

Lattice Structures for Dynamics

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Rigorous Numerics in Dynamics Minitutorial - Part 1 of 3
SIAM Conference on Applications of Dynamical Systems 2017

Iterate a map $f : X \rightarrow X$ on a locally compact metric space (not necessarily injective nor surjective).

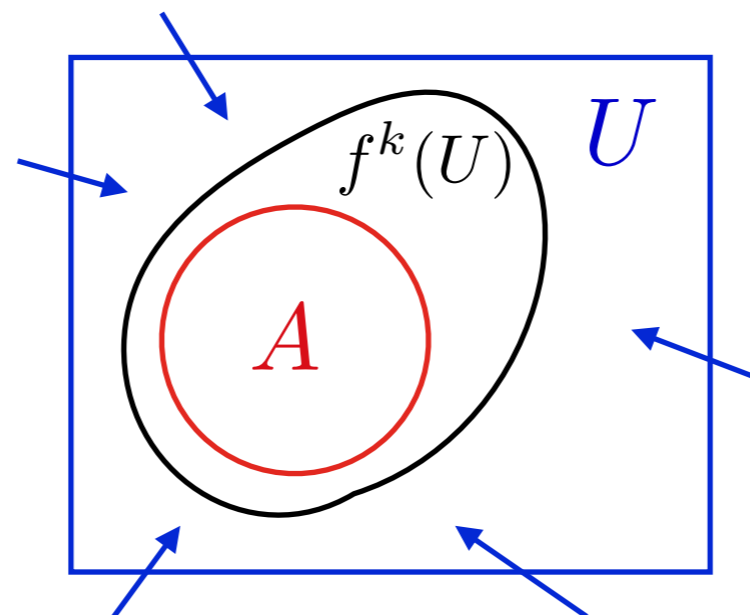
A set $U \subset X$ is an **attracting neighborhood** if there exists $k_0 > 0$ such that $f^k(\text{cl}(U)) \subset \text{int}(U)$ for all $k \geq k_0$.

A **trapping region** is forward invariant, ie. $f(U) \subset U$, attracting neighborhood.

An **attracting block** is a trapping region with $k_0 = 1$.

A set $A \subset X$ is an **attractor** if there exists an attracting neighborhood $A \subset U$ such that $A = \omega(U)$.

(Conley)



Observable dynamics: attractors

Computable dynamics: attracting blocks / trapping regions / attracting neighborhoods

$$V = U, \wedge = \cap \quad \text{ABlock}(X, f)$$

**surjective lattice homomorphism
between distributive lattices**



$$V = U, \wedge = \omega(\cdot \cap \cdot) \quad \text{Att}(X, f)$$

Lattices

A *bounded, distributive lattice* is a set L with the binary operations $\vee, \wedge : L \times L \rightarrow L$ satisfying the following axioms:

- (i) (idempotent) $a \wedge a = a \vee a = a$ for all $a \in L$,
- (ii) (commutative) $a \wedge b = b \wedge a$ and $a \vee b = b \vee a$ for all $a, b \in L$,
- (iii) (associative) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$ and $a \vee (b \vee c) = (a \vee b) \vee c$ for all $a, b, c \in L$,
- (iv) (absorption) $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ for all $a, b \in L$.
- (v) (distributive) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ for all $a, b, c \in L$.
- (vi) (neutral elements) $\exists 0, 1 \in L$ such that $0 \wedge a = 0, 0 \vee a = a, 1 \wedge a = a,$ and $1 \vee a = 1$ for all $a \in L$.

All sublattices contain 0 and 1, and all homomorphisms preserve 0 and 1.

Think sets!

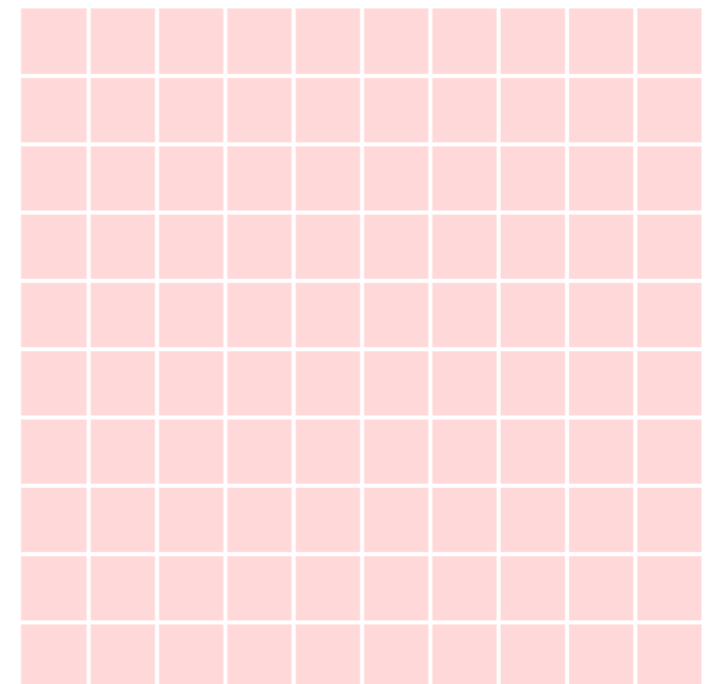
For computations:

- (1) consider a **finite** sublattices of attractors
- (2) **combinatorialize** the phase space

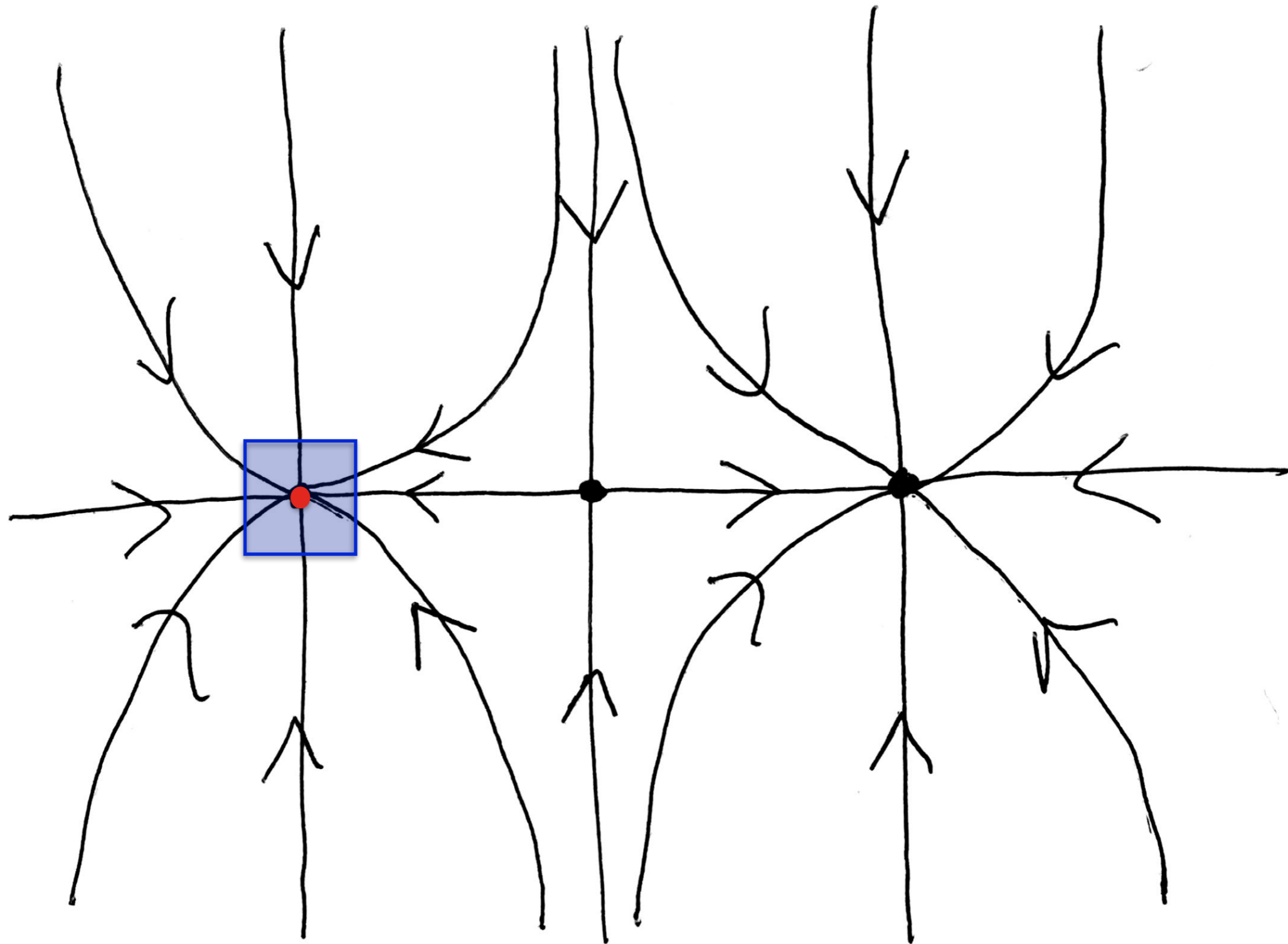
Consider the lattice of regular closed subsets, ie.

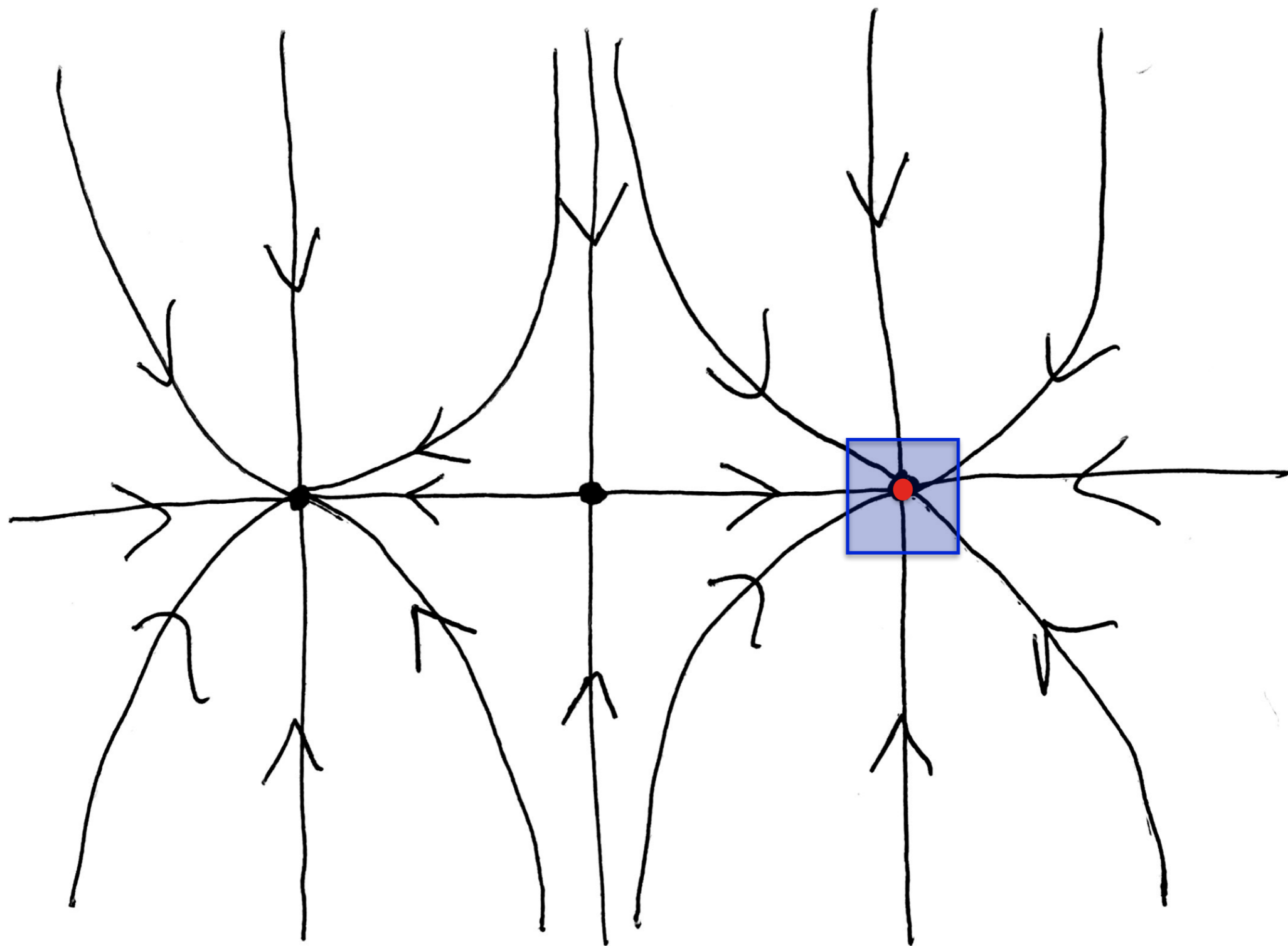
$$S = \text{cl}(\text{int}(S)) \quad \vee = \cup, \wedge = \text{cl}(\text{int}(\cdot \cap \cdot))$$

