Normal forms for mechanical systems with Lie symmetries

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Canonical Mechanical Systems

(P, Ω) = a finite dimensional symplectic manifold

$H : P \to \mathbb{R}$ determines the dynamics:

$$d\Omega_z(X_H(z), v) = dH(z) \cdot v \quad \text{for all } v \in T_zP$$

Alternatively, one can use the Poisson bracket

$$\{F, H\} := \Omega(H_F, X_H) \qquad \text{for all} \quad F, H \in \mathcal{C}^\infty(P)$$

and then X_H is determined by $\dot{F} = \{F, H\}$ for all $F, H \in \mathcal{C}^\infty(P)$.

Mainly interested in $(T^*Q, \Omega_{can}) \equiv (T^*Q, \{\cdot,\cdot\}_{can})$

$$\dot{q} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial q}$$

(P, Ω_{can}) Dynamics near Equilibria

A known study method is the Poincaré-Birkhoff normalization with respect to the canonical bracket.

Using the antihomomorphism of the Lie algebras $\mathcal{C}^\infty(P)$ and the Hamiltonian vector fields $\mathcal{X}(P)$

$$[X_F, X_G] = -X_{\{F,G\}} \quad \forall \ F, G \in \mathcal{C}^\infty(P)$$

changes of coordinates for Hamiltonians are given by:

$$H \circ X_F^t = H + t\{F, H\} + \frac{1}{2!}t^2\{F, \{F, H\}\} + \dots$$

where X_F^t is the Hamiltonian flow of F.

Let x_0 be an equilibrium (wlog $x_0 = 0$). (In particular, $DH(x_0) = 0$.)

We apply iteratively changes of coordinates $H \to \hat{H}$ such that $\hat{H}$, the k-jet of $\hat{H}$, becomes

$$j^k \hat{H} = \hat{H}^{(2)} + \hat{H}^{(3)} + \ldots + \hat{H}^{(k)}$$

so that

$$\{H^{(2)}, \hat{H}^{(m)}\} = 0 \quad \forall\ m = 2, 3, \ldots k$$

Method: we *Taylor expand* H. For the term "$H^{(m)}$" of degree m we look for a homogeneous polynomial F of degree m so that

$$H^{(m)} + \{H_2, F\} = 0 \quad \text{(as much as possible)}$$

Having F, apply a time-1 flow X_F^1 change of coordinates and obtain the new H.

Symmetries

G = a (compact) Lie group acting freely and properly on Q
Denote $\mathfrak{g}$ and $\mathfrak{g}^*$ its Lie algebra and co-algebra, respectively.

The *momentum map* is $J : P \to \mathfrak{g}^*$ such that for any $\omega \in \mathfrak{g}$ the Hamiltonian vector field of J_ω where $J_\omega(z) := \langle J(z), \omega \rangle$ satisfies

$$X_{J_\omega}(z) = \omega_P(z) = \frac{d}{dt}\Big|_{t=0} \exp(t\omega) \cdot (z)$$

E.g. N–body problems in $\mathbb{R}^3$: $G = SO(3)$, $\mathfrak{g} \simeq \mathbb{R}^3$, $\mathfrak{g}^* \simeq \mathbb{R}^3$

$$J_\omega(q,p) = \sum \langle p_i, \omega \times q_i \rangle \quad \text{and} \quad J(q \times p) = \sum q_i \times p_i$$

Theorem (Noether) If $H : P \to \mathbb{R}$ is G-invariant, then J is conserved along the motion.

Special solutions

Definition: a *relative equilibrium* is a solution that is also a group orbit; that is, there exist $\omega \in \mathfrak{g}$ and $z_0 \in T^*Q$ such that

$$z(t) = \exp(t\omega) z_0$$

is a solution.

E.g. For N-body problems,

$$(q(t), p(t)) = R(t) \cdot (q_0, p_0) \quad \text{where} \quad R(t) = \exp(t\omega)$$

for some fixed angular velocity $\omega \in \mathbb{R}^3 \simeq so(3)^*$.

Co-tangent Bundle Reduction (with $G = SO(3)$)

$(T^*Q, \Omega_{can}, SO(3))$. The momentum map is $J : T^*Q \to so(3)^*$.

N-body systems: $J(q,p) = \sum q_i \times p_i$.

