Moreau Sweeping Process on Banach Spaces

Messaoud Bounkhel

King Saud University, Saudi Arabia

SIAM Conference on Applications of Dynamical Systems DS15, Snowbird, Utah, USA, May 20, 2015.

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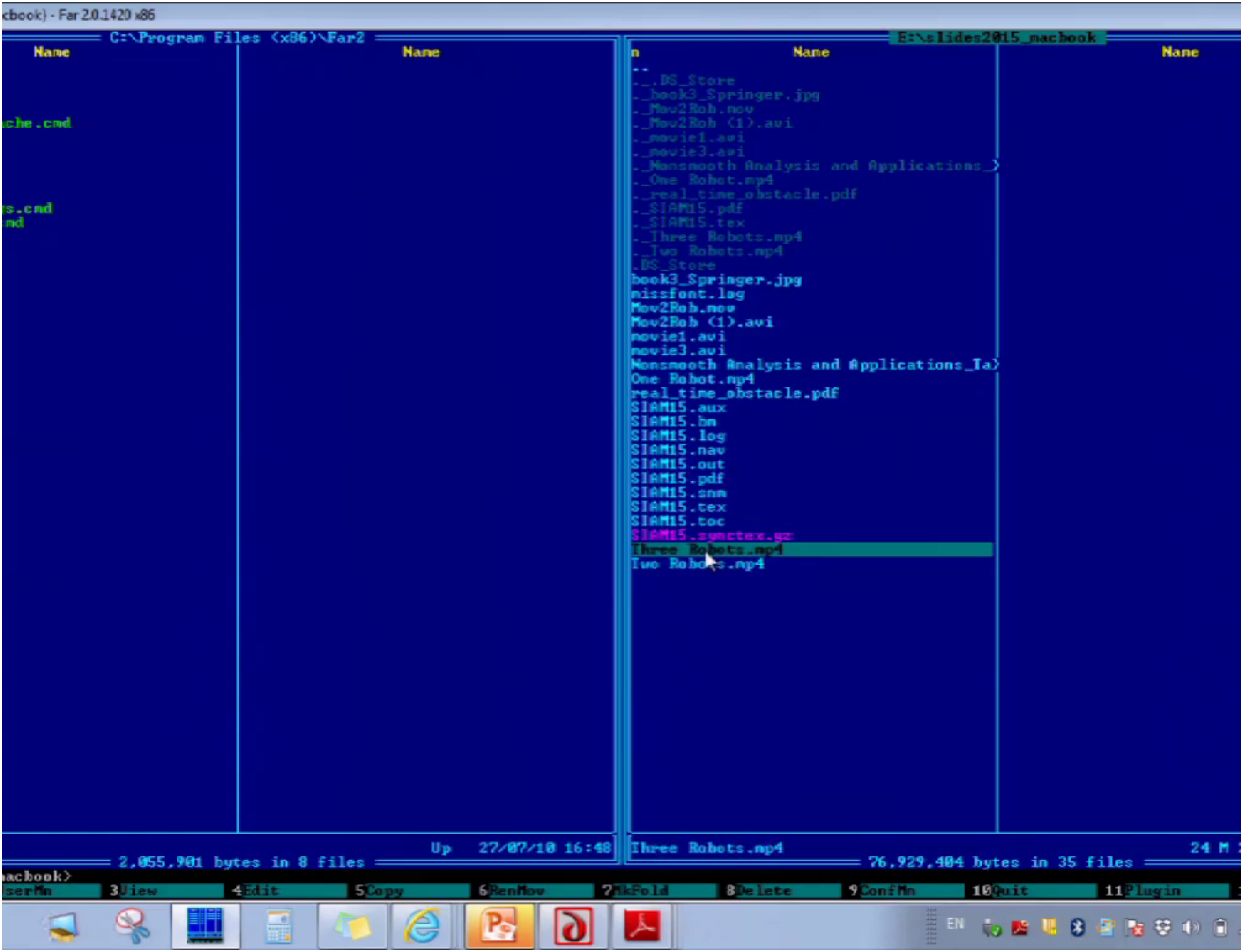
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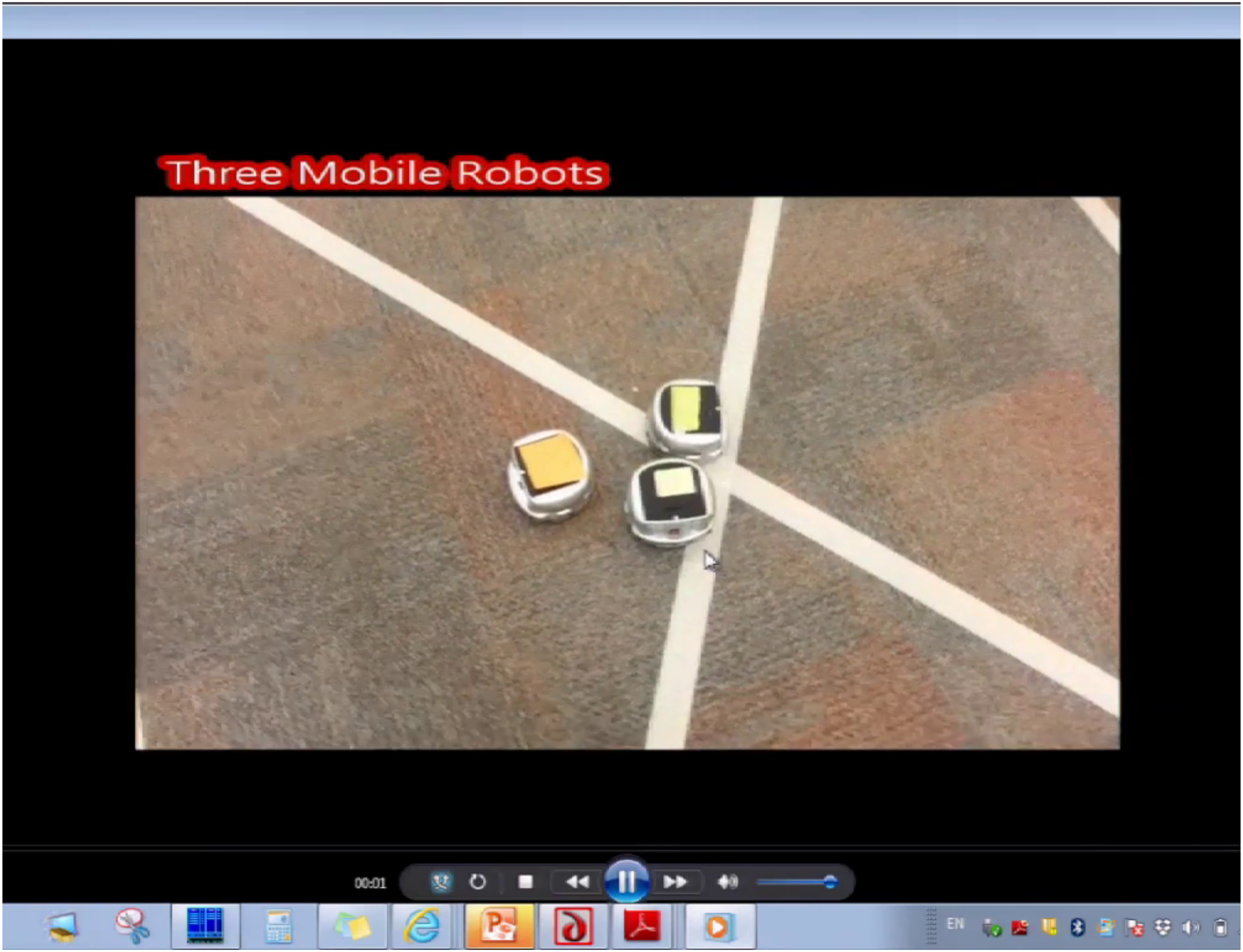