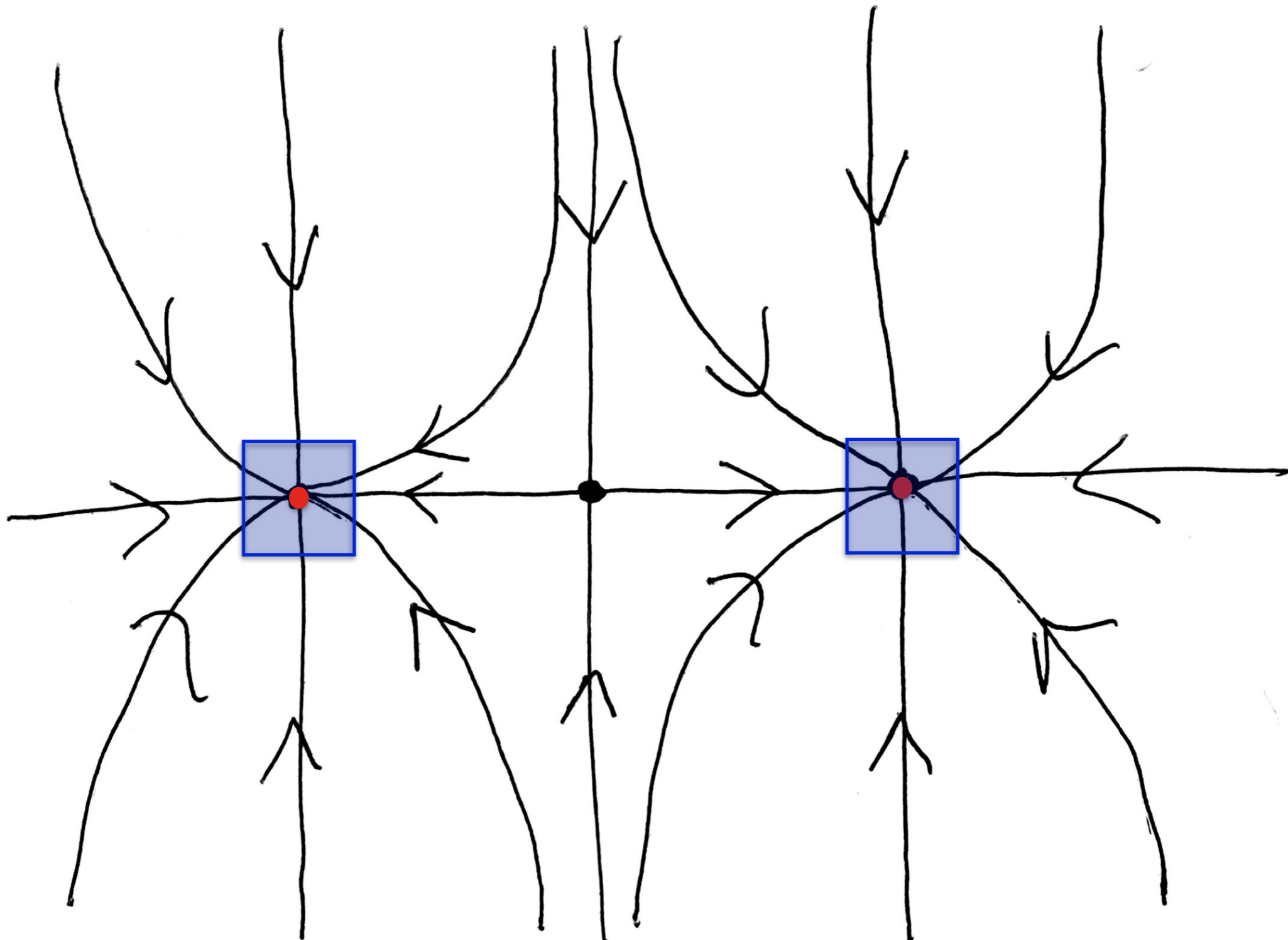
Example: the power set of a finite, full simplicial / cubical complex generates a finite sublattice of regular closed subsets.

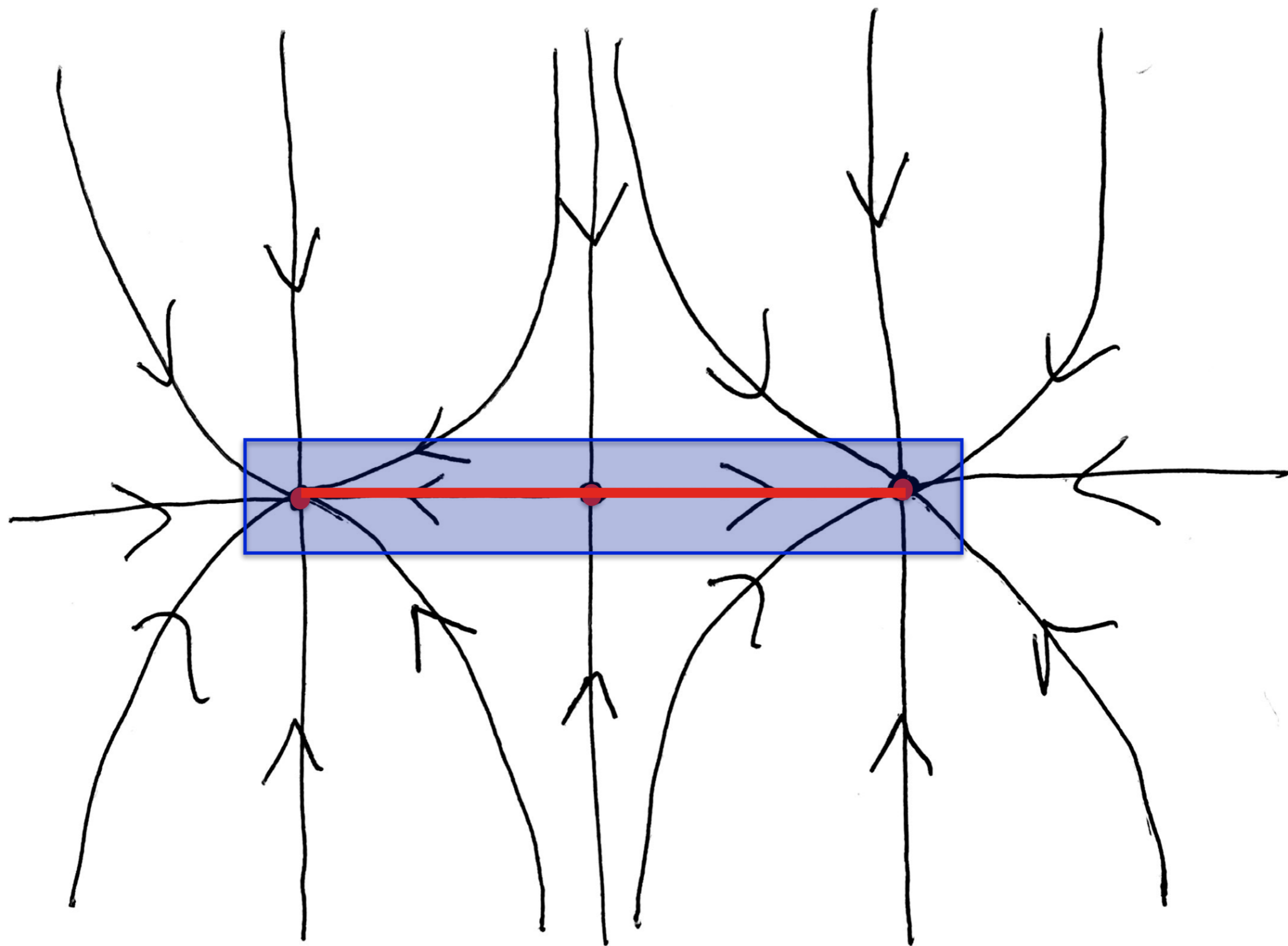


An example: time-T map of $\dot{x} = x - x^3$, $\dot{y} = -y$





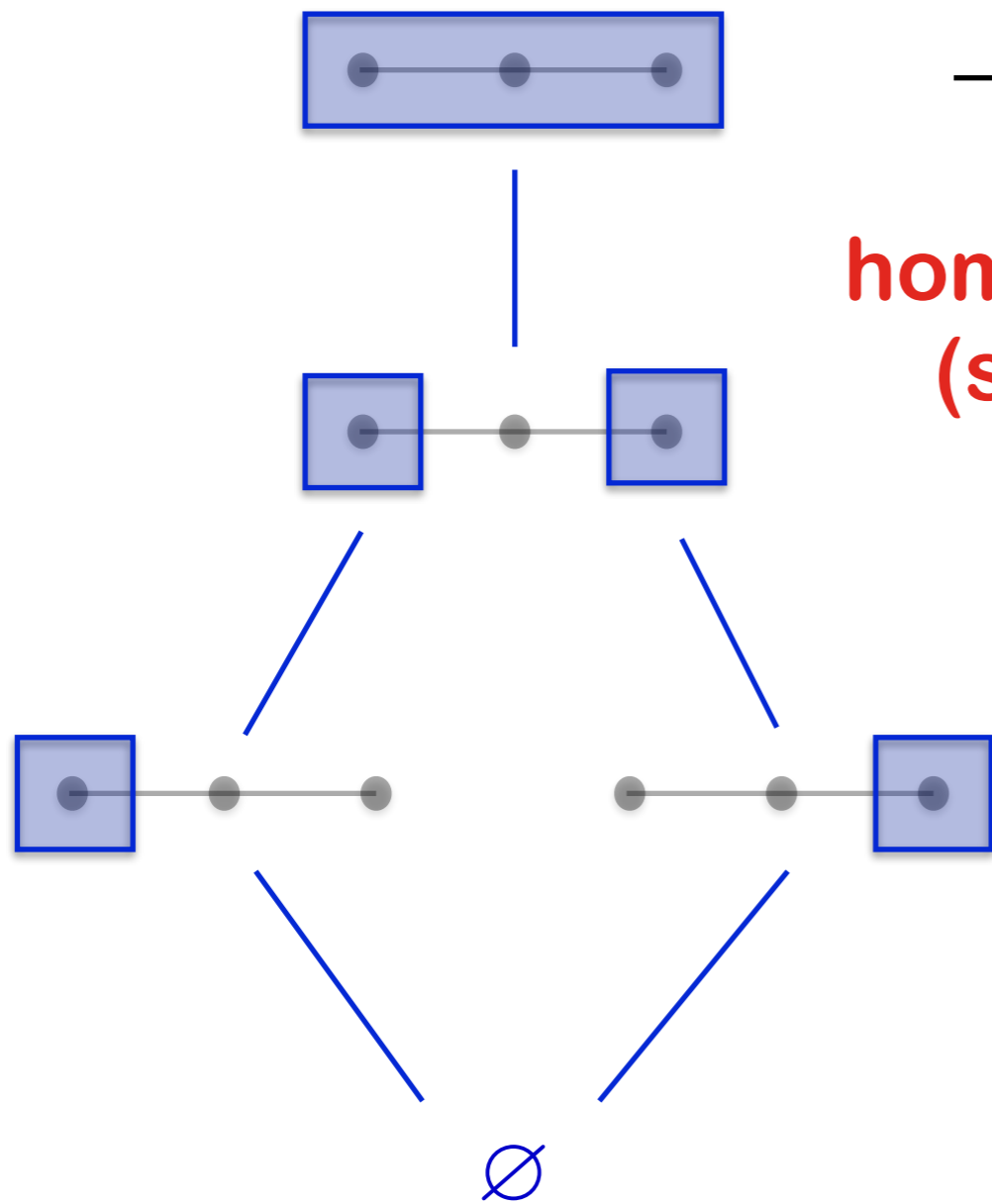




attracting block lattice (regular closed)

