Reformulating spectral problems with the Krein matrix

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My collaborators:

- Shamuel Auyeung, Eric Yu (Calvin undergraduate students)
- B. Deconinck (U. Washington), P. Miller (U. Michigan)

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- Calvin College
  - Jack and Lois Kuipers Applied Mathematics Endowment
  - Calvin Research Fellowship
- NSF DMS-1108783

Goal is to understand the dynamics associated with perturbations of (small) spatially 2π-periodic waves to Klein-Gordon-like (KG) equations:

$$\partial_t^2 u + \mathcal{M} u + f(u) = 0, \quad \mathcal{M} = \sum_{j=0}^{N} a_j \partial_x^{2j}, \text{ and } |f(u)| = \mathcal{O}(u^2).$$

Ideas also applicable to:

- KdV-like: $\partial_t u + \partial_x \left(\mathcal{M} u + f(u, \partial_x u, \partial_x^2 u, \dots) \right) = 0$
- NLS-like

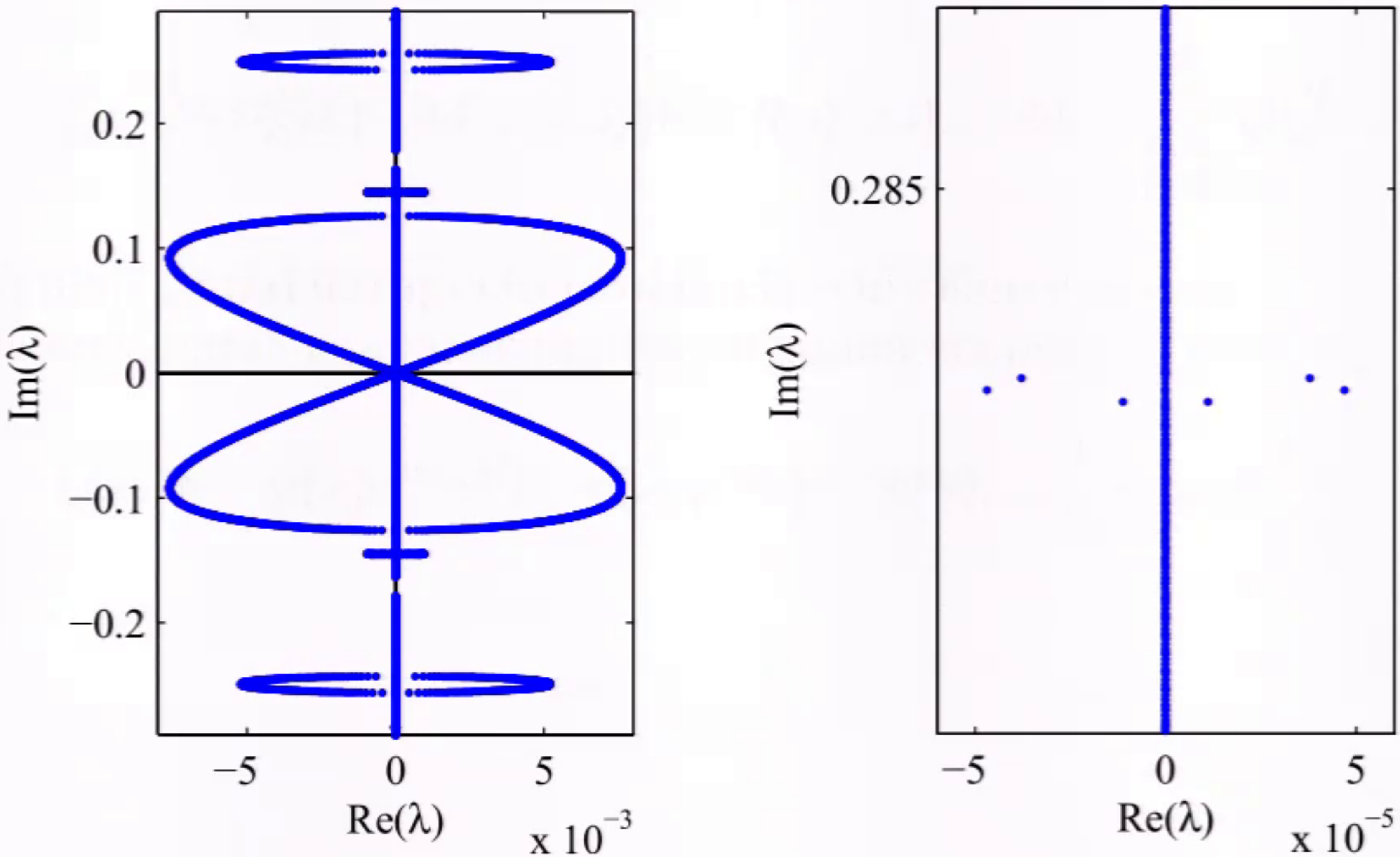
In traveling coordinates, $z = x - ct$, KG becomes

$$\partial_t^2 u - 2c\partial_{tz}^2 u + (\mathcal{M} + c^2\partial_z^2)u + f(u) = 0, \quad \mathcal{M} = \sum_{j=0}^{N} a_j \partial_z^{2j}.$$

