A Dynamical Systems Approach to the Pleistocene Climate - Part I of II

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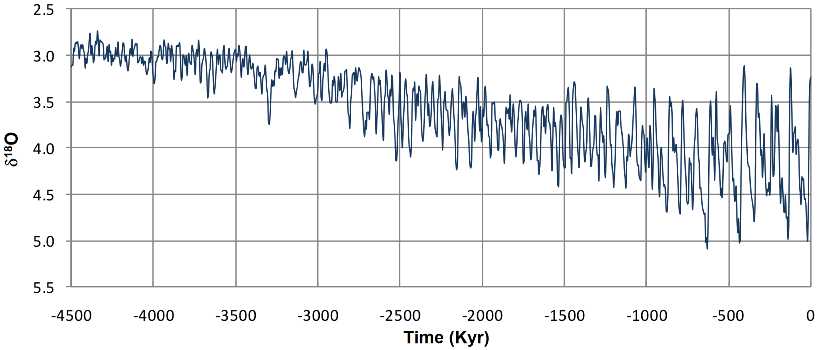
This Talk

- Background
  - Pleistocene
  - Glacial cycles
- Conceptual model
  - Maasch & Saltzman, 1990
- Dimension reduction
  - Time scales
  - Symmetry
- 2-D symmetric model
- Breaking the symmetry

Temperature Record, 4.5 Myr BP – Present



- Reconstructed from proxy data
- Oxygen isotope ratio, $\delta^{18}\mathrm{O} = \mathrm{O}^{18} / \mathrm{O}^{16}$

Pleistocene Epoch — 2.6 Myr–10K yr BP

- Early Pleistocene
  - Oscillatory behavior, period approximately 40 Kyr
  - Correlates with period of the *obliquity* of Earth's orbit
- Mid-Pleistocene Transition
  - Period changes from 40 Kyr to 100 Kyr
  - Amplitude increases
- Late Pleistocene
  - Oscillatory behavior, period approximately 100 Kyr
  - Correlates with period of the *precession* of Earth's orbit

Conceptual Models

- Saltzman and collaborators, 1988–1991
  - K.A. Maasch and B. Saltzman, J. Geophys. Res. 1990
- State variables (anomalies, dimensionless, rescaled)

x : total global ice mass

y : atmospheric CO_2 concentration

z : North Atlantic Deep Water (NADW)

- Dynamical system

$$\dot{x} = -x - y + f(t)$$
$$\dot{y} = (r - z^2)y - (p - sz)z$$
$$\dot{z} = -qx - qz$$
$$\dot{p} = \varepsilon_p, \ \dot{r} = \varepsilon_r, \quad 0 < \varepsilon_p, \varepsilon_r \ll 1$$

Dynamical System

$$
\begin{aligned}
\dot{x} &= -x - y + f(t) \\
\dot{y} &= (r - z^2)y - (p - sz)z \\
\dot{z} &= -qx - qz \\
\dot{p} &= \varepsilon_p, \;\; \dot{r} = \varepsilon_r, \quad 0 < \varepsilon_p, \varepsilon_r \ll 1
\end{aligned}
$$

- Time t measured in units of 10 Kyr
- Orbital (Milankovitch) forcing, $f(t)$
- Parameters p, q, r, s, all positive, $q > 1$
- Slowly varying parameters p and r

Internal Dynamics

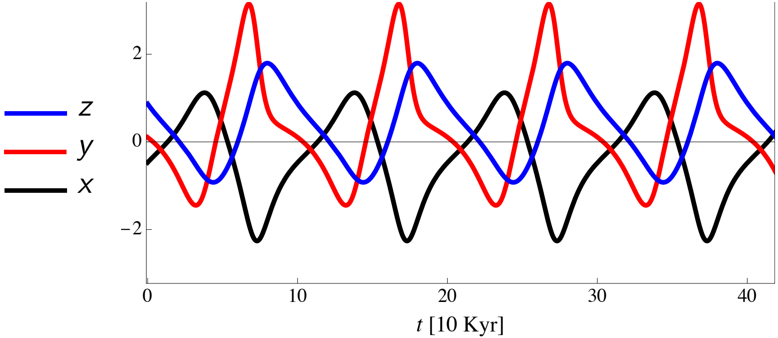
- Maasch–Saltzman model

$$\begin{aligned}
\dot{x} &= -x - y + f(t) \\
\dot{y} &= (r - z^2)y - (p - sz)z \\
\dot{z} &= -qx - qz \\
\dot{p} &= \varepsilon_p, \ \dot{r} = \varepsilon_r, \quad 0 < \varepsilon_p, \varepsilon_r \ll 1
\end{aligned}$$