$$V = U$$

$$\wedge = \text{cl}(\text{int}(\cdot \cap \cdot))$$

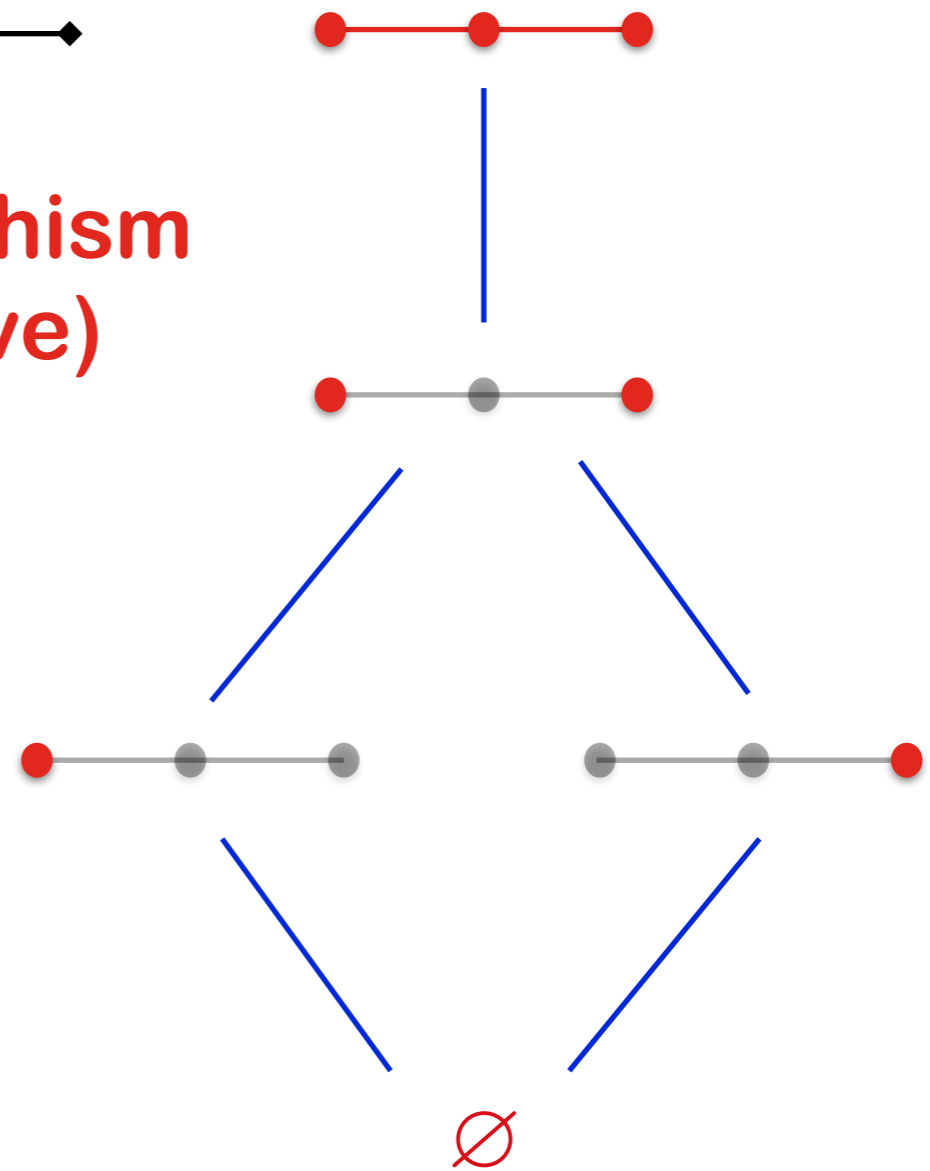


**lattice
homomorphism
(surjective)**

attractor lattice

$$V = U$$

$$\wedge = \omega \cap$$



J - functor

A lattice L has a naturally induced partial order as follows.

Given $a, b \in L$ define

$$a \leq b \iff a \wedge b = a.$$

Given a lattice L , an element $0 \neq c \in L$ is *join-irreducible* if

$$c = a \vee b \text{ implies } c = a \text{ or } c = b \text{ for all } a, b \in L.$$

The set of join-irreducible elements in L is denoted by $J(L)$.

c is join-irreducible iff there exists a unique element $a \in L$ such that $a < c$ and there is no z such that $a < z < c$.

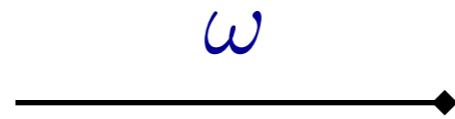
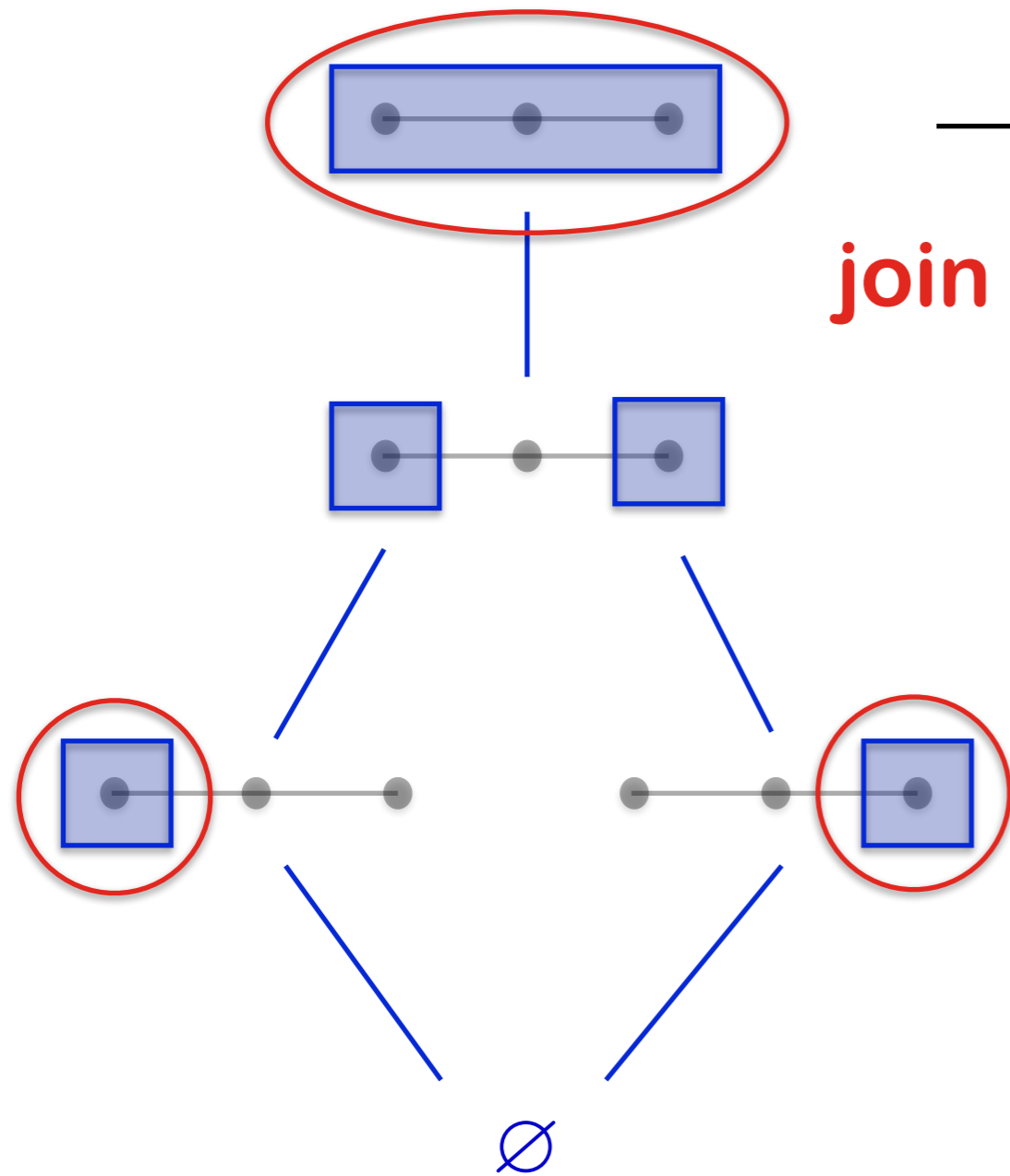
J is a contravariant functor from finite distributive lattices to finite posets.

In all lattices we consider, the order \leq is induced by inclusion.

attracting block lattice

$$V = U$$

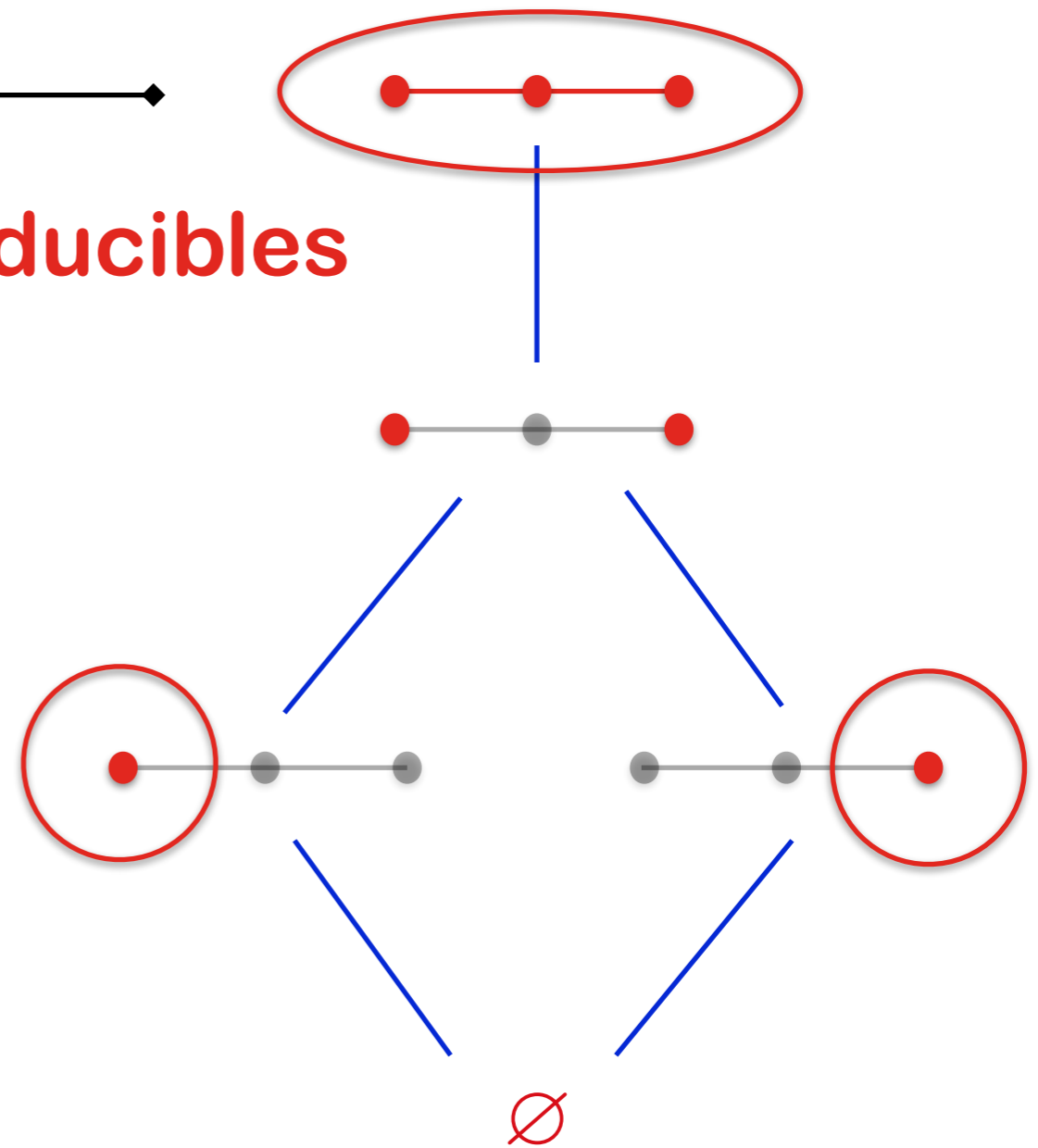
$$\wedge = \text{cl}(\text{int}(\cdot \cap \cdot))$$



