PDE Aspects of Mean Field Games

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Mean Field Games (MFG) study collective behavior of rational agents.

- collective behavior = infinitely many agents, having individually a negligible influence on the global system
- rational agents = each agent controls his state in order to minimize a cost which depends on the other agents' positions

Some references :

— Early work by Lasry-Lions (2006)
... and Caines-Huang-Malhamé (2006)

— Similar models in the economic literature : heterogeneous agent models (Aiyagari ('94), Bewley ('86), Krusell-Smith ('98),...)

Two key equations :

- The MFG system (finite dimensional)
- The Master equation (infinite dimensional)

The MFG system :

$$(MFG) \qquad \left\{ \begin{array}{ll} -\partial_t u - \nu \Delta u + H(x, Du, m) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ \partial_t m - \nu \Delta m - \text{div}\,(m D_p H(x, Du, m)) = 0 & \text{in } (0,T) \times \mathbb{R}^d \\ u(T,x) = G(x, m(T)), \; m(t_0, \cdot) = m_0 & \text{in } \mathbb{R}^d \end{array} \right.$$

where

- the unknown are $u = u(t,x)$ and $m = m(t,x)$ with $m(t,\cdot)$ a probability density for any $t \in [0,T]$,
- $H = H(x,p,m) : \mathbb{R}^d \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is a Hamiltonian (convex in p)
- $m_0 \in \mathcal{P}(\mathbb{R}^d)$ is the initial condition for m,
- $G = G(x,m) : \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$ is a terminal condition for u
- $\nu \geq 0$ is a fixed volatility.

The Master equation :

$$
\begin{cases}
-\partial_t U - (\nu+\beta)\Delta_x U + H(x, D_x U, m) \\
\qquad -(\nu+\beta)\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y\,[D_m U]\ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U, m)\ dm(y) \\
\qquad -2\beta \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x\,[D_m U]\,dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right]\ dm \otimes dm = 0 \\
\qquad\qquad \text{in } [0,T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\
U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d)
\end{cases}
$$

where

- $\nu, \beta \geq 0$ are fixed parameters,
- the unknown is $U : [0,T] \times \mathbb{T}^d \times \mathcal{P}(\mathbb{R}^d) \to \mathbb{R}$,
- $\partial_t U$, $D_x U$ and $\Delta_x U$ stand for the usual derivatives with respect to the local variables (t,x) of U,
- $D_m U$ and $D^2_{mm} U$ are the first and second order derivatives with respect to the measure m,
- H and G are the same as for the MFG system.

Aim of the talk

- Discuss the meaning and the well-posedness of the two equations.
- Show that the MFG system can be obtained as a “mean field limit".
- Discuss the numerical approximation of MFG.

Missing parts : the talk **will not present** the stochastic aspects of MFG.

(Caines-Huang-Malhamé, Carmona-Delarue, Kolokoltsov, Bensoussan-Frehse-Yam,...)

Outline

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We discuss here the meaning of the MFG system :

$$(MFG) \qquad \left\{ \begin{array}{lll} (i) & -\partial_t u - \nu \Delta u + H(x, Du, m) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ (ii) & \partial_t m - \nu \Delta m - \mathrm{div}\,(m D_p H(x, Du, m)) = 0 & \text{in } (0, T) \times \mathbb{R}^d \\ (iii) & u(T, x) = G(x, m(T)), \; m(t_0, \cdot) = m_0 & \text{in } \mathbb{R}^d \end{array} \right.$$

By fixed point argument : fix a family $(m = m(t))_{t\in[0,T]}$ of probability densities on $\mathbb{R}^d$.

An average agent controls the stochastic differential equation

$$dX_s = \alpha_s ds + \sqrt{2}\sigma dB_s, \qquad X_t = x$$

where $\sigma := \sqrt{\nu}$, (α_s) is the control and (B_s) is a standard B.M. He aims at minimizing the cost

$$J(x,(\alpha_s),m) := \mathbb{E}\left[\int_t^T L(X_s,\alpha_s,m(s))\,ds + G(X_T,m(T))\right] .$$

where $L(x,q,m) = \sup_{p\in\mathbb{R}^d}\{-\langle p,q\rangle - H(x,p,m)\}$.

His value function u is given by

$$u(t,x) = \inf_{(\alpha_s)} J(x,(\alpha_s),m) .$$

Recall that u depends on m !!!

By fixed point argument : fix a family $(m = m(t))_{t\in[0,T]}$ of probability densities on $\mathbb{R}^d$.

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His value function u is given by

$$u(t,x) = \inf_{(\alpha_s)} J(x,(\alpha_s),m) .$$

Recall that u depends on m!!!

- The value function u then satisfies the Hamilton-Jacobi equation

$$\left\{\begin{array}{lll} (i) & -\partial_t u - \nu\Delta u + H(x, Du, m) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & u(x,T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{array}\right.$$

- The optimal control is given by

$$\alpha^*(t,x) = -D_pH(x, Du(t,x), m(t))\ .$$

Proof by verification : *If u solves (i) and (iii), we have by Itô's formula,*

$$\begin{array}{l} \frac{d}{ds}\mathbf{E}\left[u(s,X_s) - \int_s^T L(X_\tau, \alpha_\tau, m(\tau))d\tau\right] \\ \quad = \mathbf{E}\left[\partial_s u(s,X_s) + \langle Du, \alpha_s\rangle + \nu\Delta u + L(X_s, \alpha_s, m(s))\right] \\ \quad \geq \mathbf{E}\left[\partial_s u(s,X_s) + \nu\Delta u - H(X_s, Du, m(s))\right] \ = \ 0 \end{array}$$

with equality only for $\alpha = \alpha^*$.

- The value function u then satisfies the Hamilton-Jacobi equation

$$\left\{\begin{array}{lll} (i) & -\partial_t u - \nu\Delta u + H(x, Du, m) = 0 & \text{in } \mathbb{R}^d \times (0, T) \\ (iii) & u(x, T) = G(x, m(T)) & \text{in } \mathbb{R}^d \end{array}\right.$$

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$$\alpha^*(t, x) = -D_p H(x, Du(t, x), m(t)) \ .$$

Proof by verification : *If u solves (i) and (iii), we have by Itô's formula,*

$$\begin{array}{l} \displaystyle \frac{d}{ds}\mathbf{E}\left[u(s, X_s) - \int_s^T L(X_\tau, \alpha_\tau, m(\tau))d\tau\right] \\ \quad = \ \mathbf{E}\left[\partial_s u(s, X_s) + \langle Du, \alpha_s\rangle + \nu\Delta u + L(X_s, \alpha_s, m(s))\right] \\ \quad \geq \ \mathbf{E}\left[\partial_s u(s, X_s) + \nu\Delta u - H(X_s, Du, m(s))\right] \ = \ 0 \end{array}$$

with equality only for $\alpha = \alpha^*$.

Integrating between 0 *and* T *:*

$$\mathbf{E}\left[u(T,X_T) - u(t,x) + \int_t^T L(X_\tau,\alpha_\tau,m(\tau))d\tau\right] \geq 0$$

with equality for $\alpha = \alpha^*$.

By (iii), $u(T,X_T) = G(X_T,m(T))$, *so that*

$$u(t,x) \leq \mathbf{E}\left[\int_t^T L(X_\tau,\alpha_\tau,m(\tau))d\tau + G(X_T,m(T))\right]$$

with equality for $\alpha = \alpha^*$.

Therefore u is the value function.