- Autonomous dynamical system
  - No external forcing
  - No variation of parameters

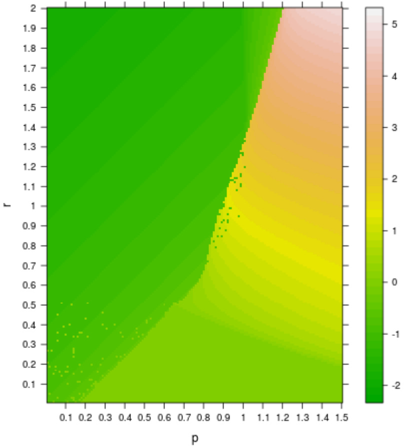
$$\begin{aligned}
\dot{x} &= -x - y \\
\dot{y} &= (r - z^2)y - (p - sz)z \\
\dot{z} &= -qx - qz
\end{aligned}$$

Computational Result



- Maasch & Saltzman: $p = 1.0$, $q = 1.2$, $r = 0.8$, $s = 0.8$
- Limit cycle, period 100 Kyr
- Approximately correct shape and order of events
  - Slow glaciation followed by rapid deglaciation
  - Deglaciation happens during temperature spike
  - Build-up of NADW during interglacial stage

Numerical Exploration – Equilibrium or Limit Cycle



- Fix $q = 1.2$, $s = 0.8$
- Vary $(p, r) \in \Omega$
- Integrate system of nonlinear ODEs, random initial data
- Color map $x^* = \limsup_{t\to\infty} x(t)$
- *Could come from equilibrium point or limit cycle*

Dimension Reduction

Autonomous MS $\xrightarrow{s=0}$ **Symmetric MS**

$$\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - (p - sz)z\\ \dot{z} &= -qx - qz\end{aligned}$$

$$\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - z^2)y - pz\\ \dot{z} &= -qx - qz\end{aligned}$$

$\downarrow \; q \gg 1$ $\qquad$ $\downarrow \; q \gg 1$

Asymmetric 2-D $\xrightarrow{s=0}$ **Symmetric 2-D**

$$\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + (p + sx)x\end{aligned}$$

$$\begin{aligned}\dot{x} &= -x - y\\ \dot{y} &= (r - x^2)y + px\end{aligned}$$

Symmetric 2-D System – Equilibrium States

Dynamical system

$$\dot{x} = -x - y$$
$$\dot{y} = (r - x^2)y + px$$

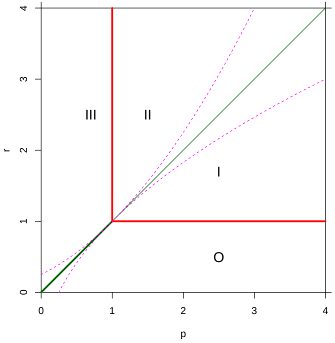
- Equivalent to Duffing–Van der Pol equation

$$\ddot{x} + g(x)\dot{x} + f(x) = 0$$

where $f(x) = x(x^2 - (r - p))$, $g(x) = x^2 - (r - 1)$

- Equilibrium states
  - Trivial state $P_0 = (0, 0)$ for all (p, r)
  - Nontrivial states $P_{1,2} = \sqrt{r - p}\,(\pm 1, \mp 1)$ if $r > p$
    - Generated in a pitchfork bifurcation along $r = p$
    - P_1 "cold" state, P_2 "warm" state

Linear Stability



- P_0 stable in O
  - $\{p > 1, r = 1\}$
  - Supercritical Hopf bifurcation
- $P_{1,2}$ stable in III
  - $\{p = 1, r > 1\}$
  - Subcritical Hopf bifurcation
- Bogdanov–Takens singularity
  - $(p, r) = (1, 1)$
  - "Organizing Center"

Focus on Organizing Center

- Blow up parameters

$$\begin{aligned} r - p &= \eta^2 \mu \\ r - 1 &= \eta^2 \lambda \end{aligned} \quad \Longrightarrow \quad r - 1 = m(p-1), \;\; m = \frac{\lambda}{\lambda - \mu}$$

- Rescale variables

$$t = \eta\tau, \quad x = \eta u, \quad -(x+y) = \eta^2 v$$

- Dynamical system near organizing center

$$\boxed{\begin{aligned} \dot{u} &= v \\ \dot{v} &= \mu u - u^3 + \eta(\lambda - u^2) v \end{aligned}}$$

