Compactness estimates for Hamilton-Jacobi equations

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Outline

1. Introduction to Hamilton-Jacobi equations
   - Hamilton-Jacobi equations
2. Compactness estimates
   - Kolmogorov's ε-entropy
   - Proof of upper bound
   - Proof of lower bound
   - Extensions and concluding remarks

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Hamilton-Jacobi equation

$$\begin{cases} u_t(t,x) + H\big(t,x,D_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

where

- $H : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of class C^2 such that

$$(a) \quad \lim_{|p|\to\infty} \frac{H(t,x,p)}{|p|} = +\infty$$

$$(b) \quad D_p^2 H(t,x,p) > 0$$

- $u_0 : \mathbb{R}^n \to \mathbb{R}$ is a Lipschitz function

A typical example

$$H(t,x,p) = f(t,x)\left(1+|p|^2\right)^{\frac{m}{2}} + g(t,x)$$

where

- $m > 1$
- $f, g \in C^2\left([0,T] \times \mathbb{R}^n\right)$ satisfy

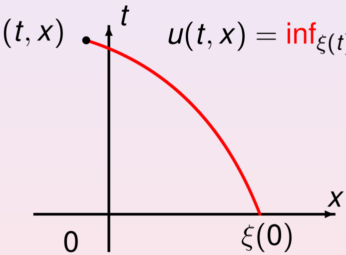
$$0 < f(t,x) < c_1 \qquad \text{and} \qquad -c_2(1+|x|) \leqslant g(t,x) \leqslant c_2$$

Associated variational problem

Denote by $L : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ the Legendre transform

$$L(t, x, q) = \max_{p \in \mathbb{R}^n} \big[\langle q, p \rangle - H(t, x, p) \big]$$

then the value function



$$u(t, x) = \inf_{\xi(t)=x} \left\{ \int_0^t L\big(s, \xi(s), \xi'(s)\big)\, dt + u_0\big(\xi(0)\big) \right\}$$

gives the viscosity solution of

$$\begin{cases} u_t(t, x) + H\big(t, x, D_x V(t, x)\big) = 0 & (t, x) \in (0, T) \times \mathbb{R}^n \\ u(0, x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

Weak solutions to Hamilton-Jacobi equations

$$\begin{cases} u_t(t,x) + H\big(x, \nabla u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases} \qquad (HJ)$$

- has no global smooth solution due to crossing of characteristics
- may have infintely many Lipschitz solutions satisfying (HJ) a.e.
  - Dacorogna and Marcellini (1999)
- has a unique viscosity solution
  - Crandall and Lions (1983), Crandall, Evans and Lions (1984)
  - Bardi and Capuzzo Dolcetta (1997), Fleming and Soner (1993)
- the viscosity solution is the unique semiconcave u satisfying (HJ) a.e.
  - Kruzhkov (1960), Douglis (1961), Krylov (1987)
  - C – Sinestrari (Birkhäuser, 2004)
  - Villani (Springer, 2009)

Three cases for irreversibility

$S_t : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$ semigroup

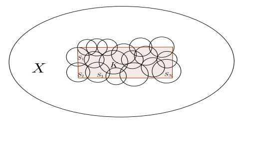
$$S_t(u_0)(\cdot) = u(t,\cdot) \qquad \begin{cases} u_t(t,x) + H\big(t,x,D_x u(t,x)\big) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

- **gain of regularity** $\rightarrow$ semiconcavity
- **compactness** $\rightarrow$ estimates for Kolmogorov's entropy
- **loss of regularity** $\rightarrow$ propagation of singularities

Outline

A compactness measure: Kolmogorov's ε-entropy

Let (X, d) be a metric space and K a totally bounded subset of X



For any $\varepsilon > 0$, let $N_\varepsilon(K)$ be the minimal number of sets in a cover of K by subsets of X having diameter no larger than 2ε

