The extremely attractive woman doesn't enjoy being pestered and leaves.)

Nash equilibria in the Princeton bar

Pure strategy Nash equilibria: $n-1$ men approach attractive women, one approaches the extremely attractive woman.

These Nash equilibria are not symmetric. There must be a symmetric one.

Let $\sigma = ps + (1-p)r$ with $0 < p < 1$. We must find p such that $(\sigma, \dots, \sigma)$ is a Nash equilibrium.

$$\pi_1(s, \sigma, \dots, \sigma) = \pi_1(r, \sigma, \dots, \sigma)$$
$$a = bp^{n-1} + 0(1 - p^{n-1})$$
$$p^{n-1} = \frac{a}{b}$$
$$p = \left(\frac{a}{b}\right)^{\frac{1}{n-1}}$$

Colonel Blotto vs. the People's Militia

There are two valuable towns. Col. Blotto has four regiments. The People's Militia has three regiments. Each decides how many regiments to send to each town.

If Col. Blotto sends m regiments to a town and the People's Militia sends n, Col. Blotto's payoff for that town is

$$\begin{aligned} 1+n &\text{ if } m>n, \\ 0 &\text{ if } m=n, \\ -(1+m) &\text{ if } m<n. \end{aligned}$$

Col. Blotto's total payoff is the sum of his payoffs for each town. The People's Militia's payoff is the opposite of Col. Blotto's.

$S_1 = \{40, 31, 22, 13, 04\}$, $S_2 = \{30, 21, 12, 03\}$

Colonel Blotto payoffs

			PM			
			q_1 **30**	q_2 **21**	q_3 **12**	q_4 **03**
	p_1	**40**	$(4, -4)$	$(2, -2)$	$(1, -1)$	$(0, 0)$
	p_2	**31**	$(1, -1)$	$(3, -3)$	$(0, 0)$	$(-1, 1)$
CB	p_3	**22**	$(-2, 2)$	$(2, -2)$	$(2, -2)$	$(-2, 2)$
	p_4	**13**	$(-1, 1)$	$(0, 0)$	$(3, -3)$	$(1, -1)$
	p_5	**04**	$(0, 0)$	$(1, -1)$	$(2, -2)$	$(4, -4)$

Gintis suggests: look for a mixed strategy Nash equilibrium with $p_1 = p_5$, $p_2 = p_4 = 0$, $q_1 = q_4$, and $q_2 = q_3$.

Colonel Blotto symmetries

			PM			
			q_1 **30**	q_2 **21**	q_3 **12**	q_4 **03**
	p_1	**40**	$(4, -4)$	$(2, -2)$	$(1, -1)$	$(0, 0)$
	p_2	**31**	$(1, -1)$	$(3, -3)$	$(0, 0)$	$(-1, 1)$
CB	p_3	**22**	$(-2, 2)$	$(2, -2)$	$(2, -2)$	$(-2, 2)$
	p_4	**13**	$(-1, 1)$	$(0, 0)$	$(3, -3)$	$(1, -1)$
	p_5	**04**	$(0, 0)$	$(1, -1)$	$(2, -2)$	$(4, -4)$

You can see a symmetry: flip across a horizontal line through the middle, flip again across a vertical line though the middle: the matrix is unchanged.

Theorem

If a game has a symmetry group, then it has a mixed strategy Nash equilibrium that is fixed by that symmetry group.

This justifies $p_1 = p_5$, $p_2 = p_4$, $q_1 = q_4$, and $q_2 = q_3$.

Rock-Paper-Scissors

		Player 2		
		r	p	s
	r	$(0,0)$	$(-1,1)$	$(1,-1)$
Player 1	p	$(1,-1)$	$(0,0)$	$(-1,1)$
	s	$(-1,1)$	$(1,-1)$	$(0,0)$

The *only* strategy profile that is fixed by the symmetry group of Rock-Paper-Scissors ($\mathbb{Z}_3$) is $\left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$.

Therefore it is a Nash equilibrium.

In the book we include a chapter on symmetry groups of normal form games before the chapter on evolutionary stability.

Replicator system

Consider a symmetric two-player game with finite strategy set $S = \{s_1, \dots, s_n\}$.

A population uses strategy s_1 with probability p_1, ..., strategy s_n with probability p_n; all $p_i \geq 0$ and $\sum p_i = 1$.

The population state is $\sigma = \sum p_i s_i$.

Replicator system:

$$\dot{p}_i = \big(\pi_1(s_i, \sigma) - \pi_1(\sigma, \sigma)\big) p_i, \quad i = 1, \dots, n.$$

A differential equation on the simplex

$$\Sigma = \{(p_1, \dots, p_n) : \text{all } p_i \geq 0 \text{ and } \sum p_i = 1\}.$$

- Nash equilibria (σ, σ) correspond to equilibria of the replicator system.
- Evolutionarily stable states are asymptotically stable equilibria of the replicator system.

Microsoft vs. Apple

In the early days of personal computing, you could buy a computer running Microsoft Windows, or one running the Apple operating system.

- Either was okay, but Apple's was better.
- Neither dealt well with files produced by the other.
- Thus if your coworker used Windows and you used Apple, not much got accomplished.

		Player 2	
		m	**a**
Player 1	**m**	$(1,1)$	$(0,0)$
	a	$(0,0)$	$(2,2)$

Microsoft vs. Apple: Nash equilibria

		Player 2	
		m	**a**
Player 1	**m**	$(1, 1)$	$(0, 0)$
	a	$(0, 0)$	$(2, 2)$

Nash equilibria: (m, m), (a, a), (σ^*, σ^*) with $\sigma^* = \frac{2}{3}m + \frac{1}{3}a$.

The mixed strategy Nash equilibrium is hard to understand. Even if people picked computers randomly, why would they choose the worse one with higher probability?

Microsoft vs. Apple: Replicator equation

		Player 2	
		m	**a**
Player 1	**m**	$(1, 1)$	$(0, 0)$
	a	$(0, 0)$	$(2, 2)$

$$\dot{p}_1 = \big(\pi_1(m, \sigma) - \pi_1(\sigma, \sigma)\big)p_1 = \big(p_1 - (p_1^2 + 2p_2^2)\big)p_1,$$
$$\dot{p}_2 = \big(\pi_1(a, \sigma) - \pi_1(\sigma, \sigma)\big)p_2 = \big(2p_2 - (p_1^2 + 2p_2^2)\big)p_2.$$

Reduce to one equation:

$$\dot{p}_1 = \Big(p_1 - \big(p_1^2 + 2(1 - p_1)^2\big)\Big)p_1 = (1 - p_1)(3p_1 - 2)p_1.$$

The worse computer has the smaller basin of attraction.