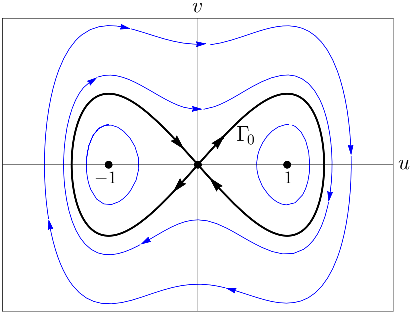
- Perturbed Hamiltonian system $(\eta > 0)$

$$H(u, v) = \tfrac{1}{2} v^2 + \tfrac{1}{4} u^4 - \tfrac{1}{2} \mu \, u^2$$

- Interesting case: $\mu > 0$ (wlog $\mu = 1$)

Melnikov Theory

- Hamiltonian (unperturbed) system
- Phase portrait

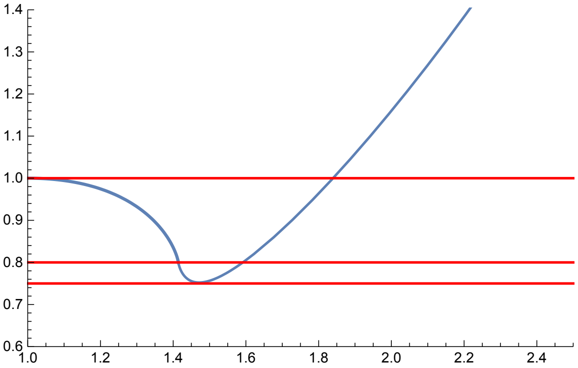


- Homoclinic and periodic orbits Γ through $(u_0, 0)$, $u_0 > 1$
- Melnikov function $M(\lambda, u_0) = \oint_\Gamma (\lambda - u^2)\, v(u)\, du$
- Global bifurcation set $\{(p, r) : r - 1 = m\,(p - 1)\}$

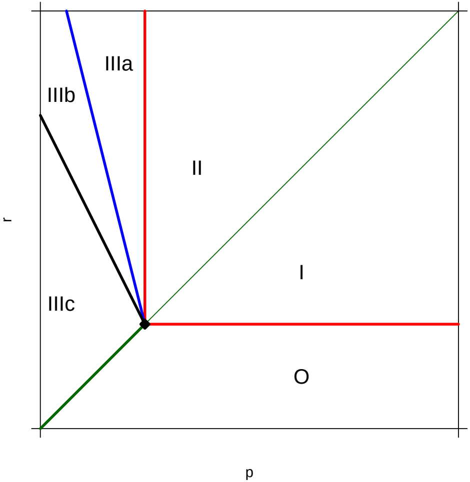
$$M(\lambda, u_0) = 0 \quad \Longrightarrow \quad \lambda = R(u_0), \quad m = \frac{\lambda}{\lambda - 1}$$

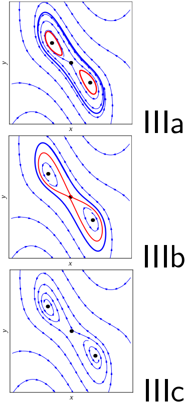
Zero Set of the Melnikov Function

- $M(\lambda, u_0) = 0 \quad \Longrightarrow \quad \lambda = R(u_0), \quad u_0 > 1$



Decomposition of Region III (sketch)

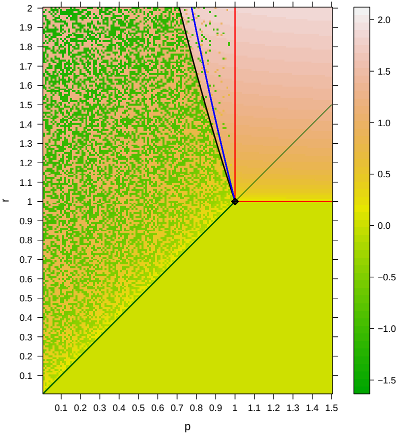




Limit Cycles

- Trivial state P_0
  - Stable in O
  - Loses stability at transition O $\rightarrow$ I
  - Supercritical Hopf bifurcation, generates limit cycles
  - Amplitude increases as (p, r) moves through I and II
  - Limit cycles persist in IIIa and IIIb
  - Limit cycles disappear at transition IIIb $\rightarrow$ IIIc
- Nontrivial states P_1, P_2
  - Emerge as (p, r) transits from I $\rightarrow$ II
  - Unstable in II, stable in III
  - Subcritical Hopf bifurcation
  - Generate unstable limit cycles in IIIa and IIIb
  - Affect the basins of attraction of stable limit cycles
- Stable limit cycles throughout O, I, II, IIIa, IIIb