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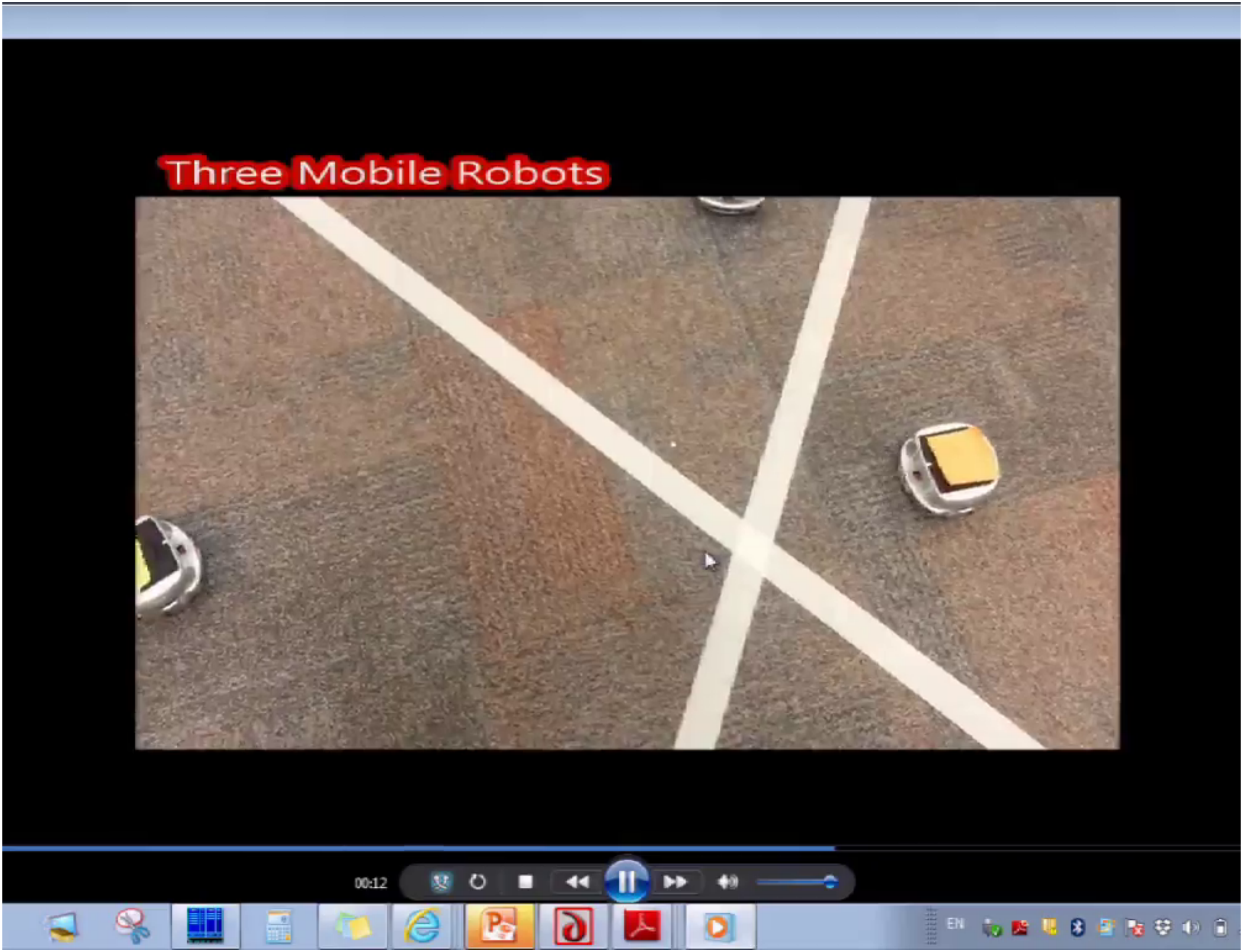
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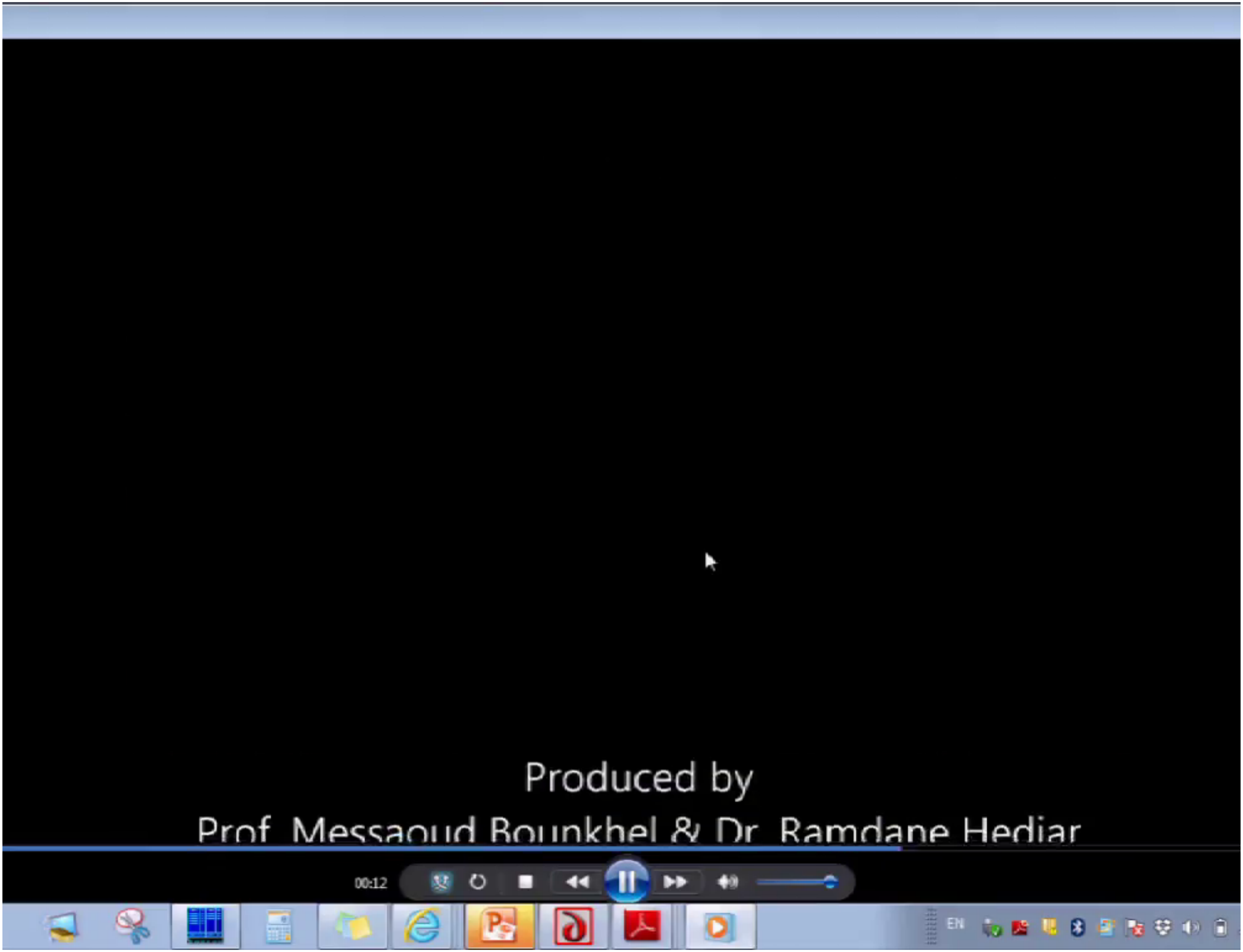