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By (*iii*), $u(T,X_T)=G(X_T,m(T))$, *so that*

$$u(t,x)\leq \mathbf{E}\left[\int_t^T L(X_\tau,\alpha_\tau,m(\tau))d\tau+G(X_T,m(T))\right]$$

with equality for $\alpha=\alpha^*$.

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By (*iii*)*,* $u(T,X_T)=G(X_T,m(T))$*, so that*

$$u(t,x)\leq \mathbf{E}\left[\int_t^T L(X_\tau,\alpha_\tau,m(\tau))d\tau+G(X_T,m(T))\right]$$

with equality for $\alpha=\alpha^*$*.*

Therefore u is the value function.

To summarize : Given a family $(m(t))_{t\in[0,T]}$ of probability densities,

- the value function u of an average agent is the solution to the HJ eq

$$\left\{\begin{array}{lll} (i) & -\partial_t u - \nu\Delta u + H(x,Du,m) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & u(x,T) = G(x,m(T)) \quad \text{in } \mathbb{R}^d & \end{array}\right.$$

- The optimal control is given by

$$\alpha^*(t,x) = -D_pH(x,Du(t,x),m(t)) \ .$$

- Therefore its optimal dynamics solves the SDE

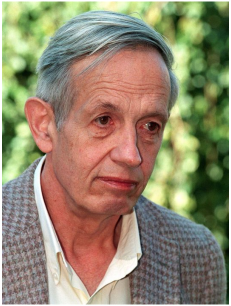
$$dX_s = -D_pH(X_s,Du(t,X_s),m(s))ds + \sqrt{2}\sigma dB_s, \qquad X_t = x$$

- Assuming that the initial distribution of the players is the probability m_0 and that the Brownian Motions of the players are all independent, the actual distribution $(\tilde{m}(t))_{t\in[0,T]}$ of the players solves the Kolmogorov equation

$$\left\{\begin{array}{lll} (ii) & \partial_t \tilde{m} - \nu\Delta\tilde{m} - \mathrm{div}(\tilde{m}\, D_pH(x,Du,m)) = 0 & \text{in } \mathbb{R}^d \times (0,T) \\ (iii) & \tilde{m}(0) = m_0 \quad \text{in } \mathbb{R}^d & \end{array}\right.$$

A solution (u,m) of the MFG system is a fixed point of the map $m \to \tilde{m}$.

Comments :



- The MFG system describes a Nash equilibrium configuration.
  $\longrightarrow$ no agent has interest to deviate unilaterally.

- It relies on two key assumptions :
  — rationality of the agents
  — the agents are infinitesimal, indistinguishable with independent noises.

Validity of the MFG systems :

- Used in many areas :
  - Economic models (Heterogeneous agent, finance,...)
  - Engineering literature (wireless power control,...)
  - Crowd motion, vaccination strategies, etc...
- MFG models as limit of Nash equilibrium configuration for finitely many agents (C.-Delarue-Lasry-Lions)
- Learning procedures (C.-Hadikhanloo)

In terms of mathematical analysis :

- The forward-backward coupling is unusual and challenging in terms of PDE
- It is related to calculus of variation and optimal control of PDEs

Outline

We now discuss the well-posedness of the MFG system :

$$(MFG) \qquad \left\{ \begin{array}{lll} (i) & -\partial_t u - \nu\Delta u + H(x,Du,m) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m - \nu\Delta m - \mathrm{div}\,(mD_pH(x,Du,m)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & u(T,x) = G(x,m(T)),\ m(t_0,\cdot) = m_0 & \text{in } \mathbb{R}^d \end{array}\right.$$

4 different regimes :

- Local/non local coupling,
  - either $(x,p)\to H(x,p,m)$ is “smooth" whatever $m\in\mathcal{P}(\mathbb{R}^d)$,
  - or $H(x,p,m) = H(x,p,m(x))$ where $m = m(x)dx$
- Uniformly parabolic/degenerate parabolic
  Namely : either $\nu > 0$ or $\nu = 0$.

Here we study :

1. Nonlocal, uniformly parabolic regime
2. Nonlocal, first order regime
3. Local, first order regime (by variational methods)

Outline

For simplicity, we work

- with periodic boundary conditions i.e., in the torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$.

- and with separate Hamiltonian :

$$H(x, p, m) = H(x, p) - F(x, m)$$

Under these conditions, the MFG system becomes :

$$(MFG) \quad \begin{cases} (i) & -\partial_t u - \nu\Delta u + H(x, Du) = F(x, m(t)) \qquad \text{in } (0, T)\times\mathbb{T}^d \\ (ii) & \partial_t m - \nu\Delta m - \mathrm{div}\,(mD_pH(x, Du)) = 0 \qquad \text{in } (0, T)\times\mathbb{T}^d \\ (iii) & u(T, x) = G(x, m(T)),\ m(t_0, \cdot) = m_0 \qquad \text{in } \mathbb{T}^d \end{cases}$$

Nonlocal, uniformly parabolic regime

Existence : We assume that

1. $\nu > 0$,
2. F and G are Lipschitz continuous in $\mathbb{T}^d \times P(\mathbb{T}^d)$.
3. $F(\cdot, m)$ and $G(\cdot, m)$ are bounded in $C^{1+\beta}(\mathbb{T}^d)$ and $C^{2+\beta}(\mathbb{T}^d)$ (for some $\beta \in (0,1)$) uniformly with respect to $m \in P(\mathbb{T}^d)$.
4. The Hamiltonian $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is locally Lipschitz continuous, D_pH exists and is continuous on $\mathbb{T}^d \times \mathbb{R}^d$, and H satisfies the growth condition
$$\langle D_x H(x,p), p\rangle \geq -C_0(1+|p|^2)$$
for some constant $C_0 > 0$.
5. The probability measure m_0 is absolutely continuous with respect to the Lebesgue measure, has a $C^{2+\beta}$ continuous density.

Theorem (Lasry-Lions '06)

Under the above assumptions, there is at least one classical solution to the MFG system.

Proof : Let $m = (m(t)) \in C^{1/2}([0,T],\mathbb{P}(\mathbb{T}^d))$ with $m(0) = m_0$ and let u solve

$$\left\{\begin{array}{ll} -\partial_t u - \nu\Delta u + H(x,Du) = F(x,m(t)) & \text{in } (0,T)\times\mathbb{T}^d \\ u(T,x) = G(x,m(T)), \quad \text{in } \mathbb{T}^d \end{array}\right.$$

Then $u \in C^{1+\alpha/2,2+\alpha}$ with bounded norm.

Let now $\tilde m$ be the solution to

$$\left\{\begin{array}{ll} \partial_t \tilde m - \nu\Delta\tilde m - \mathrm{div}\,(\tilde m D_p H(x,Du)) = 0 & \text{in } (0,T)\times\mathbb{T}^d \\ \tilde m(t_0,\cdot) = m_0 \quad \text{in } \mathbb{T}^d \end{array}\right.$$

Then $\tilde m \in C^{1+\alpha/2,2+\alpha} \cap C^{1/2}([0,T],\mathbb{P}(T^d))$ with bounded norm.

Conclusion by Schauder fixed point Theorem. □

Uniqueness : Assume in addition that

- Either $T > 0$ is "small"
- Or $H = H(x, p)$ is uniformly convex in p and F and G are monotone :

$$\int_{\mathbb{T}^d} (F(x,m) - F(x,m'))(m - m') \geq 0, \quad \int_{\mathbb{T}^d} (G(x,m) - G(x,m'))d(m - m') \geq 0,$$

for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$.