Symmetric 2-D System – Limit Cycles



- Integrate ODEs, random initial data
- Use AUTO to find bifurcation curves
  - Hopf
  - Homoclinic
  - Saddle-node of limit cycles
- Color code

$$x^* = \limsup_{t\to\infty} x(t)$$

Breaking the Symmetry

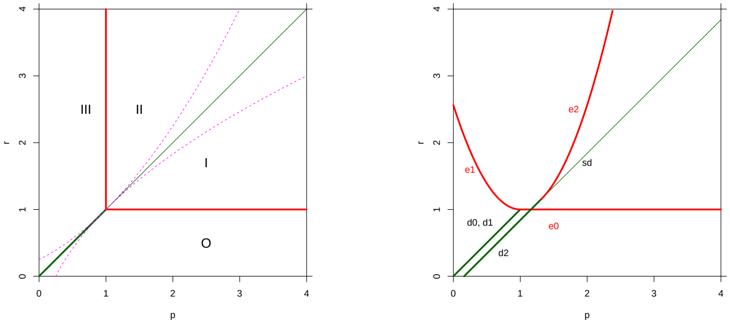
- Two-dimensional model with asymmetry ($s > 0$)

$$\dot{x} = -x - y,$$
$$\dot{y} = (p + s\,x)x + (r - x^2)y$$

- Equilibrium states
  - Trivial state $P_0 = (0,0)$ for all (p, r, s)
  - Nontrivial states if $r > p - \frac{1}{4}s^2$

$$P_1 = x_1^*\,(1,-1), \quad x_1^* = -\tfrac{1}{2}s + \tfrac{1}{2}\sqrt{s^2 + 4(r-p)}$$
$$P_2 = x_2^*\,(1,-1). \quad x_2^* = -\tfrac{1}{2}s - \tfrac{1}{2}\sqrt{s^2 + 4(r-p)}$$

Linear Stability

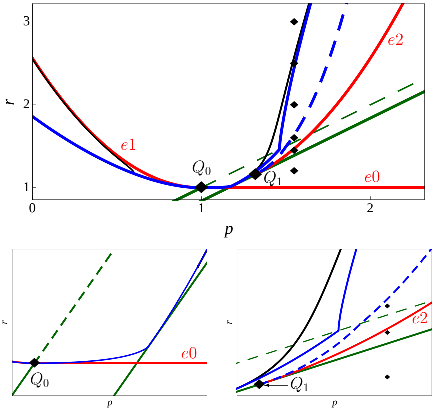


$s = 0$ $\qquad$ $s = 0.8$

- Bogdanov–Takens singularities

$$Q_0 = (1, 1), \quad Q_1 = (1 + \tfrac{1}{2}s^2, 1 + \tfrac{1}{4}s^2)$$

Stability Boundaries and Bifurcation Curves

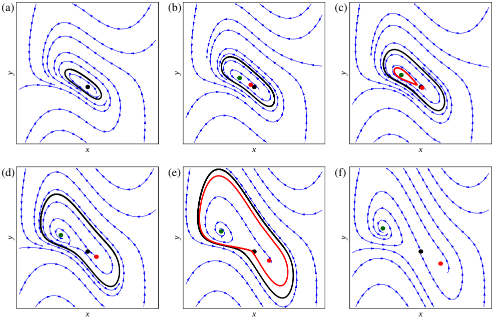


2.pdf

$s = 0.8$

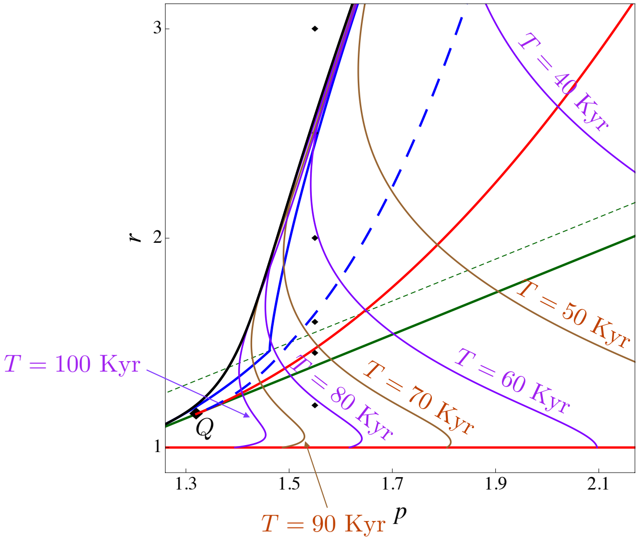
- Use AUTO to find bifurcation curves
  - Hopf
  - Homoclinic
  - Saddle-node of limit cycles

Phase Portraits



$$r = 1.2 \to 1.45 \to 1.6 \to 2.0 \to 2.5 \to 3.0$$
$$s = 0.8,\ p = 1.55$$

Isoperiod Curves



THANK YOU!