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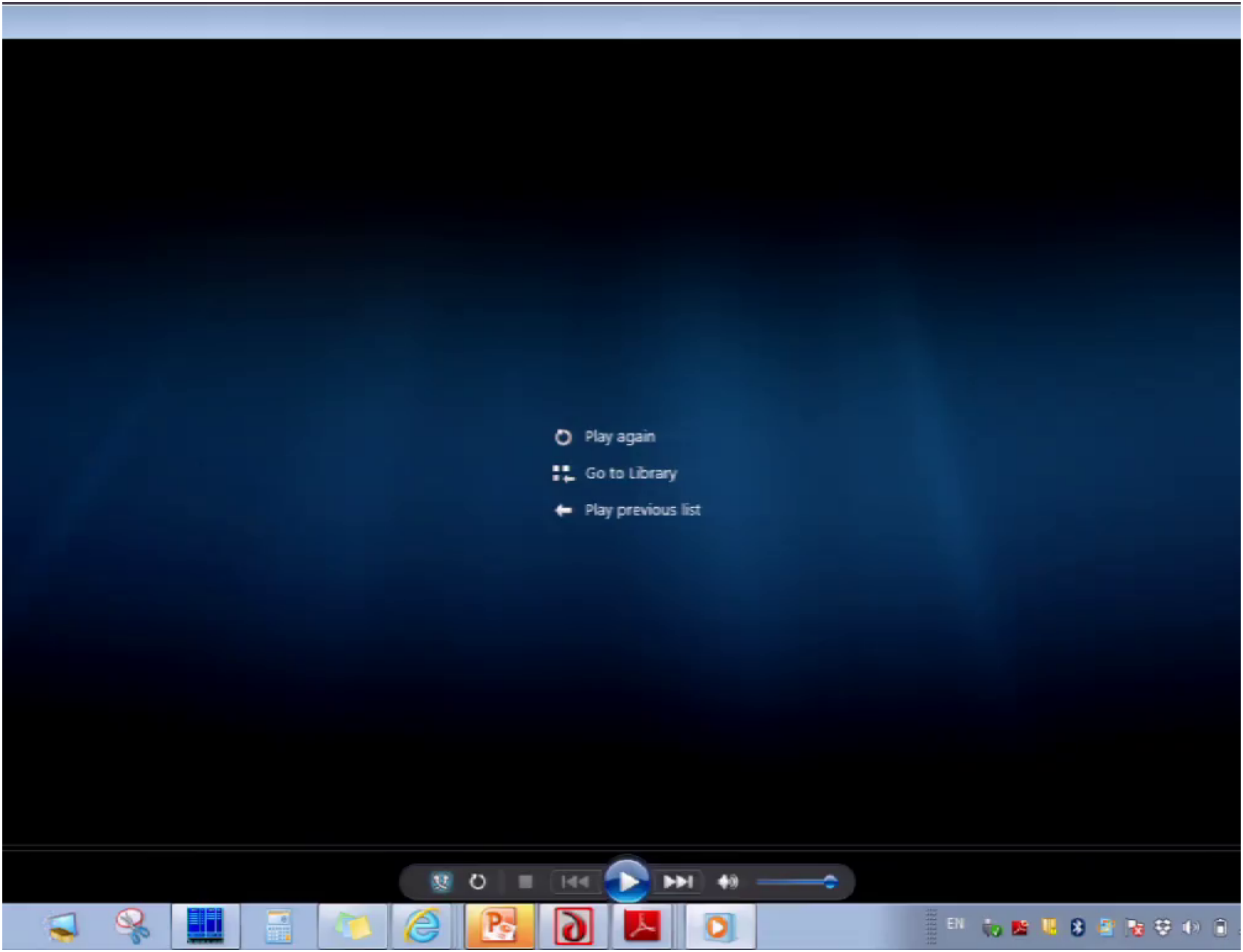
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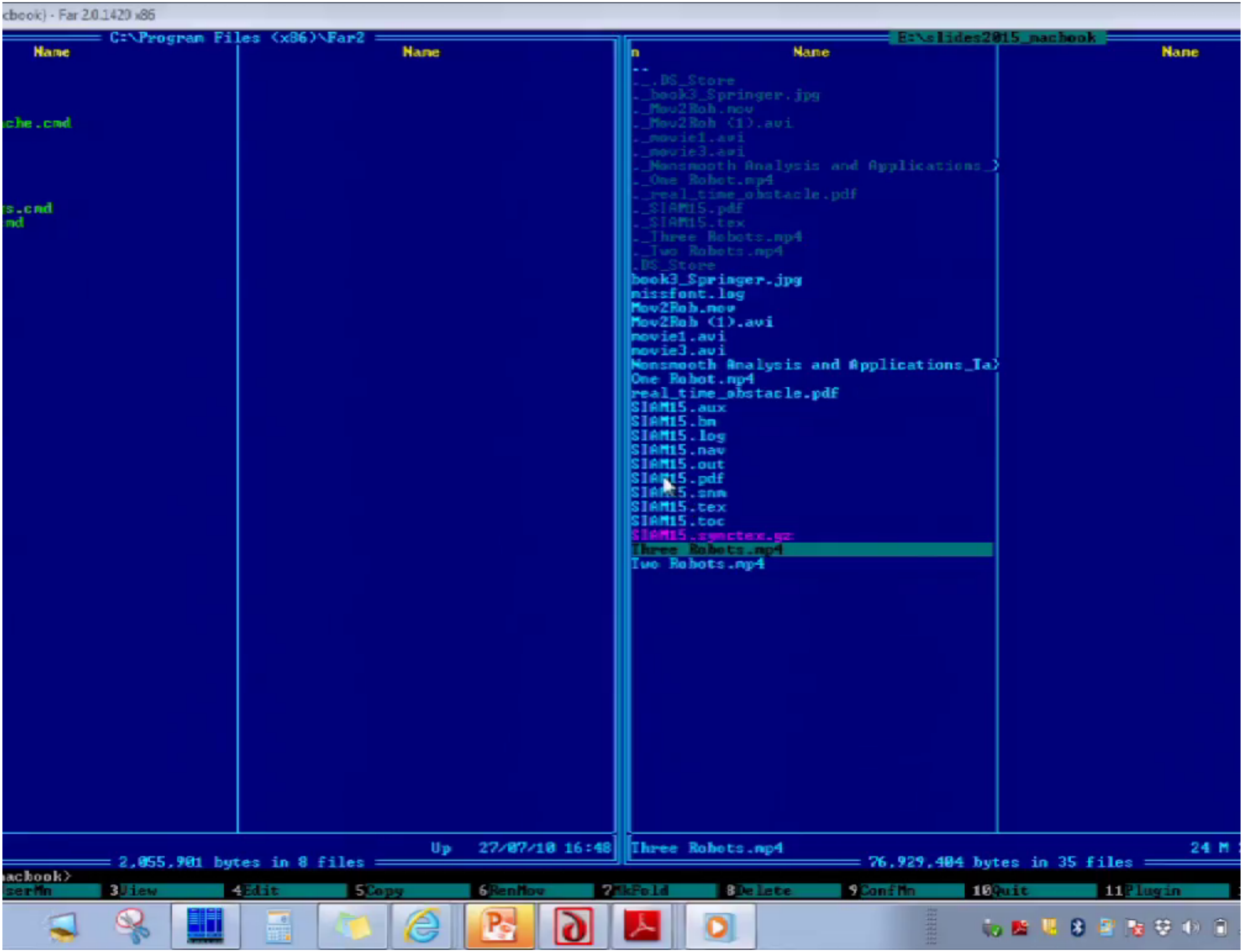