Theorem (Lasry-Lions '06)

Under the above assumptions, the solution to (MFG) is unique.

Typical example : $F(m) = (\rho \star m) \star \rho$, where ρ is smooth and symmetric. Then

$$\int_{\mathbb{T}^d} (F(m) - F(m'))(m - m') \;=\; \int_{\mathbb{T}^d} (\rho \star (m - m'))^2 \;\geq\; 0$$

Proof : Let (u_1, m_1) and (u_2, m_2) be two solutions. Set $\bar u = u_1 - u_2$ and $\bar m = m_1 - m_2$. Then

$$\begin{aligned}
\frac{d}{dt}\int_{\mathbb{T}^d} \bar u \bar m \;&=\; \int_{\mathbb{T}^d} (\partial_t \bar u)\bar m + \bar u(\partial_t \bar m)\\
&=\; \int_{\mathbb{T}^d} (-\Delta \bar u + H(x,Du_1) - H(x,Du_2) - F(x,m_1) + F(x,m_2))\bar m\\
&\qquad\qquad + \bar u(\Delta \bar m + \mathrm{div}(m_1\ D_pH(x,Du_1)) - \mathrm{div}(m_2\ D_pH(x,Du_2)))
\end{aligned}$$

Note that

$$\int_{\mathbb{T}^d} -(\Delta \bar u)\bar m + \bar u(\Delta \bar m) = 0$$

and, from the monotonicity condition on F,

$$\int_{\mathbb{T}^d} (-F(x,m_1) + F(x,m_2))\bar m = \int_{\mathbb{T}^d} (-F(x,m_1) + F(x,m_2))(m_1 - m_2) \leq 0.$$

Integrating by parts the terms in H :

$$\begin{aligned}
&\int_{\mathbb{T}^d} (H(x,Du_1) - H(x,Du_2))\bar m - \langle D\bar u\ ,\ m_1\ D_pH(x,Du_1) - m_2\ D_pH(x,Du_2)\rangle\\
&\quad = -\int_{\mathbb{T}^d} m_1\,(H(x,Du_2) - H(x,Du_1) - \langle D_pH(x,Du_1), Du_2 - Du_1\rangle)\\
&\quad\quad - \int_{\mathbb{T}^d} m_2\,(H(x,Du_1) - H(x,Du_2) - \langle D_pH(x,Du_2), Du_1 - Du_2\rangle)
\end{aligned}$$

The uniform convexity of H then implies :

$$\int_{\mathbb{T}^d}(H(x,Du_1)-H(x,Du_2))\bar m - \langle D\bar u, m_1\ D_pH(x,Du_1)-m_2\ D_pH(x,Du_2)\rangle)$$
$$\leq -\int_{\mathbb{T}^d}\frac{(m_1+m_2)}{2C}|Du_1-Du_2|^2$$

Putting the estimates together we get

$$\frac{d}{dt}\int_{\mathbb{T}^d}\bar u\bar m \ \leq -\int_{\mathbb{T}^d}\frac{(m_1+m_2)}{2C}|Du_1-Du_2|^2.$$

Integrating on $[0,T]$ and rearranging :

$$\int_0^T\int_{\mathbb{T}^d}\frac{(m_1+m_2)}{2C}|Du_1-Du_2|^2 \leq -\int_{\mathbb{T}^d}\bar u(T)\bar m(T)+\int_{\mathbb{T}^d}\bar u(0)\bar m(0) \ \leq \ 0$$

because $\bar m(0)=m_1(0)-m_2(0)=m_0-m_0=0$ and

$$\int_{\mathbb{T}^d}\bar u(T)\bar m(T)=\int_{\mathbb{T}^d}(G(x,m_1(T))-G(x,m_2(T)))(m_1(T)-m_2(T))\geq 0.$$

Therefore $Du_1=Du_2$ in $\{m_1>0\}\cup\{m_2>0\}$: m_1 and m_2 solve the Kolmogorov equation, so that $m_1=m_2$. Then, in turn, u_1 and u_2 solve the same HJ equation, and $u_1=u_2$. □

Nonlocal, first order regime

We now consider the first order MFG system

$$(MFG) \quad \left\{ \begin{array}{lll} (i) & -\partial_t u + H(x, Du) = F(x, m(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m - \mathrm{div}\,(mD_pH(x,Du)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & u(T,x) = G(x, m(T)),\ m(t_0,\cdot) = m_0 & \text{in } \mathbb{R}^d \end{array} \right.$$

Assumptions : Same as before, except we also assume that $H : \mathbb{T}^d \times \mathbb{R}^d \to \mathbb{R}$ is uniformly convex in p.

Theorem (Lasry-Lions '06)

Under the above assumptions, there is at least one solution (u, m) to the MFG system, where

- u is Lipschitz continuous and satisfies the HJ in the viscosity sense,
- $m \in L^\infty$ and satisfies the Kolmogorov equation in the sense of distribution.

If, moreover, F and G are monotone, then the solution is unique.

Proof : By viscous approximation : for $\varepsilon > 0$, let $(u^\varepsilon, m^\varepsilon)$ be the solution to

$$\left\{\begin{array}{lll} (i) & -\partial_t u^\varepsilon - \varepsilon\Delta u^\varepsilon + H(x, Du^\varepsilon) = F(x, m^\varepsilon(t)) & \text{in } (0,T)\times\mathbb{R}^d \\ (ii) & \partial_t m^\varepsilon - \varepsilon\Delta m^\varepsilon - \text{div}\,(m^\varepsilon D_pH(x, Du^\varepsilon)) = 0 & \text{in } (0,T)\times\mathbb{R}^d \\ (iii) & u^\varepsilon(T,x) = G(x, m(T)),\ m^\varepsilon(t_0,\cdot) = m_0 & \text{in } \mathbb{R}^d \end{array}\right.$$

Then

- (u^ε) is uniformly Lipschitz continuous and uniformly semi-concave :

$$\|u^\varepsilon\|_\infty + \|\partial_t u^\varepsilon\|_\infty + \|Du^\varepsilon\|_\infty + D^2u^\varepsilon \le C$$

  In particular, (Du^ε) is pre-compact in L^1.
- Because of the semi-concavity estimate, (m^ε) is uniformly bounded in L^∞.

Then any limit of the $(u^\varepsilon, m^\varepsilon)$ as $\varepsilon \to 0$ is a solution of the MFG system.

Uniqueness relies on Di Perna-Lions/Ambrosio theory on ODEs with discontinuous coefficients.

□

Outline

The first order MFG system with local coupling

We now concentrate on the first order Mean Field Game system :

$$(MFG)\quad \begin{cases} (i) & -\partial_t u + H(x, Du) = F(x, m(t,x)) \\ & \qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathbb{T}^d \\ (ii) & \partial_t m - \mathrm{div}(m\, D_pH(x,Du)) = 0 \\ & \qquad\qquad\qquad\qquad \text{in } (0,T)\times\mathbb{T}^d \\ (iii) & m(0,x) = m_0(x),\ u(T,x) = u_T(x) \qquad \text{in } \mathbb{T}^d \end{cases}$$

where

- $H = H(x,p)$ is convex in p, periodic in x,
- $F = F(x,m)$ is a local coupling, increasing in m, periodic in x
- $u_T = u_T(x)$ is a periodic terminal cost,
- m_0 is a probability density on $\mathbb{T}^d$.