Definition

The ε-entropy of K is defined as

$$\mathcal{H}_\varepsilon(K \mid X) = \log_2 N_\varepsilon(K)$$

Motivation

Kolmogorov's ε-entropy used

- to study accuracy and resolution of numerical methods
  - Lin – Tadmor, 2001: L^1-Stability and error estimates for approximate Hamilton-Jacobi equations
- to analyze computational complexity
  - derive minimum number of operations to compute solutions with error $< \varepsilon$

For conservation laws

- Lax (1954)
- De Lellis – Golse (2005)
- Ancona – Glass – Khai T. Nguyen (2012, 214)

Problem set-up

$S_t : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n)$

$$S_t(u_0)(x) = u(t,x) \qquad \begin{cases} u_t(t,x) + H\big(t,x,D_x u(t,x)\big) = 0 \\ u(0,x) = u_0(x) \end{cases}$$

Our goal

to study dependence with respect to $\varepsilon > 0$ of

$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}) \mid \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) = \log_2 N_\varepsilon\big(S_T(\mathcal{C})\big)$$

for a suitable class of initial conditions $\mathcal{C}$

$N_\varepsilon\big(S_T(\mathcal{C})\big)$ = minimal number of subsets of $\mathbf{W}^{1,1}(\mathbb{R}^n)$ with diam $\leqslant 2\varepsilon$ covering $S_T(\mathcal{C})$

A special class of Hamilton-Jacobi equations

For $H(x,p) = H(p)$ the Cauchy problem

$$\begin{cases} u_t(t,x) + H\big(D_x u(t,x)\big) = 0 & (t,x) \in [0,+\infty) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^N \end{cases}$$

admits the solution $S_t(u_0)(x) = u(t,x)$ given by

Hopf-Lax semigroup

$$\begin{cases} S_t : \mathrm{Lip}(\mathbb{R}^n) \to \mathrm{Lip}(\mathbb{R}^n) & t \geqslant 0 \\ S_t(u_0)(x) = \min_{y \in \mathbb{R}^n} \left\{ t \cdot L\left(\frac{x-y}{t}\right) + u_0(y) \right\} & x \in \mathbb{R}^n \end{cases}$$

where

$$L(q) = \max_{p \in \mathbb{R}^n} \left\{ \langle p, q \rangle - H(p) \right\} \qquad (q \in \mathbb{R}^n)$$

Properties of the Hopf-Lax semigroup

Given $K, L, M > 0$, define

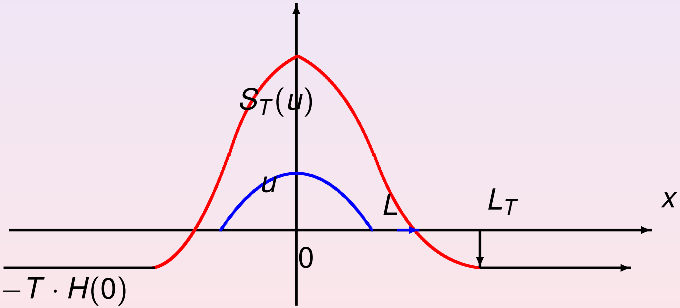
$$\begin{aligned} \mathcal{C}_{[L,M]} &= \left\{ u \in \mathrm{Lip}(\mathbb{R}^n) \ : \ \mathrm{spt}(u) \subset [-L, L]^n \,,\ \mathrm{Lip}[u] \leqslant M \right\} \\ \mathcal{SC}_{[K,L,M]} &= \left\{ u \in \mathcal{C}_{[L,M]} \ : \ u(x+h) + u(x-h) - 2u(x) \leqslant K|h|^2 \right\} \end{aligned}$$