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Introduction

- Sweeping Process: 1971 J.J. Moreau (Hilbert, Convex): Find $x : I \to H$ (I=[0,T]):

$$(SP) \quad \dot{x}(t) \in -N(C(t), x(t)), \text{ a.e. } [0, T], x(0) = x_0 \in C(0).$$

- 1988 Valadier (finite, complement of convex);
- 1993 Castaing, Monteiro-Marques, Duc Ha, and Valadier (finite, complement of convex) with perturbation (SPP):

$$\dot{x}(t) \in -N(C(t), x(t)) + F(t, x(t)), \text{ a.e. } [0, T], x(0) = x_0 \in C(0);$$

- Nonconvex case and finite dimension: Benabdellah 2000, Colombo and Goncharov 1999, Thibault 2003;

- Nonconvex case and infinite: Regular case: (Colombo-Goncharov 1999, Bounkhel and Thibault 2000)
- Banach spaces setting: Benabdellah 2004, Bounkhel and Alyusof 2009, Bernicot-Venel 2010.
- Various extensions: state dependence, differential measure, delay, assumptions on F, ...,etc.
- Second and third order (with and without perturbs.)
- Applications: Thibault *et al.* (electr. pbs), Venel-Maury (crowd motions), Bounkhel-Hedjar (robots), Bounkhel (Nanoparticule Motions).

Motivations

- The main motivation of this direction of research was: the extension of Moreau's result from Hilbert space settings to Banach space settings (even in the convex case).

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Motivations

- The main motivation of this direction of research was: the extension of Moreau's result from Hilbert space settings to Banach space settings (even in the convex case).
- Extensions of some concepts and results in Proximal Analysis from Hilbert spaces to Banach spaces (relationship in Banach spaces between proximal normal cones and the projection operator).

Geometric proximal subdifferential

Let $f : X \to \mathbf{R} \cup \{+\infty\}$ be a l.s.c. at $\bar{x} \in \operatorname{dom} f$. A vector $x^* \in X^*$ is called a generalized proximal subgradient of f at $\bar{x}$ provided that

$$(x^*, -1) \in N^{\pi}(\operatorname{epi} f; (\bar{x}, f(\bar{x}))).$$

The set of all such vectors x^* is denoted by $\partial_G^{\pi} f(\bar{x})$, and called the geometric proximal subdifferential.

Analytic proximal subdifferential

The analytic proximal subdifferential $\partial_A^\pi f(\bar{x})$ of f at $\bar{x}$ is defined by:

$$\{x^* \in X^* : \exists \sigma, \delta > 0, \langle x^*, x - \bar{x} \rangle \leq f(x) - f(\bar{x}) + \sigma V(J(\bar{x}), x), \forall x \in \bar{x} + \delta \mathbf{B}\}.$$

Geometric proximal subdifferential

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Main results

1-Extensions of Existing Results to Banach Spaces

Res. 1: $N^{\pi}(S;\bar{x}) = N^{Conv.}(S;\bar{x})$, if S is convex and X is reflexive smooth Banach spaces. Here $N^{Conv.}(S,\bar{x})$ is the convex normal cone given by:

$$N^{Conv.}(S;\bar{x}) = \{x^* \in X^* : \langle x^*, x - \bar{x}\rangle \leq 0 \text{ for all } x \in S\}.$$

Res. 2: S nonempty closed set in reflexive smooth Banach space X:

$$N^{\pi}(S;\bar{x}) = \partial_G^{\pi}\chi_S(\bar{x})$$

$$x^* \in N^{\pi}(S;\bar{x}) \Leftrightarrow \exists\sigma > 0 : \langle x^*, x - \bar{x}\rangle \leq \sigma V(J(\bar{x}), x), \forall x \in S.$$

$$\Leftrightarrow \exists\delta, \sigma > 0 : \langle x^*, x - \bar{x}\rangle \leq \sigma V(J(\bar{x}), x), \forall x \in S \cap (\bar{x} + \delta \mathbf{B}).$$

$$N^{\pi}(S;\bar{x}) = \partial_A^{\pi}\chi_S(\bar{x})$$

Res. 7: f l.s.c., If f has a local minimum at $\bar{x}$, then $0 \in \partial_A^\pi f(\bar{x})$.

Res. 8: (Density theorem)

Res. 9: Let $p \geq 2, q \in (1,2]$, X be a p-uniformly convex and q-uniformly smooth Banach space and S be a bounded subset of X, then there exist constants $\alpha, \beta > 0$ s.t.

$$\alpha\|x-y\|^p \leq V(J(x), y) \leq \beta\|x-y\|^q, \text{ for all } x, y \in S.$$

Applications to Sweeping Processes

(⋆) First Order Sweeping Processes. (SPP)

$$\begin{cases} \dot{x}(t) \in -N(C(t); x(t)) + F(t, x(t)) \quad \text{a.e. } I, \\ x(t) \in C(t), \forall t \in I, x(0) = x_0 \in C(0). \end{cases}$$

(⋆) Second Order Sweeping Processes. (SSPMP)

$$\begin{cases} \ddot{x}(t) \in -N(C(x(t)); \dot{x}(t)) + F(t, x(t), \dot{x}(t)) + G(t, x(t), \dot{x}(t)) \text{ a.e. } I, \\ \dot{x}(t) \in C(x(t)), \forall t \in I, x(0) = x_0, \dot{x}(0) = u_0 \in C(x_0). \end{cases}$$

(⋆) Variants of Sweeping Processes.

$$(\bullet)\begin{cases} \dot{x}(t) \in -N(C(t);\dot{x}(t)) + F(t,x(t)) \quad \text{a.e. } I, \\ \dot{x}(t) \in C(t), \forall t \in I, x(0) = x_0, \dot{x}(0) = u_0 \in C(0). \end{cases}$$