attractor lattice

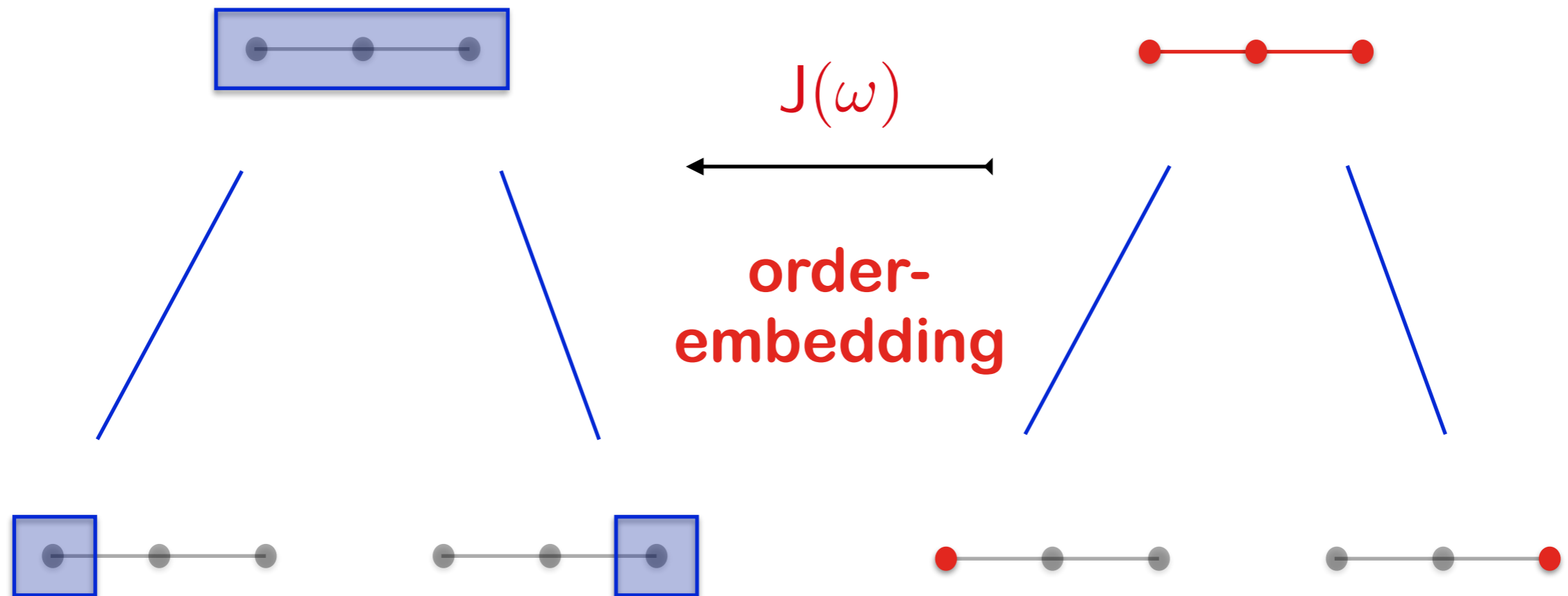
$$V = U$$

$$\wedge = \omega \cap$$



join irreducibles

Birkhoff's Representation Theorem: J induces a duality between finite distributive lattices and finite posets.



Conley form

$$N_A \wedge (\overleftarrow{N_A})^\# = \text{cl}(N_A \cap (\overleftarrow{N_A})^c)$$

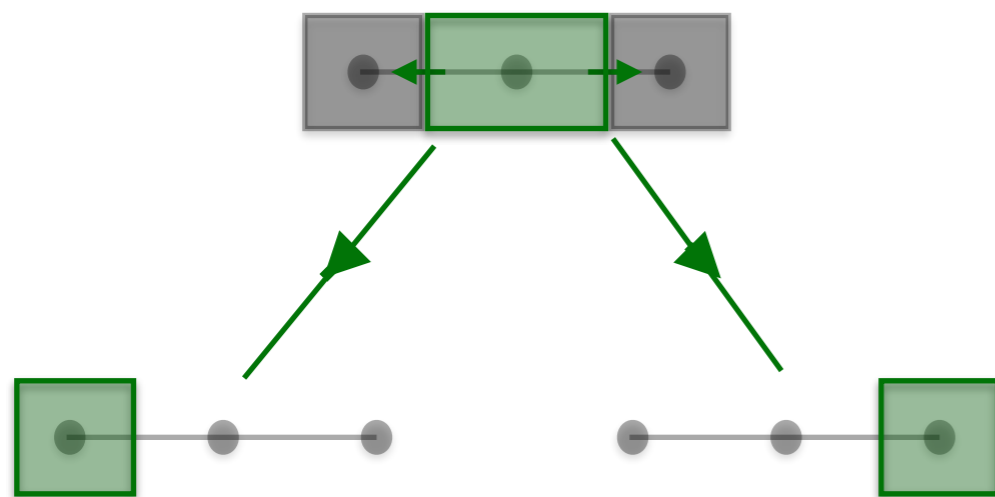
$$A \wedge (\overleftarrow{A})^* = A \cap (\overleftarrow{A})^*$$

poset of isolating neighborhoods

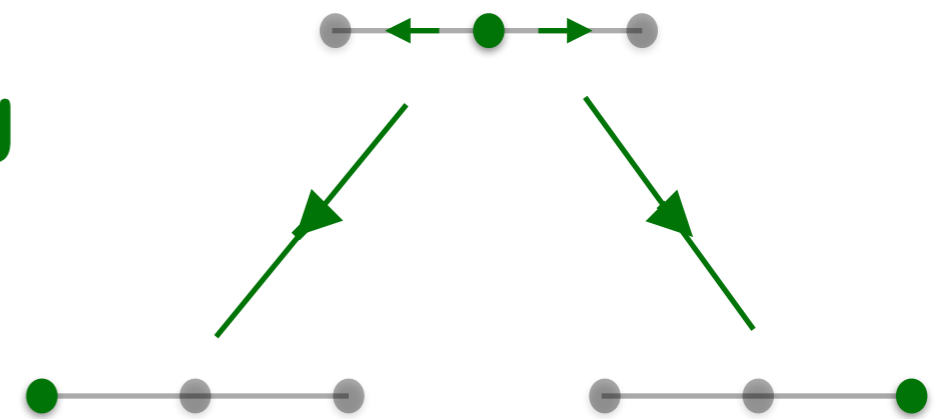
poset of invariant sets

π

order-embedding



$T(N)$

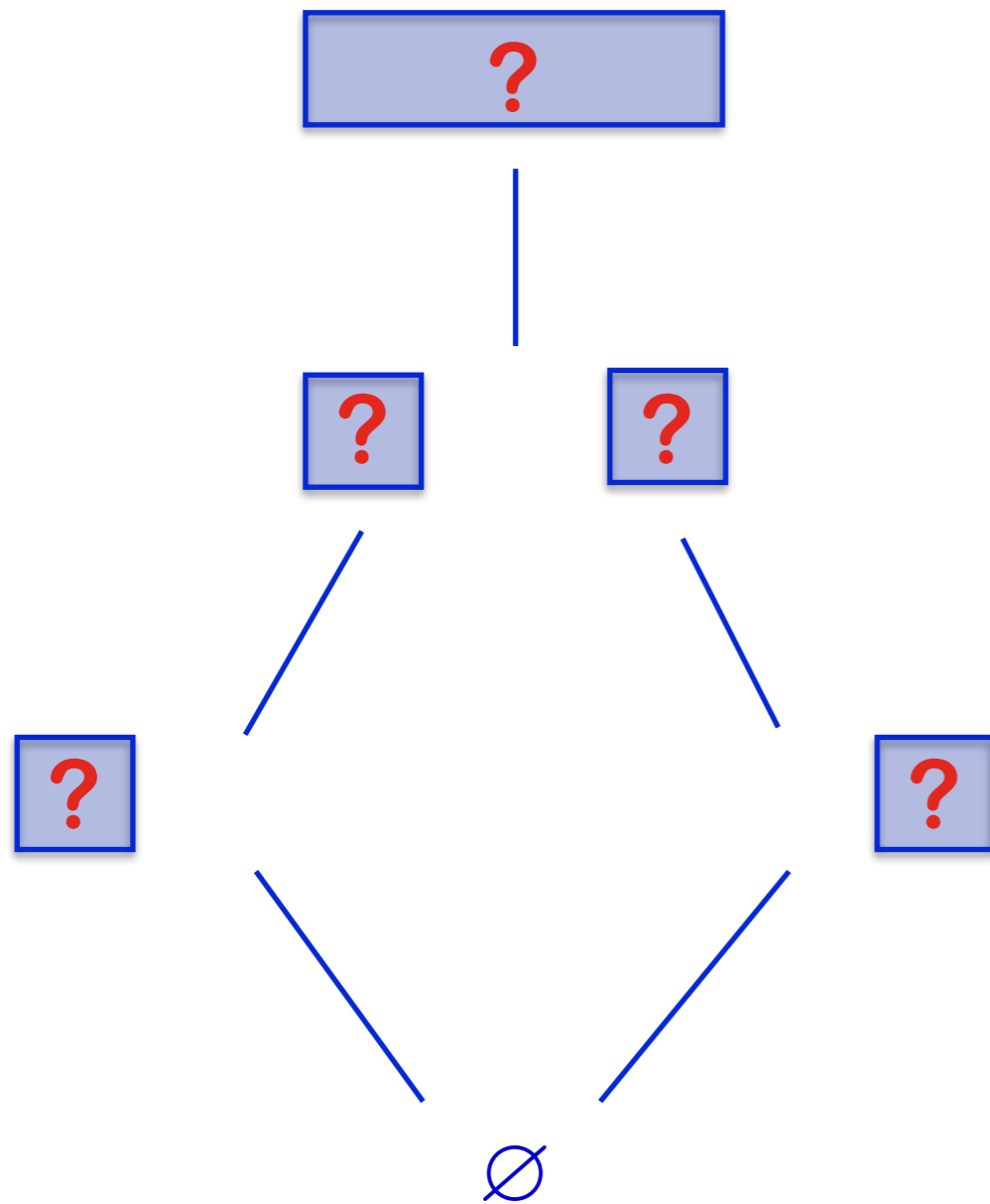


$M(A)$

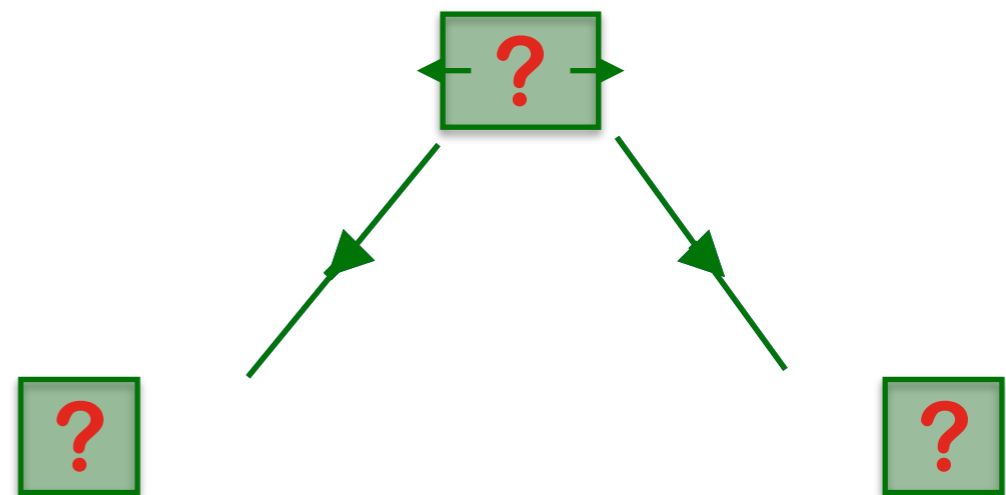
(tessellated) Morse decomposition

$$M(A) \hookrightarrow T(N)$$

attracting block lattice



poset of isolating neighborhoods

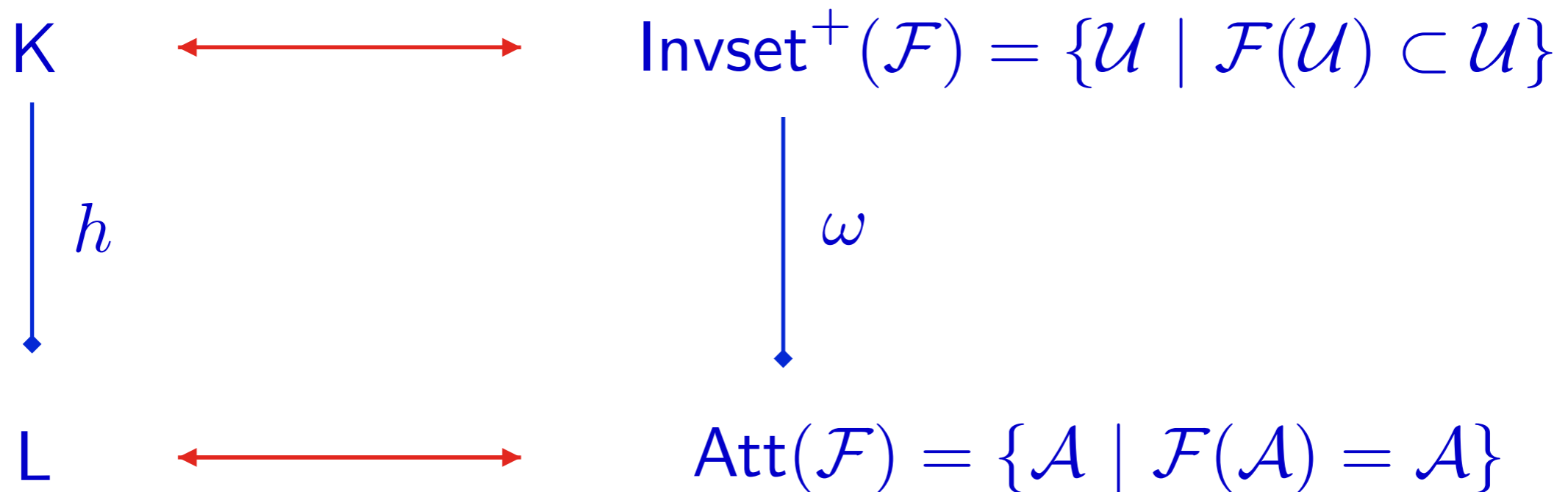


These structures are robust!