Specific difficulty : The fixed point argument **does not work.**

Two approaches :

- Reduction to a quasi-linear equation $\rightsquigarrow$ smooth solutions for some smooth data (Lasry-Lions)

  Principle : by (i),
  $$m(t,x) = F^{-1}\left(x, -\partial_t u + H(x,Du)\right).$$
  Replace m by u in (ii) : the equation
  $$\partial_t\left(F^{-1}\left(x, -\partial_t u + H(x,Du)\right)\right) - \mathrm{div}\left(F^{-1}\left(x, -\partial_t u + H(x,Du)\right) D_pH(x,Du)\right) = 0$$
  is a (singular) elliptic equation in time-space.

- MFG system as necessary conditions for two convex optimal control problems in duality
  - for the Hamilton-Jacobi equations
  - and for the continuity equations

  $\longrightarrow$ Reminiscent of optimal transport and Benamou-Brenier formulation of the Wasserstein distance.

Set :

$$\mathcal{F}(x,m) = \begin{cases} \int_1^m F(x,m')dm' & \text{if } m \geq 0 \\ +\infty & \text{otherwise} \end{cases}$$

$\mathcal{F}^*(x,a) = \sup_{m\in\mathbb{R}} (am - \mathcal{F}(x,m))$ and and $H^*(x,v) = \sup_{p\in\mathbb{R}^d} p.v - H(x,p)$.

- The optimal control of continuity equation.

$$(\mathbf{K}-\mathbf{Pb}) \qquad \inf_{(m,w)} \left\{ \int_0^T \int_{\mathbb{T}^d} mH^*(x,-v) + \mathcal{F}(x,m)\; dxdt + \int_{\mathbb{T}^d} u_T(x)m(T,x)dx \right\}$$

  where the infimum is taken over the pairs (m,v) such that

$$\partial_t m + \mathrm{div}(mv) = 0 \text{ in } (0,T)\times\mathbb{T}^d, \qquad m(0) = m_0 \qquad \text{in } \mathbb{T}^d$$

  in the sense of distributions.

- The optimal control of HJ equation

$$(\mathbf{HJ}-\mathbf{Pb}) \qquad \inf_{\alpha} \left\{ \int_0^T \int_{\mathbb{T}^d} \mathcal{F}^*(x,\alpha(t,x))\; dxdt - \int_{\mathbb{T}^d} u(0,x)m_0(x)dx \right\}$$

  where u is the solution to the HJ equation

$$-\partial_t u + H(x,Du) = \alpha \text{ in } (0,T)\times\mathbb{T}^d, \qquad u(T,\cdot) = u_T \qquad \text{in } \mathbb{T}^d.$$

Results :

- Both problems are in duality and have the MFG system as optimality condition.

  For instance, if (u, α) is optimal for $(\mathbf{HJ} - \mathbf{Pb})$, then (u, m) solves MFG with $m := F^{-1}(x, \alpha)$.

- Existence of minimizers for both problems yields weak solutions of the MFG system. (C., Graber, C.-Graber, C.-Graber-Porretta-Tonon, C.-Porretta-Tonon).

  $\longrightarrow$ Difficulty : the optimal control of HJ eq. is very singular.
  Relies on new estimates on HJ eq. with discontinuous RHS.
  (C. ('09), Cannarsa-C. ('10), C.-Rainer ('11), C.-Silvestre ('12), C.-Porretta-Tonon ('14))

- Useful for numerical computations.
  Works also for some non-local or second order MFG systems.

Conclusion on the MFG system

Well-understood :

- Existence/uniqueness of solutions in the 4 regimes
- The solution can often be obtained by variational methods
- Many extensions :
  - Fully non-linear equations,
  - Other boundary conditions,
  - Multi-population problems, etc...
- Few explicit solutions (Linear-quadratic MFG)
- Long time behavior (convergence to the ergodic MFG system)

Several open questions :

- Uniqueness issues (is the monotonicity condition necessary ?)
- Existence of classical solution in the local regime poorly understood (Gomes and al. ; Weak solutions : Lasry-Lions and Porretta)
- Degenerate equations and state-constraints
- Existence in the congestion setting

Outline

MFG systems pretty well-understood, but do not answer two key points :

- MFG with common noise or with a major agent
- Mean field limit : Convergence of Nash equilibria of $N-$player differential games as $N \to +\infty$.
  Derive the macroscopic model (=MFG system) from the microscopic one (= $N-$player differential game).

These two different issues can be understood through the **master equation**.

The Nash system for $N-$player differential games

The behavior of a $N-$person differential game is described by the Nash system

$$\left\{\begin{array}{l} -\partial_t v^{N,i}(t,\boldsymbol{x}) - \sum_{j=1}^{N} \Delta_{x_j} v^{N,i}(t,\boldsymbol{x}) - \beta \sum_{j,k=1}^{N} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i}(t,\boldsymbol{x}) + H(x_i, D_{x_i} v^{N,i}(t,\boldsymbol{x})) \\ \qquad + \sum_{j\neq i} D_p H(x_j, D_{x_j} v^{N,j}(t,\boldsymbol{x})) \cdot D_{x_j} v^{N,i}(t,\boldsymbol{x}) = F(x_i, m^{N,i}_{\boldsymbol{x}}) \\ \qquad\qquad \text{in } [0,T]\times(\mathbb{T}^d)^N,\ i=1,\dots,N, \\ v^{N,i}(T,\boldsymbol{x}) = G(x_i, m^{N,i}_{\boldsymbol{x}}) \qquad \text{in } (\mathbb{T}^d)^N,\ i=1,\dots,N, \end{array}\right.$$

where $m^{N,i}_{\boldsymbol{x}} := \frac{1}{N-1}\sum_{j\neq i} \delta_{x_j}$.

The mean field problem :

Analyse the limit of the $v^{N,i}$ as $N \to +\infty$.

Standing assumptions :

- Here the individual noise is of level $\nu := 1$, $\beta \geq 0$ is the level of the common noise,
- Ambient space $\mathbb{T}^d$,
- H smooth, globally Lipschitz continuous, with $D^2_{pp}H > 0$,
- F and G "smooth" and monotone.

Outline

- Because of the symmetry, the $v^{N,i}$ can be written as

$$v^{N,i}(t,\boldsymbol{x}) = U^N(t, x_i, m_{\boldsymbol{x}}^{N,i}), \qquad \text{where } m_{\boldsymbol{x}}^{N,i} := \frac{1}{N-1}\sum_{j\neq i}\delta_{x_j}.$$

- Following Lasry-Lions, the expected limit U of the (U^N) should formally satisfy the master equation.

$$\left\{\begin{array}{l}
-\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\
\qquad -(1+\beta)\displaystyle\int_{\mathbb{R}^d} \mathrm{div}_y\left[D_m U\right]\, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U)\, dm(y) \\
\qquad -2\beta \displaystyle\int_{\mathbb{R}^d} \mathrm{div}_x\left[D_m U\right] dm(y) - \beta\int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm}U\right]\, dm\otimes dm = F(x,m) \\
\qquad\qquad\qquad \text{in } [0,T]\times\mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d) \\
U(T,x,m) = G(x,m) \qquad \text{in } \mathbb{R}^d\times\mathcal{P}(\mathbb{R}^d)
\end{array}\right.$$

where

- the unknown is $U : [0,T]\times\mathbb{T}^d\times\mathcal{P}(\mathbb{T}^d)\to\mathbb{R}$,
- $\partial_t U$, $D_x U$ and $\Delta_x U$ stand for the usual derivatives with respect to the local variables (t,x) of U,
- $D_m U$ and $D^2_{mm} U$ are the first and second order derivatives with respect to the measure m.