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$$(\bullet)\begin{cases} \dot{x}(t) \in -N(C(t,x(t));x(t)) + F(t,x(t)) \quad \text{a.e. } I, \\ x(t) \in C(t,x(t)), \forall t \in I, x(0) = x_0 \in C(0,x_0). \end{cases}$$

Extensions to Banach spaces

($\star$) First Order Sweeping Processes. (SPP)

$$\begin{cases} \dot{x}(t) \in -N(C(t); J^*(x(t))) + F(t, J^*(x(t))) \quad \text{a.e. } I, \\ J^*(x(t)) \in C(t), \forall t \in I, J^*(x(0)) \in C(0). \end{cases}$$

($\star$) Second Order Sweeping Processes. (SSPMP)

$$\begin{cases} \ddot{x}(t) \in -N(C(J^*(x(t))); J^*(\dot{x}(t))) + F(t, x(t), \dot{x}(t)) + G(t, x(t), \dot{x}(t)) \text{ a.e. } I, \\ x(0) = x_0, \dot{x}(0) = u_0, J^*(\dot{x}(t)) \in C(J^*(x(t))), \forall t \in I. \end{cases}$$

($\star$) Variants of Sweeping Processes.

$$(\bullet)\begin{cases} \dot{x}(t) \in -N(C(t); J^*(\dot{x}(t)))) + F(t, J^*(x(t))) & \text{a.e. } I, \\ x(0) = x_0, J^*(\dot{x}(0)) \in C(0). \end{cases}$$

$$(\bullet)\begin{cases} x(t) \in -N(C(t); J^*(\dot{x}(t))) + F(t, J^*(x(t))) & \text{a.e. } I, \\ x(0) = x_0, J^*(\dot{x}(0)) \in C(0). \end{cases}$$

$$(\bullet)\begin{cases} \dot{x}(t) \in -N(C(t, J^*(x(t))); J^*(x(t))) + F(t, J^*(x(t))) & \text{a.e. } I, \\ x(0) = x_0 \in C(0, x_0). \end{cases}$$

We prove the existence of solutions for:

$$(SPP)\begin{cases} \dot{x}(t) \in -N(C(t); J^*(x(t))) + F(t, J^*(x(t))) & \text{a.e. } I \\ x(0) \in x_0, J^*(x(t)) \in C(t), \forall t \in I. \end{cases}$$

We recall some results needed in the proof.

- (•) $N(S; \bar{x}) \cap \mathbf{B}_* = \partial d_S(\bar{x})$.
- (•) Let X be a Banach space and $C : I \rightrightarrows X$ ($I = [0, T]$) be a continuous set-valued mapping with nonempty, closed and convex values. The following closedness property of the subdifferential of the distance function holds: "for any $\bar{t} \in I, \bar{x} \in C(\bar{t}), t_n \to \bar{t}$ with $t_n \in I$, $x_n \to \bar{x}$ (x_n not necessarily in $C(t_n)$), and $\varphi_n \to^w \bar{\varphi}$ with $\varphi_n \in \partial d_{C(t_n)}(x_n)$, one has $\bar{\varphi} \in \partial d_{C(\bar{t})}(\bar{x})$ ".

(•) Let X be a reflexive Banach space with dual space X^* and S be a nonempty, closed and convex subset of X. The following properties hold:

- $\pi_S(\varphi) \neq \emptyset$, for any $\varphi \in X^*$;
- $\pi_S(\varphi)$ is singleton for all $\varphi \in X^*$ if and only if X is strictly convex.

Theorem 1. Let $p \geq 2, q \in (1,2]$, X be a separable p-uniformly convex and q-uniformly smooth Banach space, and let $C : I \rightrightarrows X$ be a set-valued mapping with nonempty closed convex values satisfying for any $\varphi, \psi \in X^*$ and $t, t' \in I$

$$|\left(d^V_{C(t)}\right)^{1/q'}(\psi) - \left(d^V_{C(t')}\right)^{1/q'}(\varphi)| \leq \lambda|t' - t| + \gamma\|\varphi - \psi\|, \quad (1)$$

where $q' = \frac{q}{q-1}$, and $\lambda, \gamma > 0$.

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where $q' = \frac{q}{q-1}$, and $\lambda, \gamma > 0$. Assume that $J(C(t)) \subset K$ for any $t \in I$, for some convex compact set $K \subset X^*$. Let $F : I \times X \rightrightarrows X^*$ be an u.s.c. set-valued mapping with convex compact values in X^* such that $F(t,x) \subset L$ for all $(t,x) \in I \times X$, for some convex compact set $L \subset X^*$. Then, for any $x_0 \in C(0)$, there exists a solution of (SPP).

$$d_S^V(x^*) = \inf\{V(x^*, s) : s \in S\}.$$

In Hilbert space:

$$d_S^V(x^*) = \inf\{V(x^*, s) : s \in S\} = \inf\{\|x^* - s\|^2 : s \in S\} = d_S^2(x^*).$$

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$$(\bullet) \qquad x \in S \Longleftrightarrow d_S^V(J(x)) = 0.$$

Sketch of the proof

- Consider for every $n \in N$, the following partition of I:
  $t_{n,i} := i\frac{T}{2^n}$ $(0 \leq i \leq 2^n)$ and $I_{n,i+1} :=]t_{n,i}, t_{n,i+1}]$ if $0 \leq i \leq 2^n - 1$ and $I_{n,0} := 0$.

  Put $\mu_n := \frac{T}{2^n}$.