Then, $\forall u \in \mathcal{C}_{[L,M]}$,

1. $\mathrm{spt}\big(S_T(u) + T \cdot H(0)\big) \subset [-L_T, L_T]^n \qquad (L_T = L + T \cdot \sup_{|p| \leqslant M} |DH(p)|)$
2. $\mathrm{Lip}[S_T(u)] \leqslant M$
3. $S_T(u)$ semiconcave with constant $\frac{1}{\alpha T}$ $(D^2 H \geqslant \alpha I\,,\ \alpha > 0)$

Flow associated with H-L semigroup

$$\boxed{S_T(\mathcal{C}_{[L,M]}) + T\cdot H(0) \subset \mathcal{SC}_{[\frac{1}{\alpha T},L_T,M]}}$$



Compactness estimates for Hopf-Lax semigroup

Theorem (Ancona – C – Khai T. Nguyen, 2015)

- *$\forall L, M, T > 0$ there exist $C_0(L, M, T) > 0$ such that*

$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \Big| \, \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) \leqslant \frac{C_0}{\varepsilon^n}$$

  for all $\varepsilon > 0$ small enough

- *$\forall M, T > 0$ there exist $C_1(M, T), \Lambda(M, T) > 0$ such that*

$$\frac{C_1}{\varepsilon^n} \leqslant \mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \,\Big|\, \mathbf{W}^{1,1}(\mathbb{R}^n)\Big)$$

  for all $L > \Lambda$ and all $\varepsilon > 0$

Outline

Main steps

$$
\begin{aligned}
\mathcal{C}_{[L,M]} &= \{u \in \mathrm{Lip}(\mathbb{R}^n) \,:\, \mathrm{spt}(u) \subset [-L, L]^n \,,\ \mathrm{Lip}[u] \leqslant M\} \\
\mathcal{SC}_{[K,L,M]} &= \{u \in \mathcal{C}_{[L,M]} \,:\, u \text{ semiconcave with constant } K\}
\end{aligned}
$$

- semiconcavity of the Hopf-Lax semigroup

$$S_T\big(\mathcal{C}_{[L,M]}\big) + T \cdot H(0) \subset \mathcal{SC}_{[\frac{1}{\alpha T}, L_T, M]}$$

  where $L_T = L + T \cdot \sup_{|p| \leqslant M} |DH(p)|$

- the gradient of a semiconcave function is a decreasing set-valued map up to an additive linear map
- upper bound for the ε-entropy of semiconcave functions

$$\mathcal{H}_\varepsilon\Big(\mathcal{SC}_{[K,L,M]} \,\big|\, \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) \leqslant \frac{C(K, L, M)}{\varepsilon^n}$$

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Main ideas

1. Controllability type result: introduce a parameterized class $\mathcal{U}$ of smooth function and show that any element of such a class can be attained, at any given time $T > 0$, by the Hopf-Lax flow $S_T(u)$ for a suitable $u \in \mathcal{C}_{[L,M]}$
2. Combinatorial computation: provide an optimal (w.r.t. parameters) estimate of the maximum number of functions in $\mathcal{U}$ that can be contained in a ball of radius 2ε (with respect to the norm of $\mathbf{W}^{1,1}(\mathbb{R}^n)$)

Reachability of semiconcave functions

Theorem

Given $T, L, M > 0$*, let* $K > 0$ *be such that*

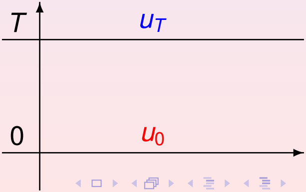
$$K\,T \leqslant \frac{1}{2\alpha_M} \quad \text{where} \quad \alpha_M = \sup_{|p| \leqslant M} \|D^2 H(p)\|$$

Then

$$\mathcal{SC}_{[K,L,M]} - T \cdot H(0) \subset S_T(\mathcal{C}_{[L_T,M]})$$

with $L_T = L + T \cdot \sup_{|p| \leqslant M} |DH(p)|$

Our goal: for any $u_T \in \mathcal{SC}_{[K,L,M]} - T \cdot H(0)$ to find $u_0 \in \mathcal{C}_{[L_T,M]}$ such that $\boxed{S_T(u_0) = u_T}$