Derivatives in the space of measures

We denote by $\mathcal{P}(\mathbb{T}^d)$ the set of Borel probability measures on $\mathbb{T}^d$, endowed for the Monge-Kantorovich distance

$$\mathbf{d}_1(m, m') = \sup_{\phi} \int_{\mathbb{T}^d} \phi(y)\, d(m - m')(y),$$

where the supremum is taken over all Lipschitz continuous maps $\phi : \mathbb{T}^d \to \mathbb{R}$ with a Lipschitz constant bounded by 1.

2 notions of derivatives :

- The directional derivative $\frac{\delta U}{\delta m}(m, y)$ (see, e.g., Mischler-Mouhot)
- The intrinsic derivative $D_m U(m, y)$ (see, e.g., Otto, Ambrosio-Gigli-Savaré, Lions)

Directional derivative

A map $U : \mathcal{P}(\mathbb{T}^d) \to \mathbb{R}$ is $\mathcal{C}^1$ if there exists a continuous map $\dfrac{\delta U}{\delta m} : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}$ such that, for any $m, m' \in \mathcal{P}(\mathbb{T}^d)$,

$$\lim_{s \to 0^+} \frac{U((1-s)m + sm') - U(m)}{s} = \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m, y) d(m' - m)(y).$$

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A computation : Let $\phi : \mathbb{T}^d \to \mathbb{R}^d$ be a Borel measurable and bounded vector field. Then

$$\begin{aligned} h^{-1}\left(U((id+h\phi)\sharp m) - U(m)\right) &\simeq h^{-1}\int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m,y)d((id+h\phi)\sharp m - m)(y) \\ &\simeq h^{-1}\int_{\mathbb{T}^d} (\frac{\delta U}{\delta m}(m,y+h\phi(y)) - \frac{\delta U}{\delta m}(m,y))dm(y) \\ &\simeq \int_{\mathbb{T}^d} D_y\frac{\delta U}{\delta m}(m,y)\cdot\phi(y)\, dm(y). \end{aligned}$$

This yields to the definition :

Intrinsic derivative

If $\dfrac{\delta U}{\delta m}$ is of class $\mathcal{C}^1$ with respect to the second variable, the intrinsic derivative $D_m U : \mathcal{P}(\mathbb{T}^d) \times \mathbb{T}^d \to \mathbb{R}^d$ is defined by

$$D_m U(m,y) := D_y \frac{\delta U}{\delta m}(m,y)$$

For instance, if $U(m) = \int_{\mathbb{T}^d} g(x)dm(x)$, then $\dfrac{\delta U}{\delta m}(m,y) = g(y) - \int_{\mathbb{T}^d} gdm$ while $D_m U(m,y) = Dg(y)$.

Second order derivatives are defined in a similar way.

Well-posedness of the master equation

The master equation is the backward equation

$$\left\{\begin{array}{l} -\partial_t U - (1+\beta)\Delta_x U + H(x, D_x U) \\ \qquad -(1+\beta)\int_{\mathbb{R}^d} \mathrm{div}_y\, [D_m U]\ dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U)\ dm(y) \\ \qquad -2\beta \int_{\mathbb{R}^d} \mathrm{div}_x\, [D_m U]\, dm(y) - \beta \int_{\mathbb{R}^{2d}} \mathrm{Tr}\left[D^2_{mm} U\right]\ dm \otimes dm = F(x, m) \\ \qquad\qquad \text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\ U(T, x, m) = G(x, m(T)) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \end{array}\right.$$

Theorem (C.-Delarue-Lasry-Lions, '15)

Under our standing assumptions, the master equation (**M**) has a unique classical solution.

Previous results : Lions ('13), Buckdahn-Li-Peng-Rainer ('14), Gangbo-Swiech ('14), Chassagneux-Crisan-Delarue ('15), Bessi ('15).

Relies on the key idea that the master equation is a nonlinear transport equation in the space of measure.

Idea of proof ($\beta = 0$)

- When $\beta = 0$, the master equation becomes

$$\begin{cases} -\partial_t U - \Delta_x U + H(x, D_x U) - F(x, m) \\ \qquad - \int_{\mathbb{R}^d} \mathrm{div}_y \left[D_m U \right] \, dm(y) + \int_{\mathbb{R}^d} D_m U \cdot D_p H(y, D_x U) \, dm(y) = 0 \\ \qquad\qquad \text{in } [0, T] \times \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \\ U(T, x, m) = G(x, m) \qquad \text{in } \mathbb{R}^d \times \mathcal{P}(\mathbb{R}^d) \end{cases}$$

- The proof relies on the method of characteristics in infinite dimension.

- Given $(t_0, m_0) \in [0, T) \times \mathcal{P}(\mathbb{T}^d)$, let $(u, m) = (u(t, x), m(t, x))$ be the solution of the MFG system :

$$(MFG) \qquad \begin{cases} -\partial_t u - \Delta u + H(x, Du) = F(x, m(t)) \\ \partial_t m - \Delta m - \mathrm{div}(m D_p H(x, Du)) = 0 \\ u(T, x) = G(x, m(T)), \ m(t_0, \cdot) = m_0 \end{cases}$$

- Under our monotonicity assumptions on F and G, the (MFG) system is well-posed. (Lasry-Lions, 2007)
- We *define* U by

$$U(t_0, \cdot, m_0) := u(t_0, \cdot)$$

  Then *formally* U solves the master equation.

- Note that, for any $h \in [0, T - t_0]$,

$$u(t_0 + h, \cdot) = U(t_0 + h, \cdot, m(t_0 + h)).$$

- So

$$\begin{aligned}
\partial_t u(t_0, x) &= \partial_t U(t_0, x, m_0) + \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(t_0, x, m_0, y)\partial_t m(t_0, y)dy \\
&= \partial_t U(t_0, \cdot, m_0) + \int_{\mathbb{T}^d} \frac{\delta U}{\delta m}(m_0, y)\left(\Delta m + \mathrm{div}(mD_pH(x, Du))\right)dy \\
&= \partial_t U(t_0, \cdot, m_0) + \int_{\mathbb{T}^d} \Delta_y \left[\frac{\delta U}{\delta m}\right](m_0, y)m_0(y)dy \\
&\qquad - \int_{\mathbb{T}^d} D_y \left[\frac{\delta U}{\delta m}\right](m_0, y) \cdot D_pH(x, Du))m_0(y)dy \\
&= \partial_t U(t_0, \cdot, m_0) + \int_{\mathbb{T}^d} \mathrm{div}_y \left[D_m U\right](m_0, y)m_0(y)dy \\
&\qquad - \int_{\mathbb{T}^d} D_m U(m_0, y) \cdot D_pH(x, Du))m_0(y)dy
\end{aligned}$$

- As

$$\begin{aligned}
\partial_t u(t_0, x) &= -\Delta u + H(x, Du) - F(x, m_0) \\
&= -\Delta_x U(t_0, x, m_0) + H(x, D_x U(t_0, x, m_0)) - F(x, m_0),
\end{aligned}$$

the map U satisfies (**M**).

Main difficulty :

- Show that U defined by
$$U(t_0, \cdot, m_0) := u(t_0, \cdot)$$
is smooth enough to perform the computation.
- This is obtained by linearization procedure (to compute the directional derivative).
- Requires to keep track of the monotonicity condition.