H invariant $\Rightarrow$ for each momentum $\mu \in so(3)^*$

$J^{-1}(\mu) := \{(q,p) \mid J(q,p) = \mu\}$ are invariant submanifolds.

Fix $\mu_0 = J(q \times p) \in so(3)^*$ (e.g. a rotation about Oz).

$J(q \times p) = \mu_0 = R_z\mu_0 = J(R_zq, R_zp) \quad \forall R_z =$ Rot. about Oz

$\Longrightarrow$ $J^{-1}(\mu_0)$ quotients by the subgroup of vertical rotations

$SO(3)_{\mu_0} := \{R \in SO(3) \mid R\mu_0 = \mu_0\}$ isotropy group of μ_0

Let $\mu_0 \in so(3)^*$ and $SO(3)_{\mu_0}$ its isotropy group. Then $J^{-1}(\mu_0)$ quotients by $SO(3)_{\mu_0}$ and, provided μ_0 is a regular value for J, the *reduced* space

$$(T^*Q)_{\mu_0} := J^{-1}(\mu_0)/SO(3)_{\mu_0}$$

is a smooth manifold.

Theorem (Meyer; Marsden-Weinstein)

There is a unique symplectic structure Ω_{μ_0} on $(T^*Q)_{\mu_0}$ such that for every G-invariant Hamiltonian H, dynamical solutions of $(T^*Q, \Omega_{can}, H, G)$ project into dynamical solutions of $((T^*Q)_{\mu_0}, \Omega_{\mu_0}, h)$ where $h \circ \pi = H$.

In general, we want to know: dynamics in the reduced space, its reconstruction to the un-reduced space, understand the mechanism of symmetry-breaking perturbations, etc.

Relative equilibria = equilibria in the reduced space.

For non-symmetric systems, the main method in use is the Poincaré-Birkhoff normalization near an equilibrium.

For symmetric co-tangent bundle systems, (local) Darboux coordinates exist for both the unreduced and the symplectic reduced spaces. We want to "embed" the reduced space $((T^*Q)_{\mu_0}, \Omega_{\mu_0})$ in (T^*Q, Ω_{can}) in a particular way.

More on reduction (with $G = SO(3)$)

$(T^*Q, \Omega_{can}, SO(3)) \longrightarrow$ the reduced space$((T^*Q)_{\mu_0}, \Omega_{\mu_0})$.

[1] $$(T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\,(SO(3))_{\mu_0} \longrightarrow T^*\left(Q/\,(SO(3))_{\mu_0}\right)$$

where one uses a *shift* map $(q,p) \to (q,p) - \mathcal{A}_{\mu_0}(q)$.

Then $\Omega_{\mu_0} = \omega_{can} - \beta_{\mu_0}$. *Non-canonical*, unless $\mu = 0$.

[2] $$(T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\,(SO(3))_{\mu_0} \simeq T^*(Q/SO(3)) \times \mathcal{O}_{\mu_0}$$

where $Q/SO(3) :=$ the shape space and

$$\mathcal{O}_{\mu_0} := \{R\mu_0 \mid R \in SO(3)\} = \text{a 2-sphere of radius } |\mu_0|$$

$(T^*(Q/G), \Omega_{can})$ and $(\mathcal{O}_{\mu_0}, \Omega_{can})$

Adopt [2], since it comes with a canonical symplectic form.

$$(T^*Q)_{\mu_0} = J^{-1}(\mu_0)/\,(SO(3))_{\mu_0} \simeq T^*\,(Q/SO(3)) \times \mathcal{O}_{\mu_0}$$

where

$$Q/SO(3) := \text{the shape space}$$

$$\mathcal{O}_{\mu_0} := \{R\mu_0 \mid R \in SO(3)\} = \text{a 2-sphere of radius } |\mu_0|$$

Easiest case: $Q = SO(3)$. In plain words, the *rigid body*.