EXTENDED Birkhoff's Representation Theorem:

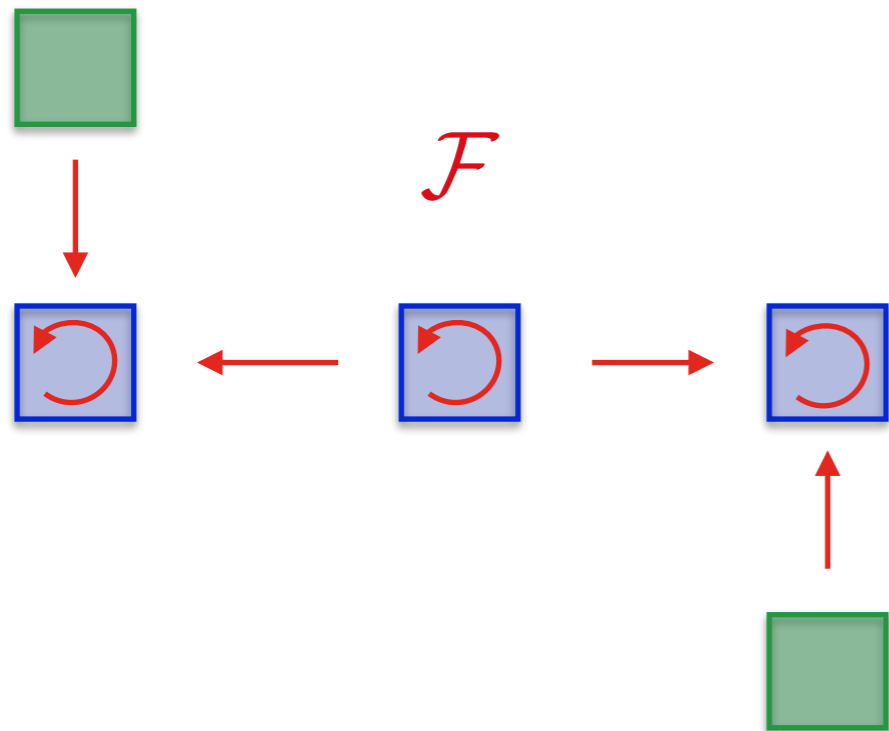
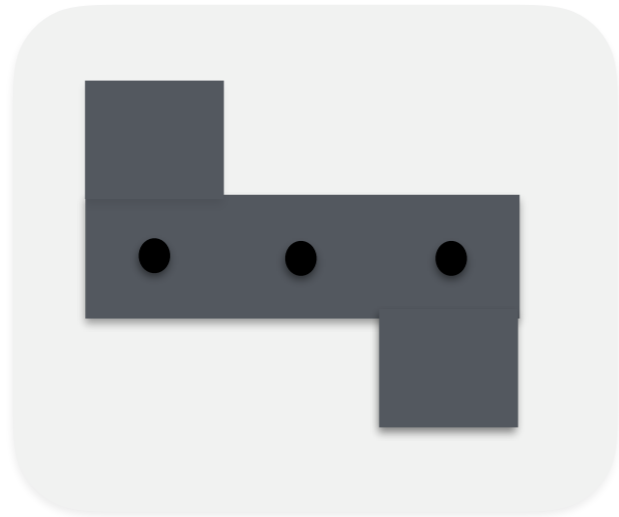
(K., Kasti, Vandervorst)

J induces a duality between surjective lattice homomorphisms on finite distributive lattices and finite binary relations / directed graphs / combinatorial multivalued maps— up to condensation and transitivity.



$$\omega(\mathcal{U}) = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \mathcal{F}^m(\mathcal{U})$$

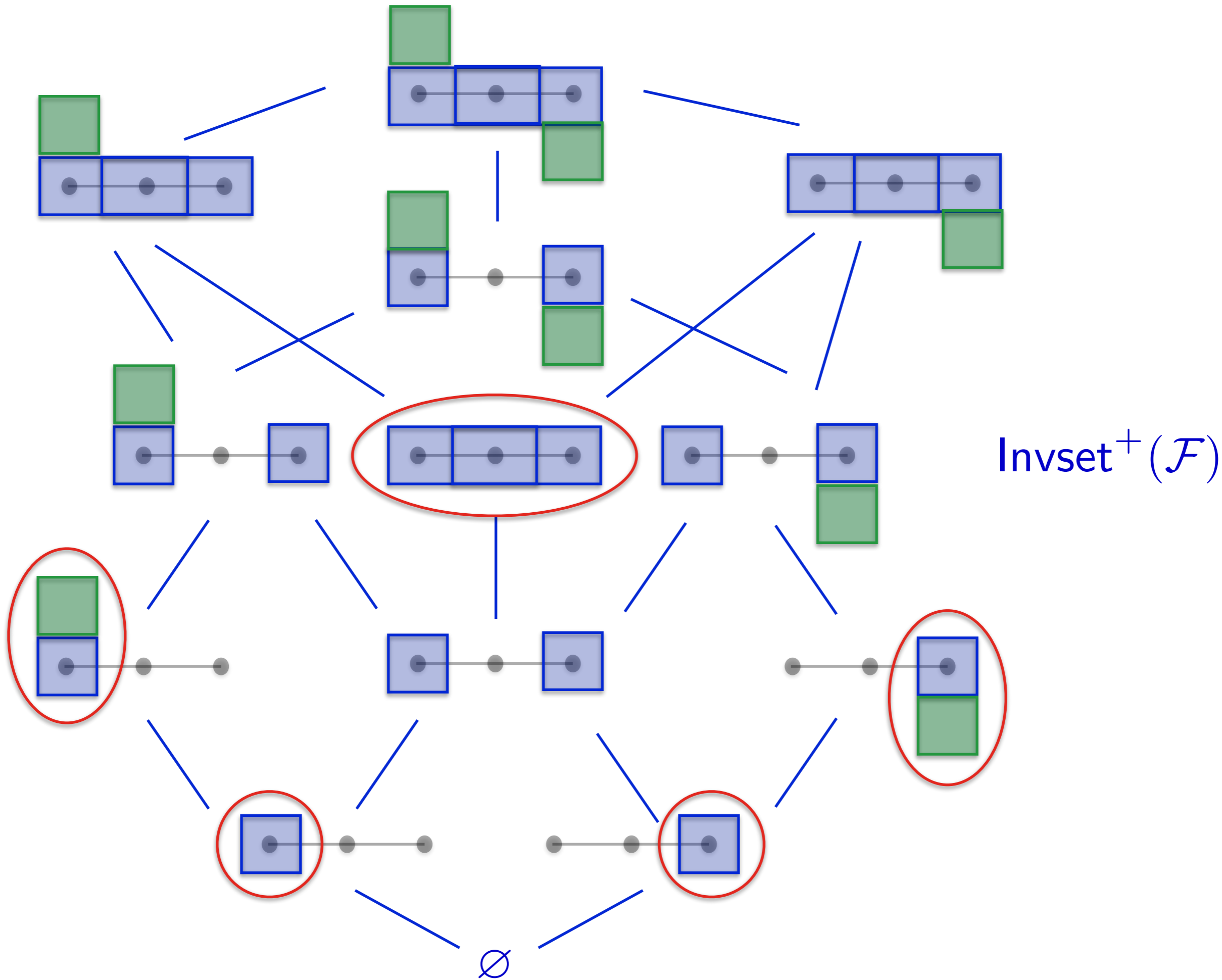
Computationally: try to represent \mathcal{F} as a **state transition graph** on regular closed subsets of X