Idea of proof ($\beta > 0$)

- Same principle, but the system of characteristics becomes the stochastic MFG system

$$(MFGs) \quad \left\{ \begin{array}{l} d_t u_t = \left\{ -(1+\beta)\Delta u_t + H(x, Du_t) - F(x, m_t) - \sqrt{2\beta}\,\mathrm{div}(v_t) \right\} dt \\ \qquad\qquad + v_t \cdot \sqrt{2\beta} dW_t \qquad \text{in } [t_0, T] \times \mathbb{T}^d, \\ d_t m_t = \left[(1+\beta)\Delta m_t + \mathrm{div}\big(m_t D_p H(m_t, Du_t) \big) \right] dt - \sqrt{2\beta}\,\mathrm{div}(m_t dW_t) \\ \qquad\qquad \text{in } [t_0, T] \times \mathbb{T}^d \\ m_{t_0} = m_0, \; u_T(x) = G(x, m_T) \qquad \text{in } \mathbb{T}^d. \end{array} \right.$$

  where (v_t) is a vector field which ensures (u_t) to be adapted to the filtration $(\mathcal{F}_t)_{t\in[t_0,T]}$ generated by the M.B. $(W_t)_{t\in[0,T]}$.

- Intermediate result : well-posedness of (MFGs).

- Proof much more difficult than for the case $\beta = 0$.

Outline

Convergence of the Nash system

We come back to the solution $(v^{N,i})$ of the $N-$player Nash system :

$$(Nash) \quad \left\{ \begin{array}{l} -\partial_t v^{N,i} - \sum_j \Delta_{x_j} v^{N,i} - \beta \sum_{j,k} \mathrm{Tr} D^2_{x_j,x_k} v^{N,i} + H(x_i, D_{x_i} v^{N,i}) \\ \qquad + \sum_{j \neq i} D_p H(x_j, D_{x_j} v^{N,j}) \cdot D_{x_j} v^{N,i} = F(x_i, m^{N,i}_{\boldsymbol{x}}) \qquad \text{in } [0,T] \times \mathbb{T}^{Nd} \\ v^{N,i}(T, \boldsymbol{x}) = G(x_i, m^{N,i}_{\boldsymbol{x}}) \qquad \text{in } \mathbb{T}^{Nd} \end{array} \right.$$

where we have set, for $\boldsymbol{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N$, $m^{N,i}_{\boldsymbol{x}} = \dfrac{1}{N-1} \sum_{j \neq i} \delta_{x_j}$.

Theorem (C.-Delarue-Lasry-Lions, '15)

Let $(v^{N,i})$ be the solution to the Nash system and U be the classical solution to the master equation (**M**). Fix $N \geq 1$ and $(t_0, m_0) \in [0,T] \times \mathcal{P}(\mathbb{T}^d)$.

(i) For any $\boldsymbol{x} \in (\mathbb{T}^d)^N$, let $m_{\boldsymbol{x}}^N := \frac{1}{N}\sum_{i=1}^N \delta_{x_i}$. Then

$$\frac{1}{N}\sum_{i=1}^N \left| v^{N,i}(t_0, \boldsymbol{x}) - U(t_0, x_i, m_{\boldsymbol{x}}^N) \right| \leq CN^{-1}.$$

(ii) For any $i \in \{1, \dots, N\}$ and $x \in \mathbb{T}^d$, let us set

$$w^{N,i}(t_0, x_i, m_0) := \int_{(\mathbb{T}^d)^{N-1}} v^{N,i}(t_0, \boldsymbol{x}) \prod_{j \neq i} m_0(dx_j) \qquad \text{where } \boldsymbol{x} = (x_1, \dots, x_N).$$

Then

$$\left\| w^{N,i}(t_0, \cdot, m_0) - U(t_0, \cdot, m_0) \right\|_{L^1(m_0)} \leq \begin{cases} CN^{-1/d} & \text{if } d \geq 3 \\ CN^{-1/2}\log(N) & \text{if } d = 2 \end{cases}$$

In (i) and (ii), the constant C does not depend on i, t_0, m_0, i nor N.

(Small) idea of proof

Let U be the solution of the master equation.

- Finite dimensional projection of U. For $N \geq 2$ and $i \in \{1, \dots, N\}$ we set

$$u^{N,i}(t, \boldsymbol{x}) = U(t, x_i, m_{\boldsymbol{x}}^{N,i}) \quad \text{where } \boldsymbol{x} = (x_1, \dots, x_N) \in (\mathbb{T}^d)^N, \; m_{\boldsymbol{x}}^{N,i} = \frac{1}{N-1} \sum_{j \neq i} \delta_{x_j}.$$

- Then, for any $N \geq 2$, $i \in \{1, \dots, N\}$, the $u^{N,i}$ are of class $\mathcal{C}^{1,2}$ and "almost" solution to the Nash system : for any $i \in \{1, \dots, N\}$,

$$\left\{\begin{array}{l} -\partial_t u^{N,i} - \sum_j \Delta_{x_j} u^{N,i} - \beta \sum_{j,k} \mathrm{Tr} D^2_{x_j,x_k} u^{N,i} + H(x_i, D_{x_i} u^{N,i}) \\ \quad + \sum_{j \neq i} D_{x_j} u^{N,i}(t, \boldsymbol{x}) \cdot D_p H(x_j, D_{x_j} u^{N,j}(t, \boldsymbol{x})) = F(x_i, m_{\boldsymbol{x}}^{N,i}) + r^{N,i}(t, \boldsymbol{x}) \\ \qquad\qquad\qquad\qquad \text{in } (0, T) \times \mathbb{T}^{Nd} \\ u^{N,i}(T, \boldsymbol{x}) = G(x_i, m_{\boldsymbol{x}}^{N,i}) \qquad \text{in } \mathbb{T}^{Nd} \end{array}\right.$$

  where $r^{N,i} \in \mathcal{C}^0([0, T] \times \mathbb{T}^d)$ with $\|r^{N,i}\|_\infty \leq \dfrac{C}{N}$.

- This is enough to conclude thanks to the symmetry on the system.

Conclusion on the mean field problem and the master equation

- Well-posedness of the master equation : understood under the monotonicity condition ensuring its continuity.

  However, the analysis of discontinuous solutions remains very challenging
  ...even in its finite dimensional version :

$$\frac{\partial U}{\partial t} + (F(U).\nabla)U = 0$$

- Field field limit : proved in a weak sense and under strong monotonicity assumption on the system.

  However
  - we do not know if a stronger convergence can be expected,
  - the problem is completely open without monotonicity condition.

Outline

Slides and numerical simulations by Y. Achdou

(Thanks a lot, Yves !)

Outline

Numerical methods :

- Used extensively by economists
  as early as Krussel-Smith ('98)

- For the MFG system :
  - Finite difference schemes
    Achdou-Camilli-Capuzzo Dolcetta ('12, '13), Camilli-Silva ('12),...
  - Variational techniques (augmented Lagrangian methods (ALG2))
    Benamou-Carlier ('14), ...

- Very little is known for the stochastic MFG system (common noise)
  ... and even less for the master equation.