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/\,(SO(3))_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

Free rigid body with a fixed point

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/(SO(3))_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

$$T^*SO(3) \to SO(3) \times so(3)^* \simeq SO(3) \times \mathbb{R}^3$$

$$(\Sigma, P) \;\to\; (\Sigma, \Sigma^{-1}P) \;=\; (\Sigma, \mu)$$

body coordinates

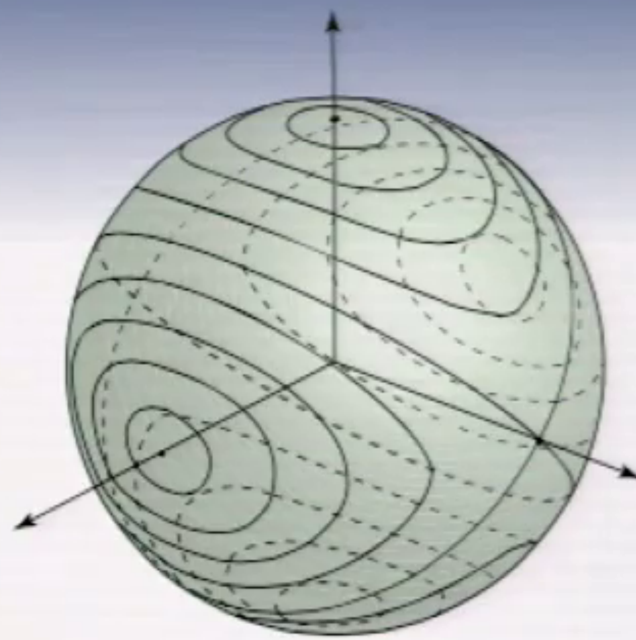
Let $\mathbb{I}_1$, $\mathbb{I}_2$, $\mathbb{I}_3$ be the principal moments of inertia of the body.

$$H(\Sigma, \mu) = H(\mu) = \frac{1}{2}\left(\frac{\mu_1^2}{\mathbb{I}_1} + \frac{\mu_2^2}{\mathbb{I}_2} + \frac{\mu_3^2}{\mathbb{I}_3}\right)$$

Spatial angular momentum is conserved $\Longrightarrow \frac{d}{dt}(\Sigma\mu) = 0 \Longleftrightarrow$

$\dot{\mu} = \mu \times (\mathbb{I}^{-1}\mu)$ Euler's equations and $|\mu| = const. =: \mu_0$

The Poisson reduced space are $(SO(3) \times so(3)^*)/SO(3) = so(3)^*$.
The symplectic leafs are given by 2-spheres.



$$\mathcal{O}_{\mu_0} = \{R\mu_0 \mid R \in SO(3)\} = \text{ sphere of radius } |\mu_0|$$

$$H(R,\mu) \equiv h(\mu) = \frac{1}{2}\left(\frac{\mu_1^2}{\mathbb{I}_1} + \frac{\mu_2^2}{\mathbb{I}_2} + \frac{\mu_3^2}{\mathbb{I}_3}\right) = const.$$

The symplectic reduced spaces

$$(T^*SO(3))_{\mu_0} = J^{-1}(\mu_0)/\,(SO(3))_{\mu_0} \simeq \mathcal{O}_{\mu_0}$$

1) Canonical coordinates on the sphere $\mathcal{O}_{\mu_0}$ may be defined alright (obviously, one needs two charts).

2) If we are interested in the dynamics in the full phase space, we can use the celestial mechanics "regularized" Serret-Andoyer-Deprit coordinates.

Note: for symmetric systems with more general Lie symmetries, we need a systematic approach.

Slice Theorems → a symmetry-adapted framework

Theorem (Symplectic Slice Theorem - free action)

Consider $\mathcal{P}$ be a symplectic manifold, $p_0 \in \mathcal{P}$ a RE with momentum μ_0, and let $\mathcal{N}$ a normal space transverse to $G \cdot p_0$ and p_0, i.e.

$$T_{p_0}P \overset{loc.}{=} T_{p_0}(G\,p_0) \oplus \mathcal{N}$$

There is a choice of $\mathcal{N}$ and coordinates such that near $G\,p_0$ we have $\mathcal{N} = \mathcal{N}_0 \oplus \mathcal{N}_1 \simeq \mathfrak{g}^*_{\mu_0} \oplus (\ker DJ(\mu_0) \cap \mathcal{N})$ s. t. $p_0 \simeq (e, 0, 0)$,

$$p \overset{loc.}{\simeq} (g, \nu, w) \in G \times \mathfrak{g}^*_{\mu_0} \times \mathcal{N}_1$$

$$\begin{aligned} \dot{g} &= g\, D\nu h(\nu, w) & \dot{R} &= R\, D\nu h(\nu, w) \\ \dot{\nu} &= ad^*_{D_\nu h(\nu, w)} \nu & \dot{\nu} &= \nu \times D_\nu h(\nu, w) \\ \dot{w} &= \mathbb{J}_{\mathcal{N}_1} D_w h(\nu, w) & \dot{w} &= \mathbb{J}\, D_w h(\nu, w) \end{aligned}$$