Backward construction

Solve the equation backwards: set $\boxed{v(t,x) = S_t(v_0)(x)}$ with

$$v_0(x) = -u_T(-x)$$

$$u(t,x) = -v(T-t,-x) \qquad (t,x) \in [0,T] \times \mathbb{R}^n$$

Then

- $u(T,\cdot) = u_T$
- $u_0 \doteq u(0,\cdot) \in \mathcal{C}_{[L_T,M]}$ by the properties of S_T
- $u_t(t,x) + H\big(\nabla u(t,x)\big) = 0$ for a.e. $(t,x) \in [0,T] \times \mathbb{R}^n$

Therefore,

$$u \text{ viscosity solution} \quad \Longrightarrow \quad u_T = S_T(u_0)$$

The viscosity property follows from the semiconvexity of $v(t,\cdot)$

A semiconvexity result

Lemma

Given $M, T > 0$, let u_0 be semiconvex with constant $-K$ such that

$$K \leqslant \frac{1}{2\alpha_M T} \quad \text{where} \quad \alpha_M := \sup_{|p|\leqslant M} \left\| D^2 H(p) \right\| \quad \text{and} \quad \mathrm{Lip}[u_0] \leqslant M$$

Then

(i) $x \mapsto S_t u_0(x)$ *is semiconvex for all* $t \in [0, T]$ *with a uniform constant*

(ii) $(t, x) \mapsto S_t u_0(x)$ *is a* $\mathcal{C}^1$ *solution of* $u_t + H(\nabla u) = 0$ *on* $(0, T) \times \mathbb{R}^N$

Lower bound for semiconcave functions

Proposition

Given $K, L, M > 0$*, for any* $\varepsilon > 0$

$$\mathcal{H}_\varepsilon\Big(\mathcal{SC}_{[K,L,M]} \mid \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) \geqslant \frac{\Gamma(K,L,M)}{\varepsilon^n}$$

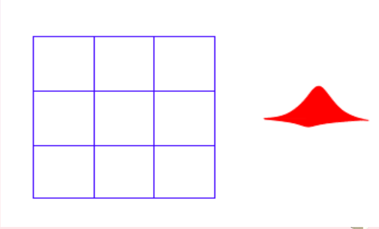
Sketch of the proof ($n = 2$):

Given $N \geqslant 1$ integer, divide $[-L, L]^2$ into N^2 squares of side $\frac{2L}{N}$

$$[-L, L]^2 = \bigcup_{i,j=1,\ldots,N} \square_{ij}$$

Construct bump functions $b_{ij} : \square_{ij} \to \mathbb{R}$ such that

- $\|\nabla b_{ij}\|_{\mathbb{L}^\infty} \leqslant \frac{KL}{12N}\ ,\ \ \|b_{ij}\|_{\mathbb{W}^{1,1}} \leqslant \frac{C}{N^3}$
- ∇b_{ij} Lipschitz with constant K



The class $\mathcal{U}_N$ of smooth functions

Let

$$\Delta_N = \left\{ \delta = (\delta_{ij})_{i,j=1}^{N} \ : \ \delta_{ij} \in \{-1, 1\} \right\}$$

Consider the class of smooth functions

$$\mathcal{U}_N = \left\{ u_\delta = \sum_{i,j=1}^{N} \delta_{ij} \cdot b_{ij} \ : \ \delta \in \Delta_N \right\}$$

Then $\#(\mathcal{U}_N) = 2^{N^2}$. Also, one can show that

- $\mathcal{U}_N \subset \mathcal{SC}_{[K,L,M]}$
- $\|u_{\delta'} - u_\delta\|_{W^{1,1}(\mathbb{R}^2)} \leqslant \varepsilon$ if $\#\{(i,j) \ : \ \delta'_{ij} \neq \delta_{ij}\} \leqslant C_{K,L} N^3 \varepsilon$