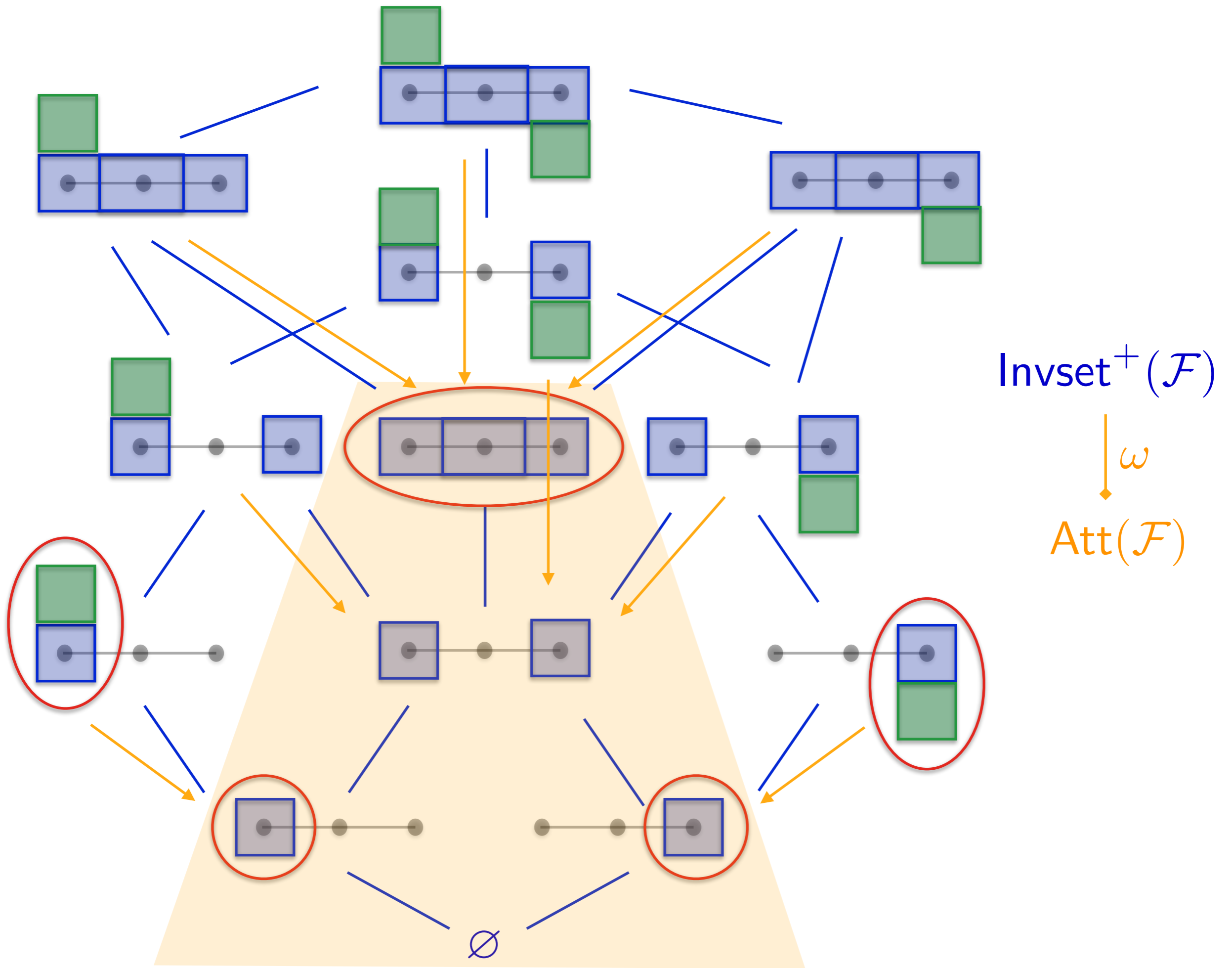


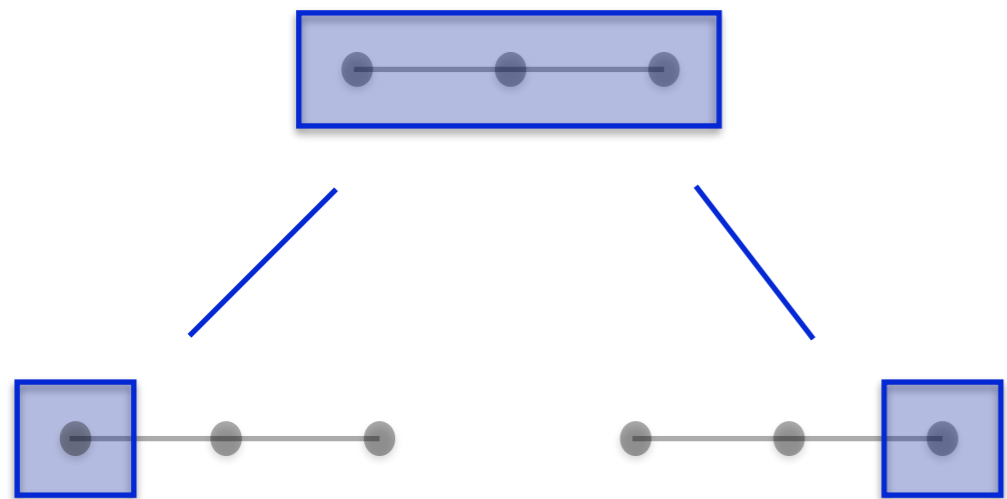
$$\text{Invset}^+(\mathcal{F}) \overset{?}{\longleftrightarrow} N \hookrightarrow \text{ABlock}_R(X, f)$$




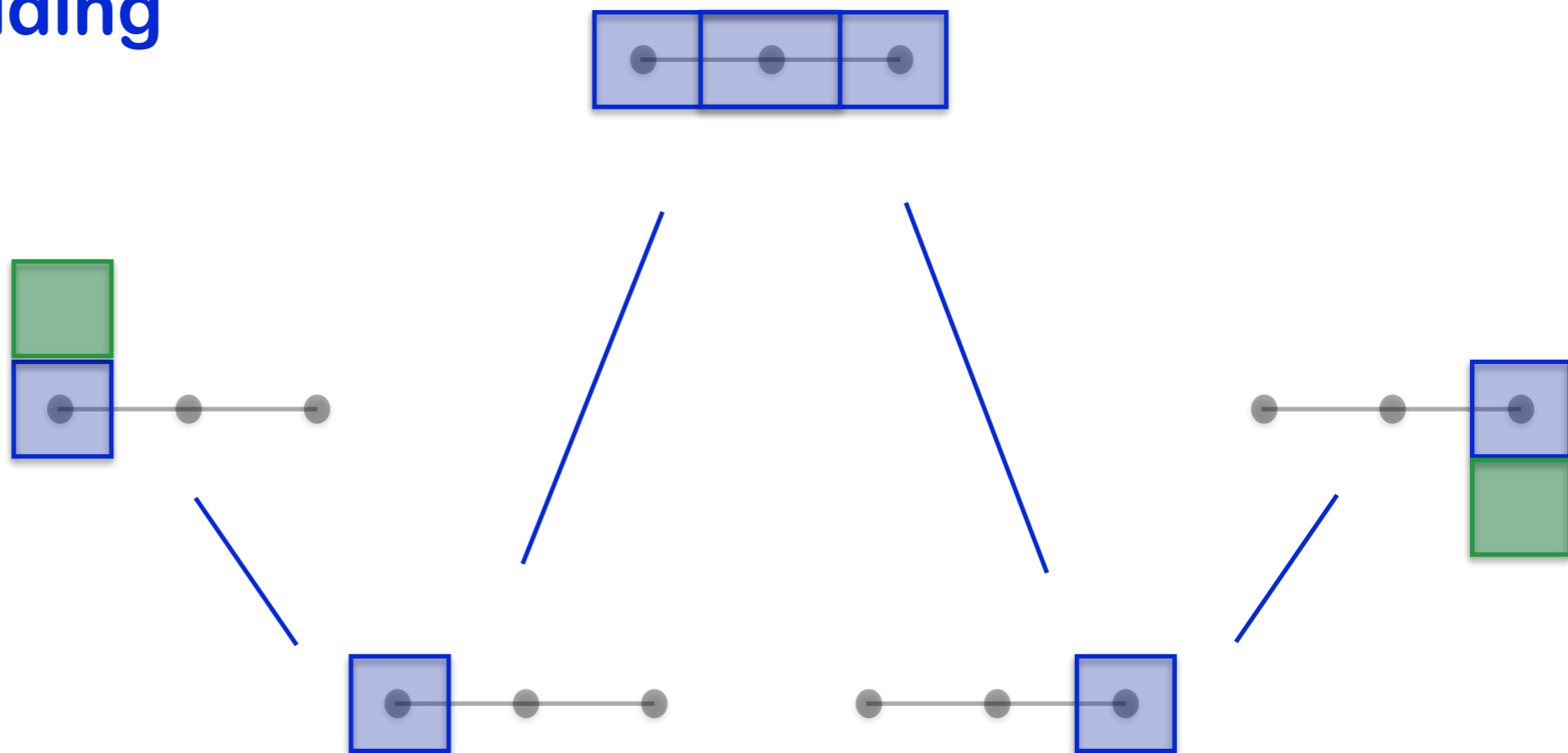
$$\text{Att}(\mathcal{F}) \overset{?}{\longleftrightarrow} A \hookrightarrow \text{Att}(X, f)$$