In this framework there is quite a body of work - long bibliography - . Some relevant papers (the free case only!):

M. Roberts and de M.E.R. Sousa Dias: *Bifurcations from relative equilibria of Hamiltonian systems*, Nonlinearity, 10, 1997

J.P. Ortega and T. Ratiu: Stability of Hamiltonian relative equilibria, Nonlinearity, 12, 1999

C. Wulff, A. Schebesch: Numerical continuation of Hamiltonian relative periodic orbits, J. Nonl. Science 18, 2008.

C. Wulff and F. Schilder: *Numerical bifurcation of Hamiltonian relative periodic orbits*, SIAM J. Appl. Dyn. Syst., 8, 2009.

For co-tangent bundles, the slice framework is great at the theoretical level, but there are no *constructive* slice theorems (even for free actions) except for

- abelian groups - no problems - essentially go into “rotating” coordinates

- for compact groups at zero momentum $\rightarrow$ T. Schmah: *A cotangent bundle slice theorem*, Diff. Geom. Appl. 25, 2007

- for $SO(3)$ $\rightarrow$ T. Schmah & C.S.: *Normal forms for Lie Symmetric Cotangent Bundle Systems with Free and Proper Actions*, in Fields Institute Communications series, Vol. “Geometry, Mechanics and Dynamics: the Legacy of Jerry Marsden”, Springer 2015

The rigid body case

$$
\begin{array}{ccc}
T^*SO(3) & \to & SO(3) \times so(3)^* \\
(R, P) & \to & (R, \mu)
\end{array}
$$

$$
\begin{array}{ccccc}
so(3)^* = (so(3)^*)_{\mu_0} \times so(3)^{\perp}_{\mu_0} \simeq (so(3)^*)_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0} \\
\mu \quad \leftrightarrow \quad (\nu, \qquad (\eta_x, \eta_y))
\end{array}
$$

Look for a $SO(3)$-equivariant symplectic diffeomorphism

$$
(SO(3) \times so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0}, \Omega_Y) \longrightarrow (SO(3) \times so(3)^*, \Omega_{can}) ,
$$

$$
\text{such that } (\mathrm{Id}, 0, 0) \longrightarrow (\mathrm{Id}, \mu_0),
$$

Theorem (A constructive symplectic tube for $SO(3)$, Schmah 2007)

The following is an $SO(3)$-equivariant symplectic local diffeomorphism with respect to the symplectic form

$$\Omega_Y(R,\nu,\eta)\left((\xi_1,\dot\nu_1,\eta_1),(\xi_2,\dot\nu_2,\eta_2)\right)$$
$$:= \langle \mu_0+\nu,[\xi_1,\xi_2]\rangle + \langle \dot\nu_2,\xi_1\rangle - \langle \dot\nu_1,\xi_2\rangle - \langle \mu_0,[\eta_1,\eta_2]\rangle$$

in a neighbourhood of $(Id,0,0)$:

$$\phi : SO(3)\times so(3)^*_{\mu_0}\times so(3)^{\perp}_{\mu_0} \longrightarrow SO(3)\times so(3)^*,$$
$$(R,\nu,\eta)\longrightarrow \left(RF(\nu,\eta)^{-1}, F(\nu,\eta)(\mu_0+\nu)\right)$$

$$F(\nu,\eta) := \exp\left(\theta\frac{\hat\eta}{\|\eta\|}\right), \quad \sin\left(\frac{\theta}{2}\right) := \frac{\|\eta\|}{2}\sqrt{\frac{\|\mu_0\|}{\|\mu_0+\nu\|}}.$$