Finite Differences

Take $d = 1$.

$$\left\{\begin{array}{ll} \dfrac{\partial u}{\partial t} + \nu \Delta u - H(x, \nabla u) = -\Phi[m] & \text{in } [0, T) \times \mathbb{T} \\ \dfrac{\partial m}{\partial t} - \nu \Delta m - \mathrm{div}\left(m \dfrac{\partial H}{\partial p}(x, \nabla u) \right) = 0 & \text{in } (0, T] \times \mathbb{T} \\ u(t = T) = \Phi_0[m(t = T)] & \\ m(t = 0) = m_\circ & \end{array}\right. \qquad (*)$$

- Let $\mathbb{T}_h$ be a uniform grid on the torus with mesh step h, and x_i be a generic point in $\mathbb{T}_h$
- Uniform time grid : $\Delta t = T/N_T$, $t_n = n\Delta t$
- The values of u and m at (x_i, t_n) are approximated by u_i^n and m_i^n

Notation

- The discrete Laplace operator :

$$(\Delta_h w)_i = \frac{1}{h^2}(w_{i+1} - 2w_i + w_{i-1})$$

- Right and left sided finite difference formula for $\frac{\partial w}{\partial x}(x_i)$

$$\frac{\partial w}{\partial x}(x_i) \approx \frac{w_{i+1} - w_i}{h}, \quad \frac{\partial w}{\partial x}(x_i) \approx \frac{w_i - w_{i-1}}{h}$$

- The collection of the 2 first order finite difference formulas at x_i

$$[D_h w]_i = \left\{ \frac{w_{i+1} - w_i}{h}, \frac{w_i - w_{i-1}}{h} \right\} \in \mathbb{R}^2$$

For the Bellman equation, a semi-implicit monotone scheme

$$\frac{\partial u}{\partial t} + \nu \Delta u - H(x, \nabla u) = -\Phi[m]$$

$$\downarrow$$

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} + \nu(\Delta_h u^n)_i - g(x_i, [D_h u^n]_i) = -\left(\Phi_h[m^{n+1}]\right)_i$$

where $[D_h u]_i \in \mathbb{R}^2$ is the collection of the two first order finite difference formulas at x_i for $\partial_x u$.

$$g(x_i, [D_h u^n]_i) = g\left(x_i, \frac{u_{i+1}^n - u_i^n}{h}, \frac{u_i^n - u_{i-1}^n}{h}\right)$$

Assumptions on the discrete Hamiltonian $g : (q_1, q_2) \to g(x, q_1, q_2)$.

- **Monotonicity :**
  - g is nonincreasing with respect to q_1
  - g is nondecreasing with respect to q_2
- **Consistency :**

$$g(x, q, q) = H(x, q), \quad \forall x \in \mathbb{T}, \forall q \in \mathbb{R}$$

- **Differentiability :** g is of class $\mathcal{C}^1$
- **Convexity (for uniqueness and stability) :**

$$(q_1, q_2) \to g(x, q_1, q_2) \text{ is convex}$$

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The approximation of the Fokker-Planck equation

$$\frac{\partial m}{\partial t} - \nu \Delta m - \mathrm{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \qquad (\dagger)$$

It is chosen so that

- each time step leads to a linear system for m with a matrix
  - whose diagonal coefficients are positive
  - whose off-diagonal coefficients are nonpositive

  in order to hopefully get a **discrete maximum principle**
- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian** g **should be used for** $(\dagger)$ **as well**

The approximation of the Fokker-Planck equation

$$\frac{\partial m}{\partial t} - \nu \Delta m - \operatorname{div}\left(m \frac{\partial H}{\partial p}(x, \nabla v) \right) = 0. \qquad (\dagger)$$

It is chosen so that

- each time step leads to a linear system for m with a matrix
  - whose diagonal coefficients are positive
  - whose off-diagonal coefficients are nonpositive

  in order to hopefully get a **discrete maximum principle**
- The argument for uniqueness should hold in the discrete case, so **the discrete Hamiltonian** g **should be used for** $(\dagger)$ **as well**

Principle

Discretize
$$-\int_{\mathbb{T}} \mathrm{div}\left(m\frac{\partial H}{\partial p}(x,\nabla u)\right) w = \int_{\mathbb{T}} m\frac{\partial H}{\partial p}(x,\nabla u)\cdot \nabla w$$

by
$$-h\sum_{i,j}\mathcal{T}_i(u,m)w_i \equiv h\sum_i m_i\nabla_q g(x_i,[D_hu]_i)\cdot[D_hw]_i$$

Discrete version of $\mathrm{div}(mH_p(x,\nabla u))$:

$$\mathcal{T}_i(u,m) = \frac{1}{h}\left(\begin{array}{c} m_i\frac{\partial g}{\partial q_1}(x_i,[D_hu]_i) - m_{i-1}\frac{\partial g}{\partial q_1}(x_{i-1},[D_hu]_{i-1}) \\ +m_{i+1}\frac{\partial g}{\partial q_2}(x_{i+1},[D_hu]_{i+1}) - m_i\frac{\partial g}{\partial q_2}(x_i,[D_hu]_i) \end{array}\right)$$

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Semi-implicit scheme

$$\left\{\begin{array}{l} \dfrac{u_i^{n+1}-u_i^n}{\Delta t}+\nu(\Delta_h u^n)_i-g(x_i,[D_h u^n]_i)=-\left(\Phi_h[m^n]\right)_i \\ \dfrac{m_i^{n+1}-m_i^n}{\Delta t}-\nu(\Delta_h m^{n+1})_i-\mathcal{T}_i(u^n,m^{n+1})=0 \end{array}\right.$$

The operator $m \mapsto -\nu(\Delta_h m)_i - \mathcal{T}_i(u,m)$ is the adjoint of the linearized version of $u \mapsto -\nu(\Delta_h u)_i + g(x_i,[D_h u]_i)$.

The discrete MFG system has the same structure as the continuous one.

Semi-implicit scheme

$$\left\{\begin{array}{l} \dfrac{u_i^{n+1}-u_i^n}{\Delta t}+\nu(\Delta_h u^n)_i - g(x_i,[D_h u^n]_i) = -\left(\Phi_h[m^{n+1}]\right)_i \\[2ex] \dfrac{m_i^{n+1}-m_i^n}{\Delta t}+\nu(\Delta_h m^{n+1})_i - \mathcal{T}_i(u^n,m^{n+1}) = 0 \end{array}\right.$$

Well known discrete Hamiltonians g can be chosen.

For example, if the Hamiltonian is of the form $H(x,\nabla u)=\psi(x,|\nabla u|)$, a possible choice is the **upwind scheme** :

$$g(x,q_1,q_2)=\psi\left(x,\sqrt{(q_1^-)^2+(q_2^+)^2}\right).$$

Theoretical results on the finite difference scheme

- Existence and a priori bounds (Lipschitz estimate if Φ is a smoothing operator)
- Uniqueness
- Convergence as $\Delta t, h \to 0$
- Solvers (a crucial issue)

Some references :

- Achdou and Capuzzo-Dolcetta ('10)
- Camilli-Silva ('14) : first order MFG.
- Achdou-Porretta ('15) : convergence in the local case.