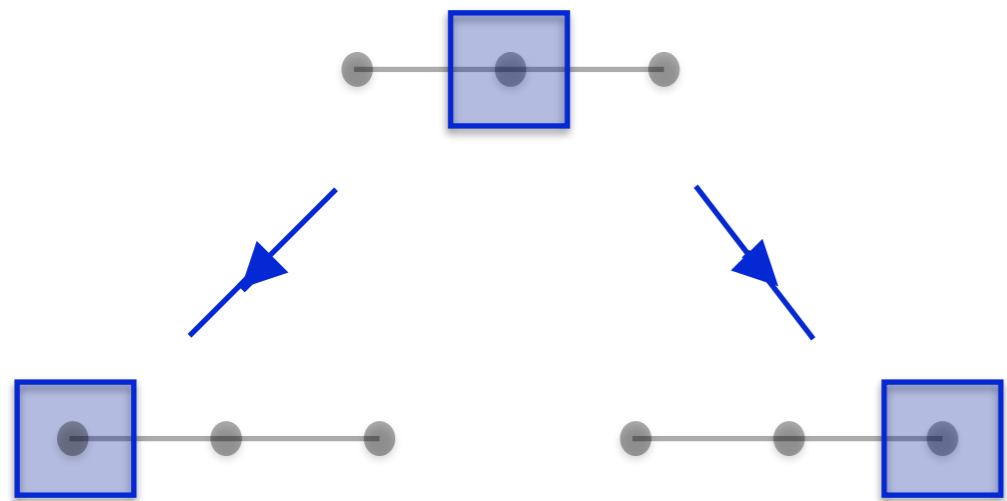




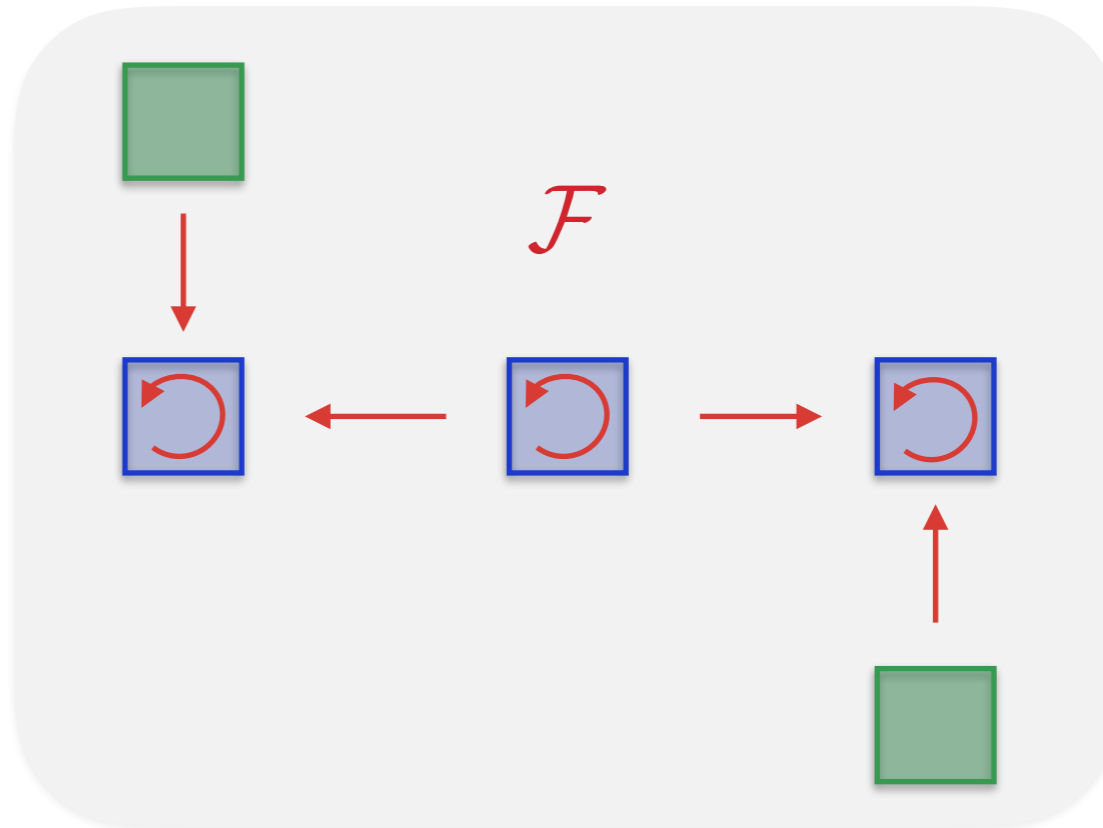


$J(\omega)$

order-embedding

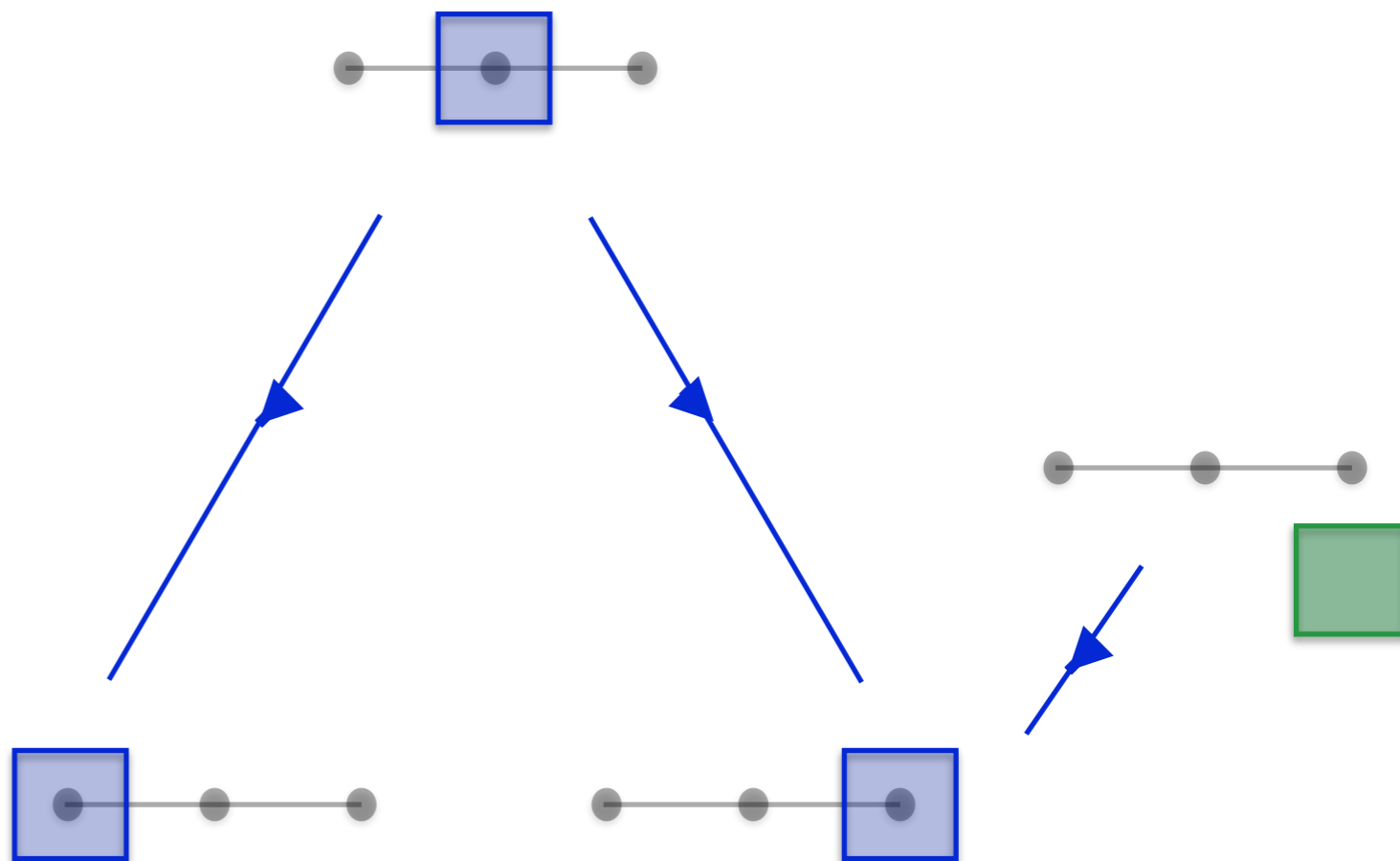




recurrent components



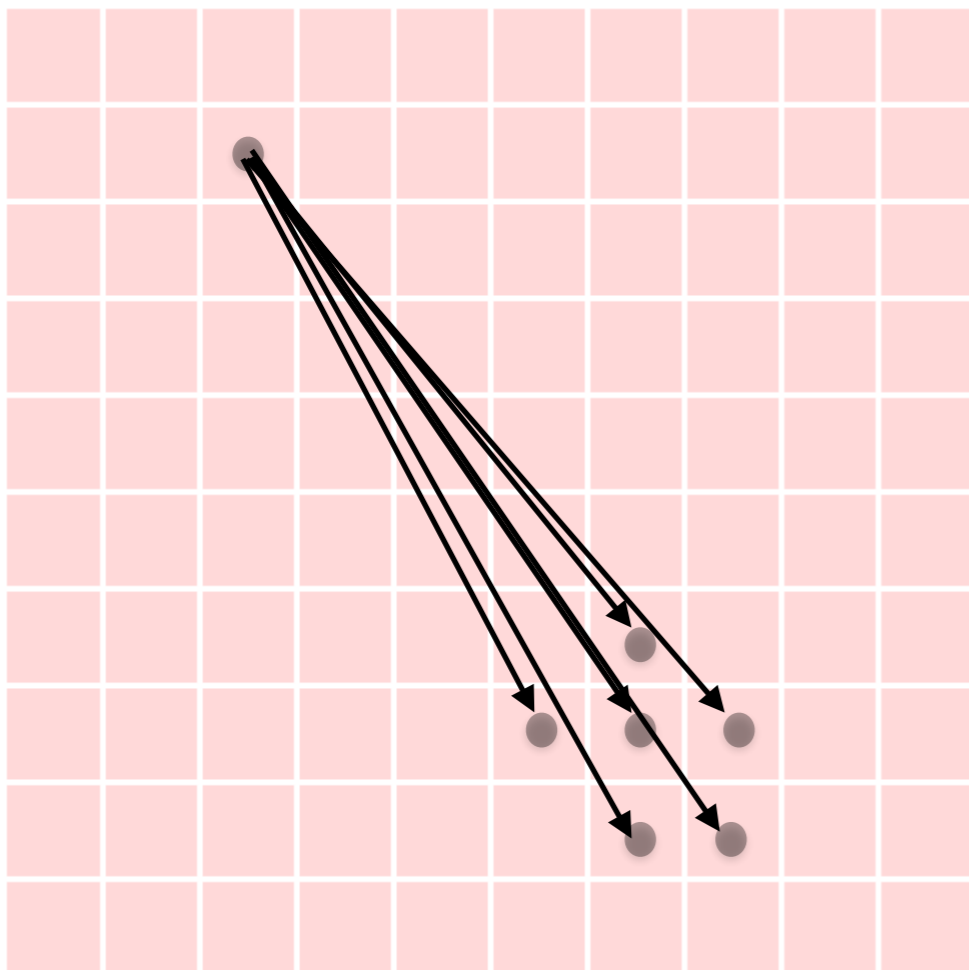
order-embedding
(via Conley form)



strongly connected
components

There are linear time graph algorithms for computing the recurrent and strong components.

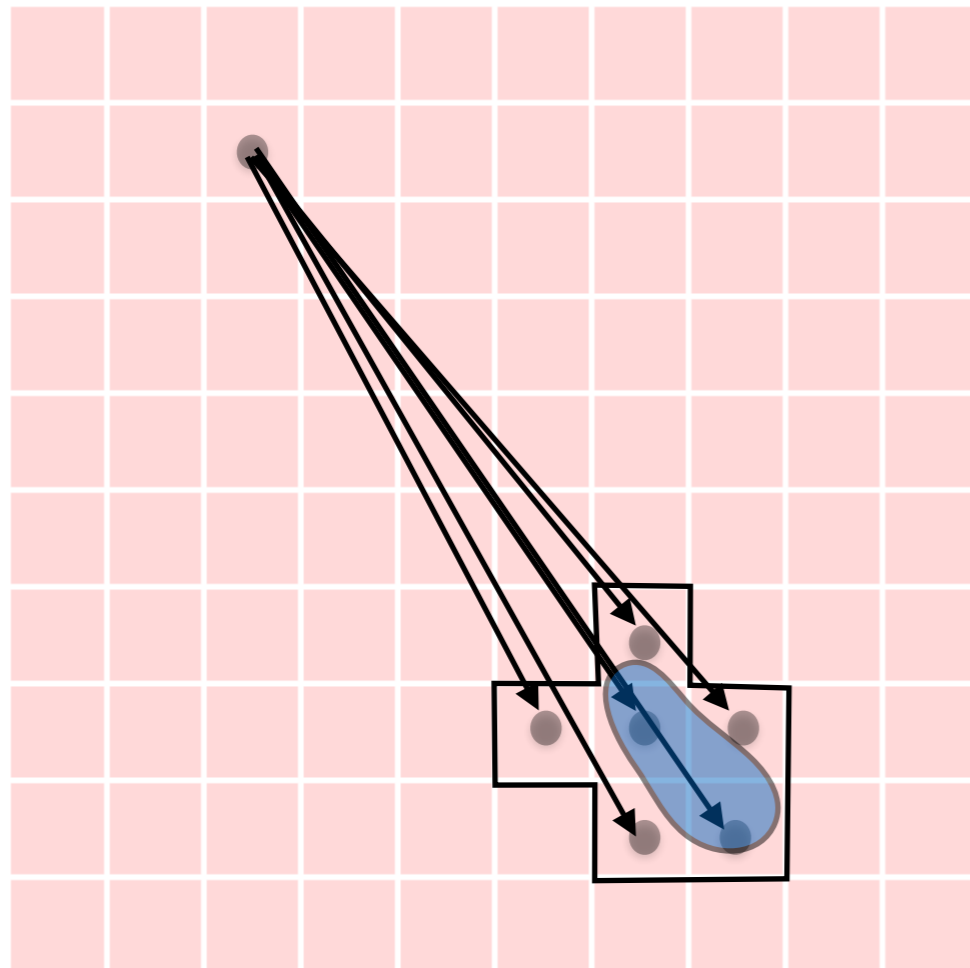
SO — **reverse the question** — if we start with an **appropriate** state transition graph, can we recover a lattice of attracting blocks that is **isomorphic** to a lattice of attractors?



$$\begin{array}{ccc}
 \text{Invset}^+(\mathcal{F}) & \xleftrightarrow[\cup]{\text{natural}} & \mathbb{N} \hookrightarrow \text{ABlock}_R(X, f) \\
 \downarrow \omega & \text{?} \approx & \downarrow \omega \\
 \text{Att}(\mathcal{F}) & \xleftrightarrow{\text{?}} & A \hookrightarrow \text{Att}(X, f)
 \end{array}$$

A state transition graph $\mathcal{F} : \mathcal{X} \rightrightarrows \mathcal{X}$ is an **outer approximation** of $f : X \rightarrow X$ if

$$f(G) \subset \text{int}(|\mathcal{F}(G)|) \quad \forall G \in \mathcal{X}$$



If x_n is an orbit of f ,
then there exists a
walk G_n of \mathcal{F} with

$$x_n \in G_n$$

Hence an outer
approximation does
not mask any
recurrent behavior.

$$\begin{array}{ccc}
\text{Invset}^+(\mathcal{F}) & \xleftrightarrow{\cup} \text{N} \hookrightarrow \text{ABlock}_R(X, f) & \text{SC}(\mathcal{F}) \\
\text{lift} \uparrow & \downarrow \omega \quad ? \approx \downarrow \omega & \text{order retraction} \downarrow \uparrow \\
\text{Att}(\mathcal{F}) & \xrightarrow{\omega \cup} \text{A} \hookrightarrow \text{Att}(X, f) & \text{RC}(\mathcal{F})
\end{array}
\iff$$

Generally N is quite large compared to A . Is A isomorphic to a sublattice of N ?
Such an **index lattice** is equivalent to the existence of a isomorphic tessellated Morse decomposition $M(A) \leftrightarrow T(N)$

Strategy: combine states to obtain a smaller sublattice via an **order retraction**.

For a specific computation an order retraction / lift may not exist.

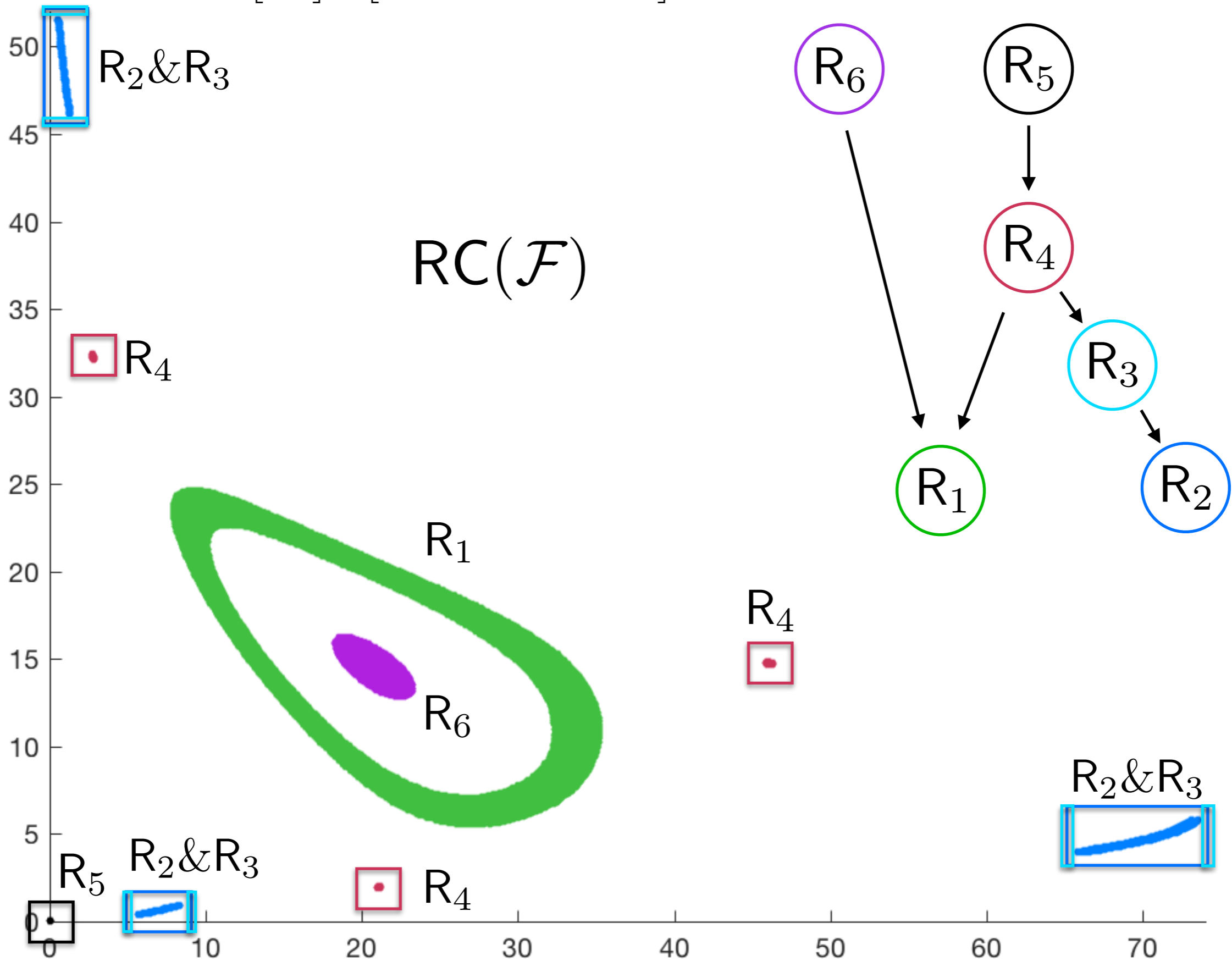
We have developed an algorithm to determine existence and the compute of an order retraction. (K. Kasti, Vandervorst)