$(R, \nu, \eta) \in SO(3) \times so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0}$ in Darboux coordinates:

$$\Omega_Y(R, \nu, \eta) = \begin{pmatrix} -(\mu_0 + \nu)\mathbb{J} & 0 & 0 \\ 0 & \mathbb{J} & 0 \\ 0 & 0 & -\mu_0\mathbb{J} \end{pmatrix}$$

$$\dot{R} = R\left[\begin{pmatrix} 0 & -(\mu_0 + \nu) & 0 \\ (\mu_0 + \nu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\left(R^{-1}\frac{\partial H}{\partial R}\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{\partial H}{\partial \nu} \end{pmatrix}\right],$$

$$\dot{\nu} = -\left(R^{-1}\frac{\partial H}{\partial R}\right)_z, \qquad \begin{pmatrix} \dot{\eta}_x \\ \dot{\eta}_y \end{pmatrix} = -\frac{1}{\mu_0}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \partial_{\eta_x} H \\ \partial_{\eta_y} H \end{pmatrix}$$

The spatial angular momentum: $J^S(\Sigma, \mu) = \Sigma\mu = R(\mu_0 + \nu)$.

SO(3)-symmetric systems on $T^*SO(3)$

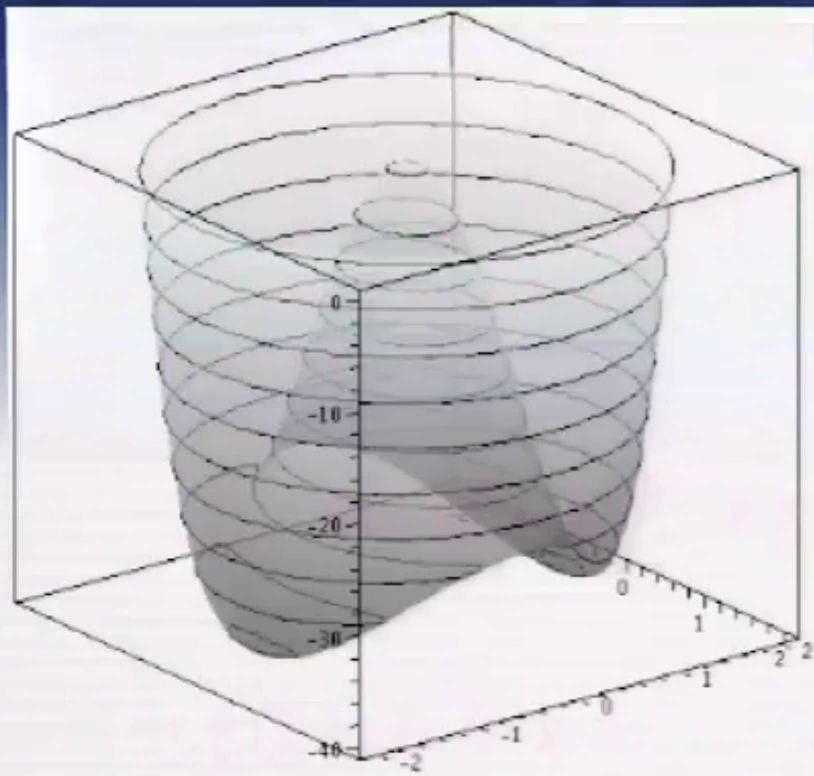
Let $H : T^*SO(3) \to \mathbb{R}$ be $SO(3)$-invariant.

$$h : so(3)^*_{\mu_0} \times T_{\mu_0}\mathcal{O}_{\mu_0} \simeq so(3)^* \to \mathbb{R}\,, \quad h = h(\nu, \rho) = h(\mu)$$

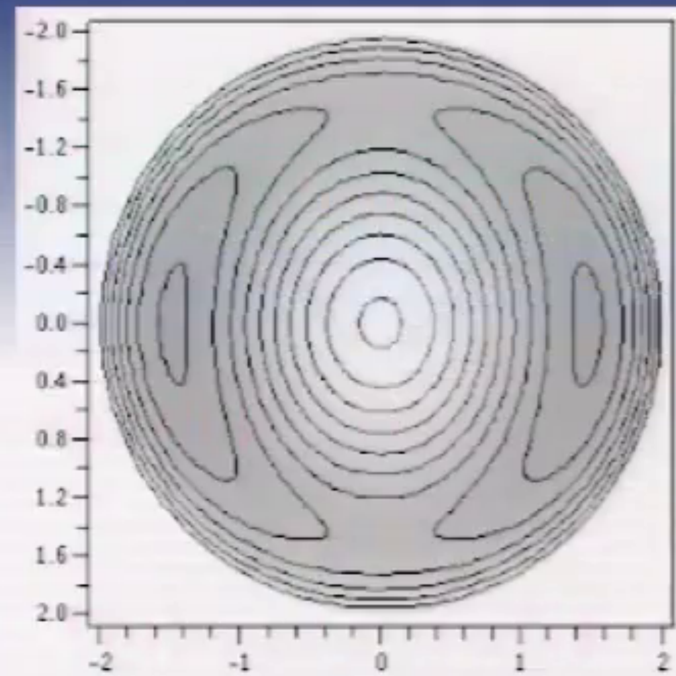
$$\xi_z = \partial_\nu h\Big|_{(\nu=\nu_0\,,\,\eta(t))}$$