Outline

Main Purpose

- Many models for crowd motion are inspired by statistical mechanics
- microscopic models : pedestrians = particles with more or less complex interactions (e.g. B. Maury et al)
- macroscopic models were recently proposed by T. Hughes et al
- in all these models, rational anticipation is not taken into account
- mean field games may lead to crowd motion models including rational anticipation

A Model for Congestion

$$
\begin{aligned}
-\frac{\partial u}{\partial t} - \nu \Delta u + H(x, Du, m) &= F(m) && \text{in } (0,T)\times\Omega \\
\frac{\partial m}{\partial t} - \nu \Delta m - \operatorname{div}\left(m \frac{\partial H}{\partial p}(\cdot, Du, m)\right) &= 0 && \text{in } (0,T)\times\Omega \\
\frac{\partial u}{\partial n} = \nu \frac{\partial m}{\partial n} + m \frac{\partial H}{\partial p}(\cdot, Du, m)\cdot n &= 0 && \text{on walls} \\
u = k \qquad m &= 0 && \text{at exits}
\end{aligned}
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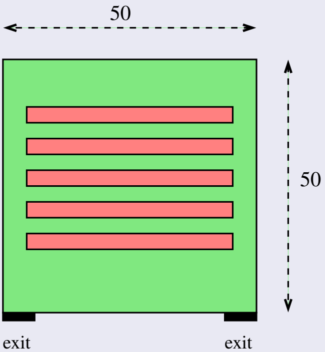
Congestion

$$H(x,p,m) = \mathcal{H}(x) + \frac{|p|^\beta}{(c_0 + c_1 m)^\alpha}$$

with $c_0 > 0$, $c_1 \geq 0$, $\beta > 1$ and $0 \leq \alpha < 4(\beta - 1)/\beta$

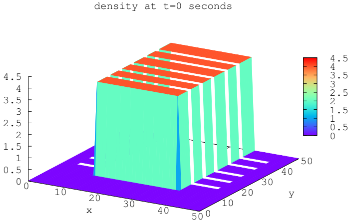
A Prototypical Case : Exit from a Hall with Obstacles

The geometry



The Data

- The initial density m_0 is piecewise constant and takes two values 0 and 4 people/m^2. There are 3300 people in the hall.



- $\nu = 0.012$
- $H(x,p,m) = \frac{8}{(1+m)^{\frac{3}{4}}}|p|^2 - \frac{1}{3200}$
- $F(m) \sim 0$

which leads to the following HJB equation

$$-\frac{\partial u}{\partial t} - \frac{6}{500}\,\Delta u + \frac{8}{(1+m)^{\frac{3}{4}}}\,|Du|^2 = \frac{1}{3200}$$

The Results

The horizon is $T = 40$ min. The two doors stay open from $t = 0$ to $t = T$.

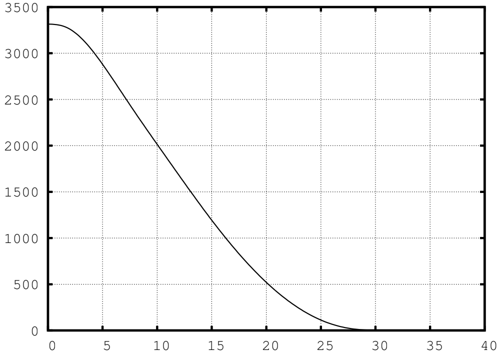


FIGURE: The number of people in the room vs. time

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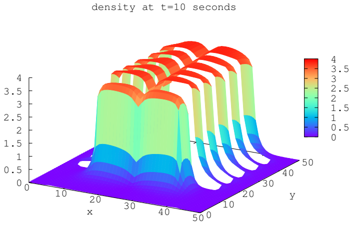
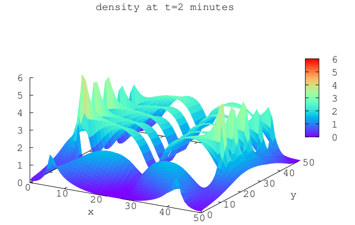
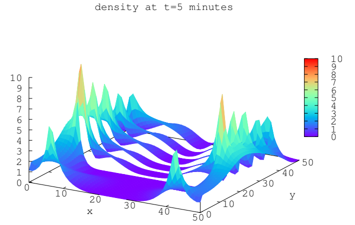
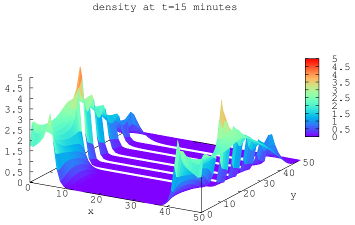
FIGURE: The evolution of the density

Evolution of the Distribution









(the scale varies w.r.t. t)

Exit from a Hall with a Common Uncertainty

Same geometry. The horizon is T.

- Before $t = T/2$, the two doors are closed.

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- At $T/2$, the probability that a given door be opened is $1/2$.

Hence the model involves three pairs of unknown functions

- (u^C, m^C) is defined on $(0, T/2) \times \Omega$ and corresponds to the situation when the room is closed.
- (u^L, m^L) and (u^R, m^R) are defined on $(T/2, T) \times \Omega$ and resp. correspond to the case when the left (resp. right) door is open.

The Boundary Value Problem

The systems of PDEs : for $j = C, L, R$,

$$-\frac{\partial u^j}{\partial t} - \nu \Delta u^j + H(m^j, Du^j) = F(m^j),$$

$$\frac{\partial m^j}{\partial t} - \nu \Delta m^j - \mathrm{div}\left(m^j \frac{\partial H}{\partial p}(m^j, Du^j)\right) = 0,$$

in $(0, T/2) \times \Omega$ for $j = C$ and in $(T/2, T) \times \Omega$ for $j = L, R$.

The boundary conditions

$$\frac{\partial u^C}{\partial n} = \frac{\partial m^C}{\partial n} = 0 \quad \text{on } (0, \frac{T}{2}) \times \partial\Omega,$$

and for $j = L, R$,
$$\begin{cases} \dfrac{\partial u^j}{\partial n} = \dfrac{\partial m^j}{\partial n} & = & 0 & \text{on } (\frac{T}{2}, T) \times \Gamma_N^j, \\ u^j = m^j & = & 0 & \text{on } (\frac{T}{2}, T) \times \Gamma_D^j \end{cases}$$

Transmission conditions at $t = T/2$

$$m^L(\frac{T}{2},x) = m^R(\frac{T}{2},x) = m^C(\frac{T}{2},x) \quad \text{in } \Omega,$$

$$u^C(\frac{T}{2},x) = \frac{1}{2}\left(u^L(\frac{T}{2},x) + u^R(\frac{T}{2},x)\right) \quad \text{in } \Omega.$$

Results

$T = 40$ min.

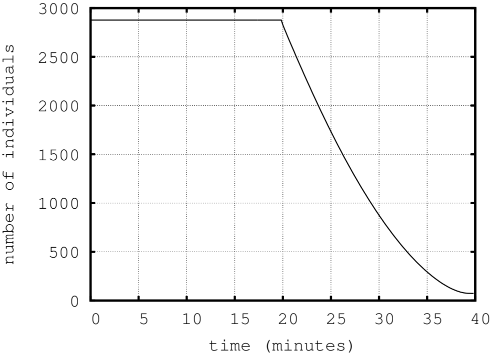

FIGURE: The number of people in the room vs. time

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- Such a behavior cannot be predicted by mechanical models
- Other examples with two populations (segregation, ...)

General conclusion

- MFG is a very active area in math, economics and engineering.
- Main equations :
  - The MFG system : the basic models are well-understood
    ... but not the more realistic ones,
    ... nor the stochastic MFG systems
  - The master equation remains very challenging
    ... as well as its finite dimensional analogue
- The mean field issue
- Still relatively little work on the numerical analysis
- Where to learn ?
  - Lasry-Lions papers, Lions' courses at the College de France (in French)
  - Few lecture notes and monographs : Achdou, C., Gomes and al., Bensoussan-Frehse-Yam...

Thank you !