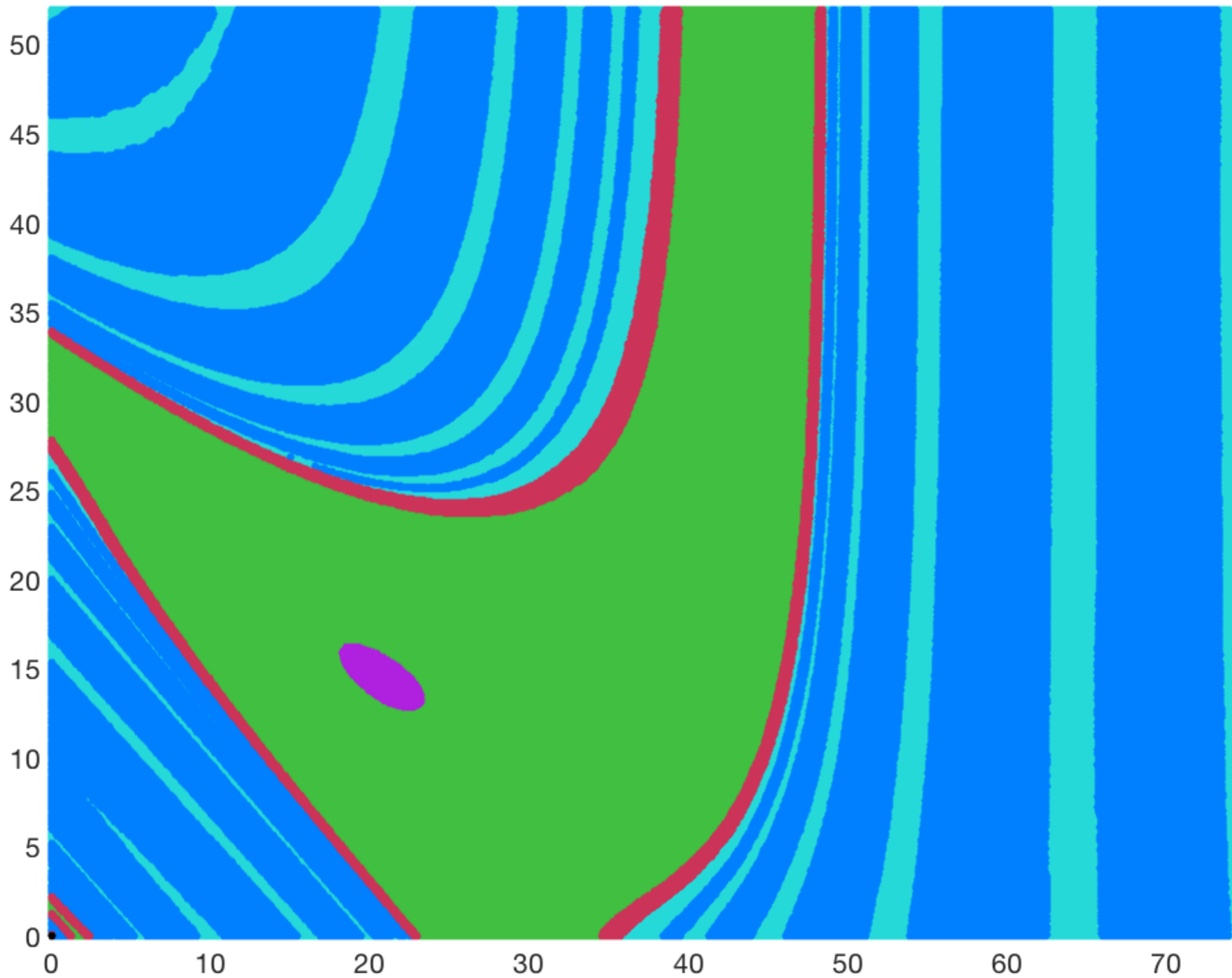
Also, theoretically if the state grid is fine enough and the outer approximation is close enough to f , then an order retraction / lift exists.
(K., Mischaikow, Vandervorst)

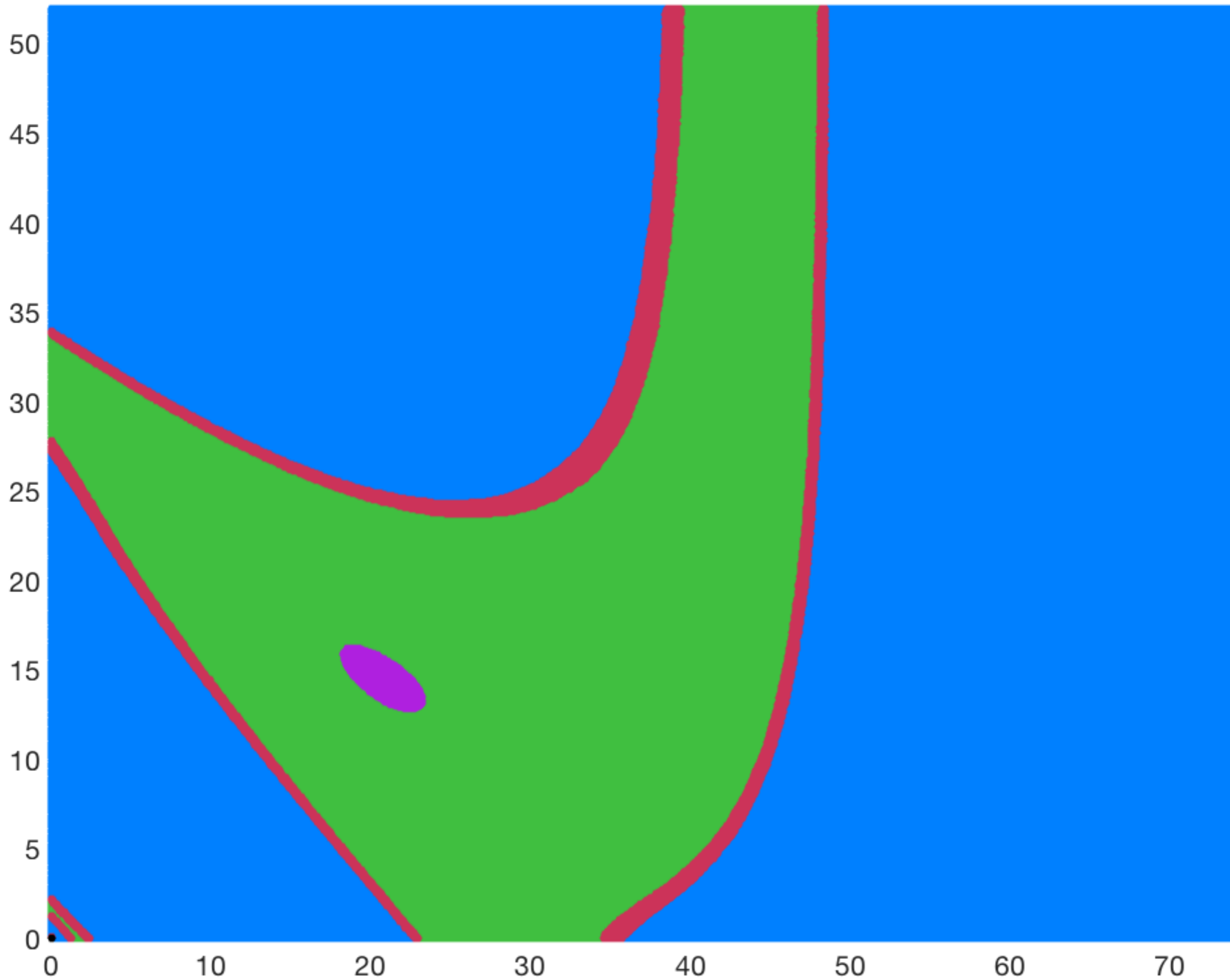
A may not be known! $M(A) \hookrightarrow T(N) \approx \text{RC}(\mathcal{F})$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto \begin{bmatrix} (\theta_1 x_1 + \theta_2 x_2) e^{-\phi(x_1+x_2)} \\ p x_1 \end{bmatrix}$$

$\theta_1 = 20.0, \theta_2 = 20.0, \phi = 0.1, \text{ and } p = 0.7.$







$$\begin{array}{ccc}
\text{Invset}^+(\mathcal{F}) & \xleftrightarrow{\cup} & \mathbf{N} \hookrightarrow \text{ABlock}_R(X, f) & & \text{SC}(\mathcal{F}) \\
\text{lift} \uparrow & & \downarrow \omega & \Leftrightarrow & \text{order retraction} \downarrow \uparrow \\
\text{Att}(\mathcal{F}) & \circlearrowleft & \mathbf{A} \hookrightarrow \text{Att}(X, f) & & \text{RC}(\mathcal{F})
\end{array}$$

Suppose \mathcal{F} is not a outer approximation.

$$\begin{array}{ccc}
\text{Invset}^+(\mathcal{F}_\ell) & \xleftrightarrow{\cup} & \mathbf{N}_\ell \hookrightarrow \text{ABlock}_R(X, f) & & \text{SC}(\mathcal{F}) \\
\updownarrow \ell & & \downarrow \omega & \Leftrightarrow & \downarrow \uparrow \\
\text{Att}(\mathcal{F}) & \longrightarrow & \mathbf{A}_\ell \hookrightarrow \text{Att}(X, f) & & \text{RC}(\mathcal{F})
\end{array}$$

$$M(\mathbf{A}_\ell) \hookrightarrow T(\mathbf{N}_\ell) \approx \text{RC}(\mathcal{F})$$

Example: polygonal grid where vector field is transverse to the boundaries of the grid elements
(Bozcko, K., Mischaikow)

Parabolic recurrence vector fields
(Ghrist, van den Berg, Vandervorst)

Future work

Extract these structures from data?

Computational Conley theory

An algorithmic approach to chain recurrence (FoCM 2005)

**Konstantin Mischaikow
Robert Vandervorst**

An computational approach to Conley's decomposition theorem (JCND 2006)

Hyunju Ban

A database schema for the analysis of global dynamics of multi parameter systems (SIADS 2009)

**Zin Arai, Hiroshi Kokubu, Konstantin Mischaikow,
Hiroe Oka, and Pawel Pilarczyk**

Lattice structures

Lattice structures of attractors I - (J. Comp. Dyn. 2014)

Lattice structures of attractors II - (FoCM 2016)

Lattice structures of attractors III - (in preparation)

Konstantin Mischaikow
Robert Vandervorst

Dynamics and order theory - (in preparation)

Dinesh Kasti
Robert Vandervorst

Efficient computation of Lyapunov functions for Morse decompositions - (DCDS 2015)

Arnaud Goulet, Shaun Harker, Dinesh Kasti,
and Konstantin Mischaikow

Software

CHomp — <http://chomp.rutgers.edu>

Konstantin Mischaikow, Shaun Harker, ...

CDS - Computational Dynamics Software

Kalies

Thank You!

**Konstantin Mischaikow (Rutgers)
Robert Vandervorst (VU Amsterdam)
Dinesh Kasti (FAU)**