$$\dot{\nu} = 0 \implies \nu = const. = \nu_0 \implies h = h(\eta\,;\nu_0)$$

$$\dot{\eta} = -\frac{1}{\mu_0}\mathbb{J}\,\partial_\eta h$$



(a) 3-d view



(b) top view

$$h(\eta_x,\eta_y;\nu_0) = \frac{1}{2}\mu_0(\mu_0+\nu_0)\left(1-\frac{\mu_0}{4(\mu_0+\nu_0)}\right)\left(\eta_x^2+\eta_y^2\right)\left(\frac{\eta_y^2}{\mathbb{I}_1}+\frac{\eta_x^2}{\mathbb{I}_2}\right)$$

$$+\frac{(\mu_0+\nu_0)^2}{2\mathbb{I}_3}\left(\;\; {}^2 \;\; {}^2 \right)$$

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Coupled systems (e.g. N-body problems)

Applying a slice theorem $\implies S \overset{loc.}{\simeq} Q/SO(3)$ shape space (or internal space)

$$SO(3) \times S \overset{loc.}{\simeq} SO(3) \times Q/SO(3) \simeq Q$$

$$SO(3) \times S \overset{loc.}{\simeq} Q \Rightarrow \ \dots \ \Rightarrow T^*SO(3) \times T^*S \overset{loc.}{\simeq} T^*Q$$

Local coordinates ("body" coordinates)

$$(\Sigma, \mu, (s, \sigma)) \in SO(3) \times so^*(3) \times T^*S \simeq T^*SO(3) \times T^*S$$

Symplectic slice coordinates: $(\Sigma, \mu, (s, \sigma)) \to (R, \nu, \eta, (s, \sigma))$

$$(R,\nu,\eta,(s,\sigma)) \in \left(SO(3)\times so(3)^*_{\mu_0}\times T_{\mu_0}\mathcal{O}_{\mu_0}\right)_{\Omega_Y}\times T^*S_{\Omega_{can}} \overset{loc.}{\simeq} T^*Q_{\Omega_{can}}$$

$$\dot{R} = R\left[\begin{pmatrix} 0 & -(\mu_0+\nu) & 0 \\ (\mu_0+\nu) & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\left(R^{-1}\frac{\partial H}{\partial R}\right) + \begin{pmatrix} 0 \\ 0 \\ \frac{\partial H}{\partial \nu} \end{pmatrix}\right]$$

$$\dot{\nu} = -\left(R^{-1}\frac{\partial H}{\partial R}\right)_z, \quad \dot{\eta} = -\frac{1}{\mu_0}\mathbb{J}\,\partial_\eta H, \quad \begin{pmatrix} \dot{s} \\ \dot{\sigma} \end{pmatrix} = \mathbb{J}\begin{pmatrix} \frac{\partial H}{\partial s} \\ \frac{\partial H}{\partial \sigma} \end{pmatrix}$$

If $H(R,\nu,\eta,s,\sigma) \equiv h(\nu,\eta,s,\sigma) \implies$

$$\nu(t) = \nu_0 \text{ and } h = h(\eta,(s,\sigma);\nu_0).$$

Further, one can look at symmetric "kinetic+potential"...

The general case: free and proper Lie group actions

We want a constructive method to find a G-equivariant symplectic diffeomorphism, called the *tube*,

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y \right) \longrightarrow (G \times \mathfrak{g}^*, \Omega_{can}) \,,$$

$$\text{such that } (e, 0, 0) \to (e, \mu_0)$$

Lucky to find this map in general ! $SO(3)$ is quite special and the calculations lead to the (regularized) Serret-Andoyer-Deprit celestial mechanics coordinates.