Choosing $N \approx \frac{1}{\varepsilon}$, by a combinatorial argument one can show that

$$\mathcal{H}_\varepsilon\Big(\mathcal{U}_N \mid \mathbf{W}^{1,1}(\mathbb{R}^2)\Big) \geqslant \frac{\Gamma}{\varepsilon^2}$$

Therefore,

$$\mathcal{H}_\varepsilon\Big(\mathcal{SC}_{[K,L,M]} \mid \mathbf{W}^{1,1}(\mathbb{R}^2)\Big) \geqslant \frac{\Gamma}{\varepsilon^2}$$

End of the proof

want to show

Let $M > 0$ be fixed. Then, $\forall T > 0$ there exist constants $\Gamma_T > 0$ and $\Lambda_T \geqslant 0$ such that

$$\mathcal{H}_\epsilon\Big(S_T(\mathcal{C}_{[L,M]}) + T \cdot H(0) \mid \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) \geqslant \frac{\Gamma_T}{\varepsilon^n} \qquad \forall L > \Lambda_T\,,\ \forall \varepsilon > 0$$

- Choose $0 < h \leqslant M$ such that $\sup_{\|p\| \leqslant h} \|DH^2(p)\| \leqslant 2 \cdot \|DH^2(0)\|$ and define

$$\Lambda_T = 2T \cdot \sup_{\|p\| \leqslant h} |DH(p)| \quad \text{and} \quad K_T = \frac{1}{4T|D^2H(0)|}$$

- By the reachability of semiconcave functions we have that, $\forall L \geqslant \Lambda_T$,

$$\mathcal{SC}_{\left[K_T, \frac{L}{2}, h\right]} \subset S_T\big(\mathcal{C}_{[L,h]}\big) + T \cdot H(0) \subset S_T\big(\mathcal{C}_{[L,M]}\big) + T \cdot H(0)$$

- Recalling the lower bound for the ε-entropy of semiconcave functions

$$\mathcal{H}_\varepsilon\Big(\mathcal{SC}_{[K,L,M]} \mid \mathbf{W}^{1,1}(\mathbb{R}^n)\Big) \geqslant \frac{\Gamma_T}{\varepsilon^n}$$

Outline

State-dependent Hamilton-Jacobi equations

Previous approach extends to

$$\begin{cases} u_t(t,x) + H\big(x, D_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

Solutions are no longer constant outside a compact subset of $\mathbb{R}^n$!

Assumptions

$$(a) \quad \lim_{|p| \to \infty} \frac{H(x,p)}{|p|} = +\infty$$

$$(b) \quad D_p^2 H(x,p) > 0$$

$$(c) \quad H(x,p) \geqslant -c_1(1 + |x|)$$

$$\begin{aligned} (d) \quad \langle p, D_p H(x,p) \rangle - H(x,p) &\geqslant c_2 |D_p H(x,p)|^{\alpha} - c_3 \\ &\geqslant c_4 |D_x H(x,p)|^{\alpha} - c_5 \qquad (\alpha > 1) \end{aligned}$$

Thank you for your attention!

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$$\begin{cases} u_t(t,x) + H\big(x, D_x u(t,x)\big) = 0 & (t,x) \in (0,T) \times \mathbb{R}^n \\ u(0,x) = u_0(x) & x \in \mathbb{R}^n \end{cases}$$

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&(c) \quad H(x,p) \geqslant -c_1(1+|x|) \\
&(d) \quad \langle p, D_p H(x,p)\rangle - H(x,p) \geqslant c_2 |D_p H(x,p)|^\alpha - c_3 \\
&\qquad\qquad \geqslant c_4 |D_x H(x,p)|^\alpha - c_5 \qquad (\alpha > 1)
\end{aligned}$$

Thank you for your attention!