- For every $n \geq 0$, we define by induction :

$$u_{n,0} := x_0, \quad z^*_{n,0} \in F(t_{n,0}, u_{n,0}),$$

$$z^*_{n,i} \in F(t_{n,i}, u_{n,i}),$$

$$u_{n,i+1} := \pi_{C(t_{n,i+1})}\left(J(u_{n,i}) + \mu_n z^*_{n,i}\right). \tag{2}$$

The equality (2) is well defined?

⋆ For every $n \geq 0$, these points $(u_{n,i})_{(0 \leq i \leq 2^n)}$ and $(z^*_{n,i})_{(0 \leq i \leq 2^n)}$ are used to construct two mappings z^*_n and u^*_n from I to X^* by defining their restrictions to each interval $I_{n,i}$ as follows:

(•) For $t = 0$, set $z^*_n(t) := z^*_{n,0}$ and

$$u^*_n(t) := J(u_{n,0}) = J(x_0),$$

(•) For all $t \in I_{n,i}(0 \leq i \leq 2^n)$, set $z^*_n(t) := z^*_{n,i}$, and

$$u^*_n(t) := J(u_{n,i}) + \frac{t - t_{n,i}}{\mu_n}\left(J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z^*_{n,i}\right) + (t - t_{n,i}) z^*_{n,i}.$$

⋆ For every $n \geq 0$, the mappings u_n are given by

$$u_n(t) := J^*(u^*_n(t)), \text{ for all } t \in I. \tag{3}$$

$$
\begin{aligned}
\alpha\|J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z^*_{n,i}\|^{q'} &\leq \|J^*(J(u_{n,i+1}))\|^2 \\
&- 2\langle J^*(J(u_{n,i+1})), J(u_{n,i}) + \mu_n z^*_{n,i}\rangle \\
&+ \|J(u_{n,i}) + \mu_n z^*_{n,i}\|^2 \\
&= \|u_{n,i+1}\|^2 - 2\langle u_{n,i+1}, J(u_{n,i}) + \mu_n z^*_{n,i}\rangle \\
&+ \|J(u_{n,i}) + \mu_n z^*_{n,i}\|^2 \\
&= V(J(u_{n,i}) + \mu_n z^*_{n,i}, u_{n,i+1})
\end{aligned}
$$

⋆ For every t, t' in $I_{n,i}$ $(0 \leq i \leq 2^n)$ one has

$$u_n^*(t') - u_n^*(t) = \frac{t'-t}{\mu_n}\left(J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z_{n,i}^*\right) + (t'-t)z_{n,i}^*.$$

Thus

$$\|u_n^*(t') - u_n^*(t')\| \leq |t'-t|\left(\frac{\|J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z_{n,i}^*\|}{\mu_n} + \|z_{n,i}^*\|\right)$$

⋆ Note that X^* is p'-unif. smooth and q'-unif. convex where p' and q' are the conjugate numbers of p and q resp. Also, $J(u_{n,i+1})$, $J(u_{n,i}) - \mu_n z_{n,i}^* \in R\mathbf{B}_*$, where $R := k + Tl$, k, l satisfy $K \subset k\mathbf{B}^*$ and $L \subset l\mathbf{B}^*$.

Then, there exists a constant $\alpha > 0$ so that

$$\alpha\|J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z_{n,i}^*\|^{q'} \leq V_*(J^*(J(u_{n,i+1})), J(u_{n,i}) + \mu_n z_{n,i}^*)$$

where $V_* : X^{**} \times X^* \to [0, \infty[$ defined by

$$V_*(\xi, \varphi) = \|\xi\|^2 - 2\langle \xi, \varphi \rangle + \|\varphi\|^2, \forall \xi \in X^{**}, \forall \varphi \in X^*.$$

$$
\begin{aligned}
\alpha\|J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z^*_{n,i}\|^{q'} &\leq \|J^*(J(u_{n,i+1}))\|^2 \\
&- 2\langle J^*(J(u_{n,i+1})), J(u_{n,i}) + \mu_n z^*_{n,i}\rangle \\
&+ \|J(u_{n,i}) + \mu_n z^*_{n,i}\|^2 \\
&= \|u_{n,i+1}\|^2 - 2\langle u_{n,i+1}, J(u_{n,i}) + \mu_n z^*_{n,i}\rangle \\
&+ \|J(u_{n,i}) + \mu_n z^*_{n,i}\|^2 \\
&= V(J(u_{n,i}) + \mu_n z^*_{n,i}, u_{n,i+1}) \\
&= d^V_{C(t_{n,i+1})}(J(u_{n,i}) + \mu_n z^*_{n,i})
\end{aligned}
$$

On the other hand we have

$$
\begin{aligned}
\left(d^V_{C(t_{n,i+1})}\right)^{\frac{1}{q'}}(J(u_{n,i}) + \mu_n z^*_{n,i}) &= \left(d^V_{C(t_{n,i+1})}\right)^{\frac{1}{q'}}(J(u_{n,i}) + \mu_n z^*_{n,i}) - \left(d^V_{C(t_{n,i})}\right)^{\frac{1}{q'}}(J(u_{n,i})) \\
&\leq \lambda(t_{n,i+1} - t_{n,i}) + \gamma\mu_n\|z^*_{n,i}\| \leq (\lambda + \gamma l)\mu_n,
\end{aligned}
$$

Thus

$$
\begin{aligned}
\|J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z^*_{n,i}\| &\leq \alpha^{-\frac{1}{q'}}\left(d^V_{C(t_{n,i+1})}\right)^{\frac{1}{q'}}(J(u_{n,i}) + \mu_n z^*_{n,i}) \\
&\leq \alpha^{-\frac{1}{q'}}(\lambda + \gamma l)\mu_n
\end{aligned}
$$

⋆ Then, one obtains for every $0 \leq i \leq 2^n$

$$\|J(u_{n,i+1}) - J(u_{n,i})) - \mu_n z^*_{n,i}\| \leq \left[\frac{\lambda + \gamma l}{\alpha^{\frac{1}{q'}}}\right] \mu_n.$$