Key relation

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y\right) \longrightarrow (G \times \mathfrak{g}^*, \Omega_{can}) \,,$$

$$\text{such that } (e, 0, 0) \to (e, \mu_0)$$

$$\phi^*\Omega_{can} = \Omega_Y \Rightarrow \ldots \Rightarrow \phi(g, \nu, \eta) = \left(gF(\nu, \eta)^{-1}, \, \mathrm{Ad}^*{}_{F(\nu,\eta)}(\mu_0 + \nu)\right)$$

for some $F : \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0} \to G$. Moreover, F must be of the form

$$F(\nu, \eta) = \exp\left(h(\nu, \eta)\, \frac{\eta}{\|\eta\|}\right)$$

for some $h : \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0} \to \mathbb{R}$.

$$F(\nu,\eta) = \exp\left(h(\nu,\eta)\,\frac{\eta}{\|\eta\|}\right),\ \nu \in \mathfrak{g}^*_{\mu_0} \simeq \mathcal{M}(\mathbb{R}^?),\ \eta \in \mathfrak{g}^{\perp}_{\mu_0} \simeq \mathcal{M}(\mathbb{R}^?)$$

must satisfy

$$\begin{aligned}&\left\langle \mu_0 + \nu, \left[F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_1,\zeta_1)\right), F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_2,\zeta_2)\right)\right]\right. \\ &+ \left\langle \dot\nu_2, F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_1,\zeta_1)\right)\right\rangle \\ &- \left\langle \dot\nu_1, F(\nu,\eta)^{-1}\left(DF(\nu,\eta)\cdot(\dot\nu_2,\zeta_2)\right)\right\rangle = \langle \mu_0, [\zeta_1,\zeta_2]\rangle\,.\end{aligned}$$

One may compute: $DF(\nu,\eta)\Big|_{(0,0)}$. Then take the derivative of the above and compute $D^2F(\nu,\eta)\Big|_{(0,0)}$, and so forth...

Unlikely to find F globally, but one can calculate the its derivatives at $(0,0)$.

So, while it is unlikely to "guess" a general formula for the tube

$$\phi : \left(G \times \mathfrak{g}^*_{\mu_0} \times \mathfrak{g}^{\perp}_{\mu_0}, \Omega_Y\right) \longrightarrow (G \times \mathfrak{g}^*, \Omega_{can}),$$
$$(e,0,0) \to (e,\mu_0)$$

$$\phi(g,\nu,\eta) = \left(gF(\nu,\eta)^{-1}, \mathrm{Ad}^*{}_{F(\nu,\eta)}(\mu_0+\nu)\right)$$

one may compute its derivatives at the (relative equilibrium) base point.

Poincaré-Birkhoff normal forms

...recall that it is a method based on canonical changes of coordinates which are applied to a *truncated Taylor expansion* at the equilibrium of the Hamiltonian.

At each step $H \to \hat{H}$ the k-jet of $\hat{H}$ at the equilibrium becomes

$$j^k \hat{H} = \hat{H}^{(2)} + \hat{H}^{(3)} + \ldots + \hat{H}^{(k)}$$

so that $\{H^{(2)}, \hat{H}^{(i)}\} = 0 \quad \forall\ i = 2, 3, \ldots k$.

$$H_{\text{tube}}(R, \nu, \eta) = (H \circ \phi)(\Sigma, \mu)$$

Knowing the derivatives at $(e, 0, 0)$ of the tube ϕ (*and these can be calculated!*) is sufficient for calculating the normal form near a relative equilibrium.

Observations and speculations

[1] The geometric splitting here is not the same with the one on the Reduced-Energy Momentum for stability of Marsden & co-workers.

[2] Conjecture: there is an explicit formula for the change of coordinates map (the tube) for all super-integrable systems on Lie group which accept (global) action-angle coordinates of super-integrable systems (e.g. Toda-lattice; see Tony Bloch's talk here).

[3] Conjecture: in all rotationally-invariant cotangent-bundle systems, (non-linearly) stable relative equilibria are Nekhoroshev long term stable. This is suggested by the results of Benettin & al. on the stability of the perturbed free rigid-body near a relative equilibrium.