⋆ Coming back to the definition of u_n^*, one observes that for $0 \leq i \leq 2^n$

$$\dot{u}_n^*(t) = \frac{1}{\mu_n}(J(u_{n,i+1}) - J(u_{n,i}) - \mu_n z_{n,i}^*) + z_{n,i}^*, \text{ a.e. } I_{n,i}.$$

Then one obtains, for a.e. $t \in I$

$$\|\dot{u}_n^*(t) - z_n^*(t)\| \leq \left[\frac{\lambda + \gamma l}{\alpha^{\frac{1}{q'}}}\right] := \delta.$$

Then, the construction of u_n^* and z_n^*, we have for a.e. $t \in I$

$$z_n^*(t) \in F(\rho_n(t), u_n(\rho_n(t)))$$

and

$$\dot{u}_n^*(t) - z_n^*(t) \in -N(C(\theta_n(t); u_n(\theta_n(t)))).$$

⋆ This inclusion entails for a.e. $t \in I$

$$\dot{u}_n^*(t) - z_n^*(t) \in -\delta\partial d_{C(\theta_n(t))}(u_n(\theta_n(t))).$$

⋆ Observe that $\|\dot{u}_n^*(t)\| \leq \delta + l$, and

$$u_n^*(t) = \left(1 - \frac{(t - t_{n,i})}{\mu_n}\right) J(u_{n,i}) + \frac{(t - t_{n,i})}{\mu_n} J(u_{n,i+1}) \in K.$$

Thus, for every $t \in I$, the set $\{u_n^*(t) : n \geq 0\}$ is relatively compact in X^*. Therefore, the previous estimates and Ascoli-Arzela theorem ensure the existence of a Lipschitz mapping $u^* : I \to X^*$ such that $u_n^* \to u^*$ uniformly on I and $\dot{u}_n^*$ in $L^1(I, X^*)$. Hence $J^* u_n^* = u_n$ converges uniformly to $J^* u^* := u$, because J^* is uniformly continuous on the compact set K and $u_n^*(t) \in K$ for all $t \in I$.

$\star$ Moreover, for a.e. $t \in I$, by definition of $\theta_n(t)$ one has $|\theta_n(t) - t| \leq \mu_n$, and so we have

$$\|u_n^*(\theta_n(t)) - u^*(t)\| \leq \|u_n^*(t) - u^*(t)\| + (\delta + l)\mu_n.$$

So

$$\lim_{n\to\infty} \theta_n(t) = t, \quad \lim_{n\to\infty} u_n^*(\theta_n(t)) = u^*(t).$$

⋆ Now, let us define $Z_n^*(t) := \int_0^t z_n^*(s)ds$. Observe that for all $t \in I$ the set $\{Z_n^*(t) : n \geq 0\}$ is contained in the compact set TL and so it is relatively compact in X^*. Therefore, as $\|z_n^*(t)\| \leq l$, Ascoli-Arzela theorem ensures the existence of an absolutely continuous mapping $Z^* : I \to X^*$ such that $Z_n^* \to Z^*$ uniformly on I and $\dot{Z}_n^* = z_n^*$ converges weakly to $\dot{Z}^* = z^*$ in $L^1(I, X^*)$.

⋆ The weak convergence in $L^1(I, X^*)$ of $\{\dot{u}_n^*\}_n$ and $\{z_n^*\}_n$ to $\dot{u}^*$ and z^* respectively entail for almost all $t \in I$ (by Mazur's lemma)

$$\dot{u}^*(t) - z^*(t) \in \bigcap_n \overline{co}\{\dot{u}_j^*(t) - z_j^*(t) : j \geq n\}.$$

⋆ By the upper semicontinuity of F and the convexity of its values and with the same techniques used above we can prove that $z^*(t) \in F(t, u(t))$ and so we get

$$\dot{u}^*(t) \in -N(C(t); u(t)) + F(t, u(t)),$$

that is,

$$\dot{x}(t) \in -N(C(t); J^*(x(t))) + F(t, J^*(x(t))),$$

with $x(t) = J(u(t))$. This completes the proof.

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with $x(t) = J(u(t))$. This completes the proof.

Nonconvex Case

Theorem 2. Let $p \geq 2, q \in (1,2]$, X be a separable p-uniform. convex and q-uniform. smooth Banach space, and let $C : I \rightrightarrows X$ satisfy for any $\varphi, \psi \in X^*$ and $t, t' \in I$

$$|\left(d^V_{C(t)}\right)^{1/q'}(\psi) - \left(d^V_{C(t')}\right)^{1/q'}(\varphi)| \leq \lambda |t' - t| + \gamma \|\varphi - \psi\|, \quad (4)$$

where $q' = \frac{q}{q-1}$, and $\lambda, \gamma > 0$. Assume that $J(C(t)) \subset K$ for any $t \in I$, for some convex compact set $K \subset X^*$. Let $F : I \times X \rightrightarrows X^*$ be an u.s.c. set-valued mapping with convex compact values in X^* such that $F(t,x) \subset L$ for all $(t,x) \in I \times X$, for some convex compact set $L \subset X^*$. If furthermore C satisfies:

(•)The generalized projs. exist uniform. around the images of C;

(•) $(t,x) \mapsto \partial^C d_{C(t)}(x)$ is scalarly u.s.c. on $I \times X$.

Then, for any $x_0 \in C(0)$, there exists a solution of (NSPP).

Open Problems

- ⋆ The existence of solutions for all variants of SP in the general Banach spaces.
- ⋆ The existence of solutions for all variants of SP in the general closed nonconvex sets.
- ⋆ Relationships between the proposed extensions of (SP) and the ones studied in terms of primal proximal normal cone in Bernicot-Venel and al.