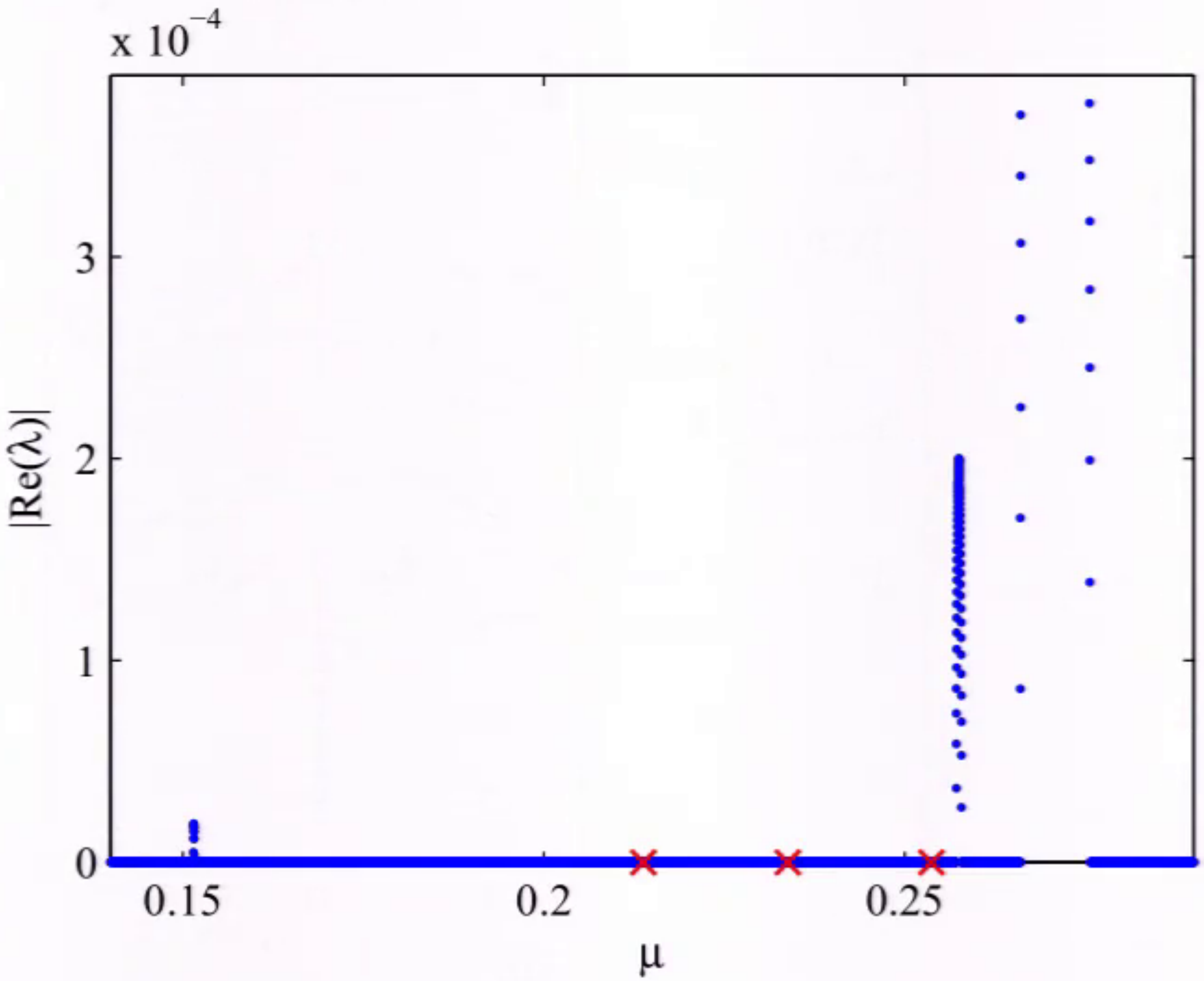
To understand the spectrum of the linearization use a Bloch-wave decomposition of the eigenfunctions:

$$u(z,t) = \psi(z)\mathrm{e}^{\mathrm{i}\mu z}\mathrm{e}^{\lambda t}; \quad \psi(z+2\pi) = \psi(z),\ -\frac{1}{2} < \mu < \frac{1}{2}.$$

KG spectrum with wave amplitude ~ 0.16:



KG spectrum as a function of Bloch-wave parameter μ with wave amplitude ~ 0.11:



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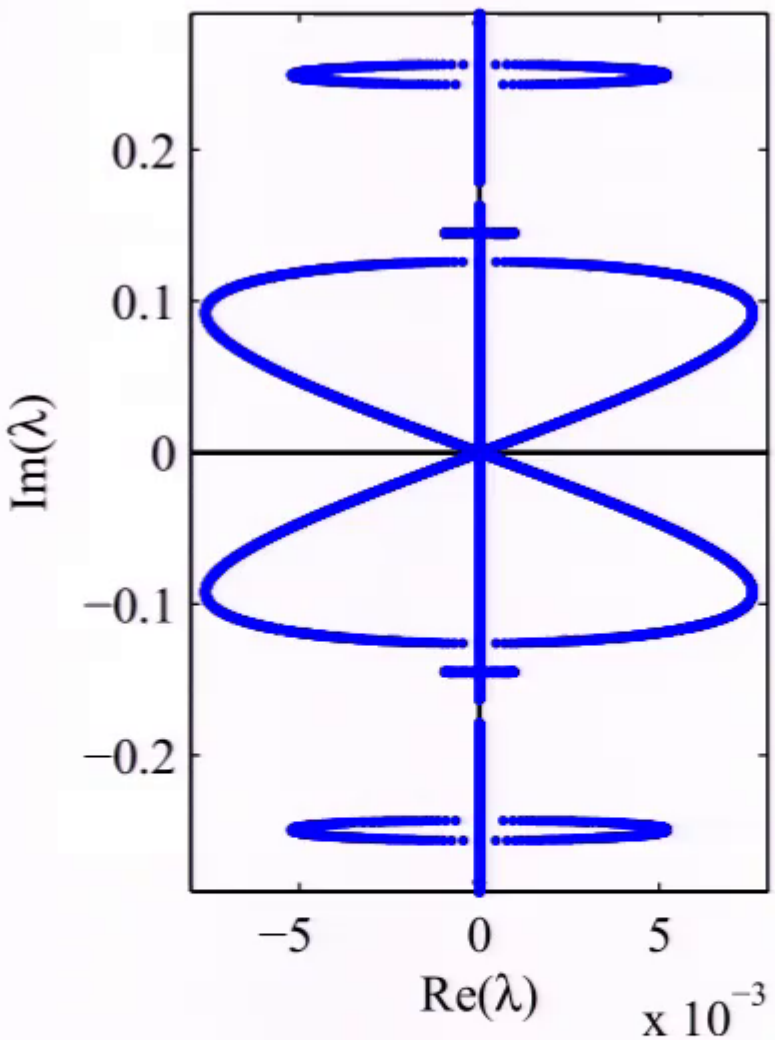
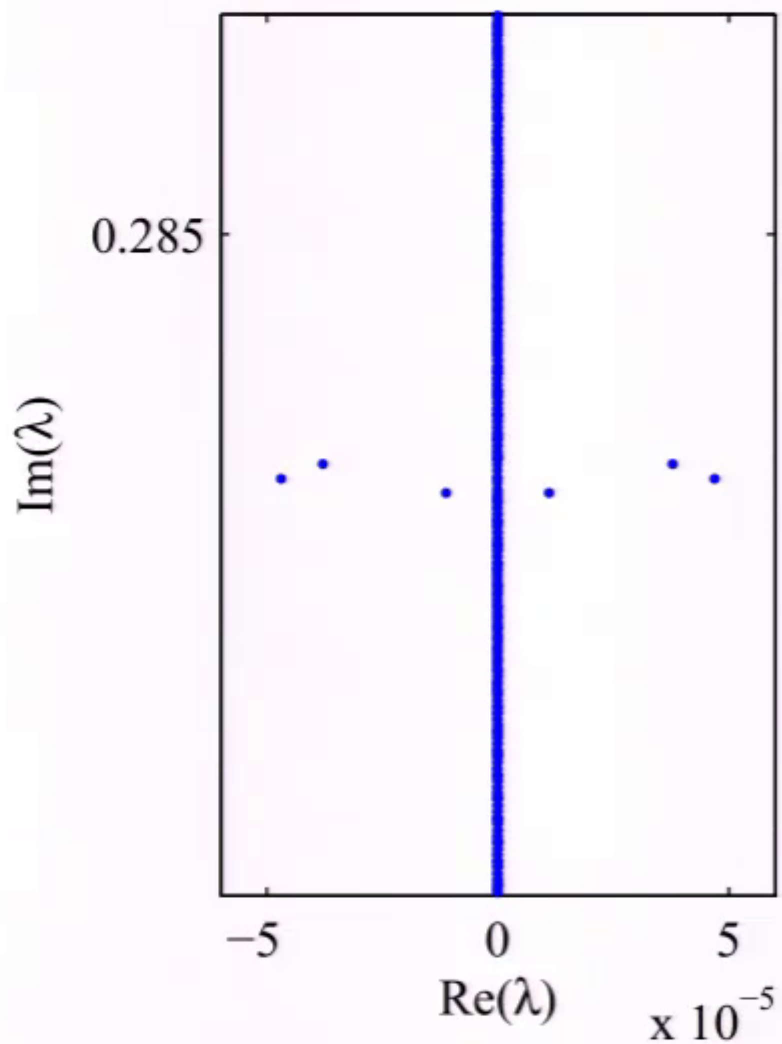
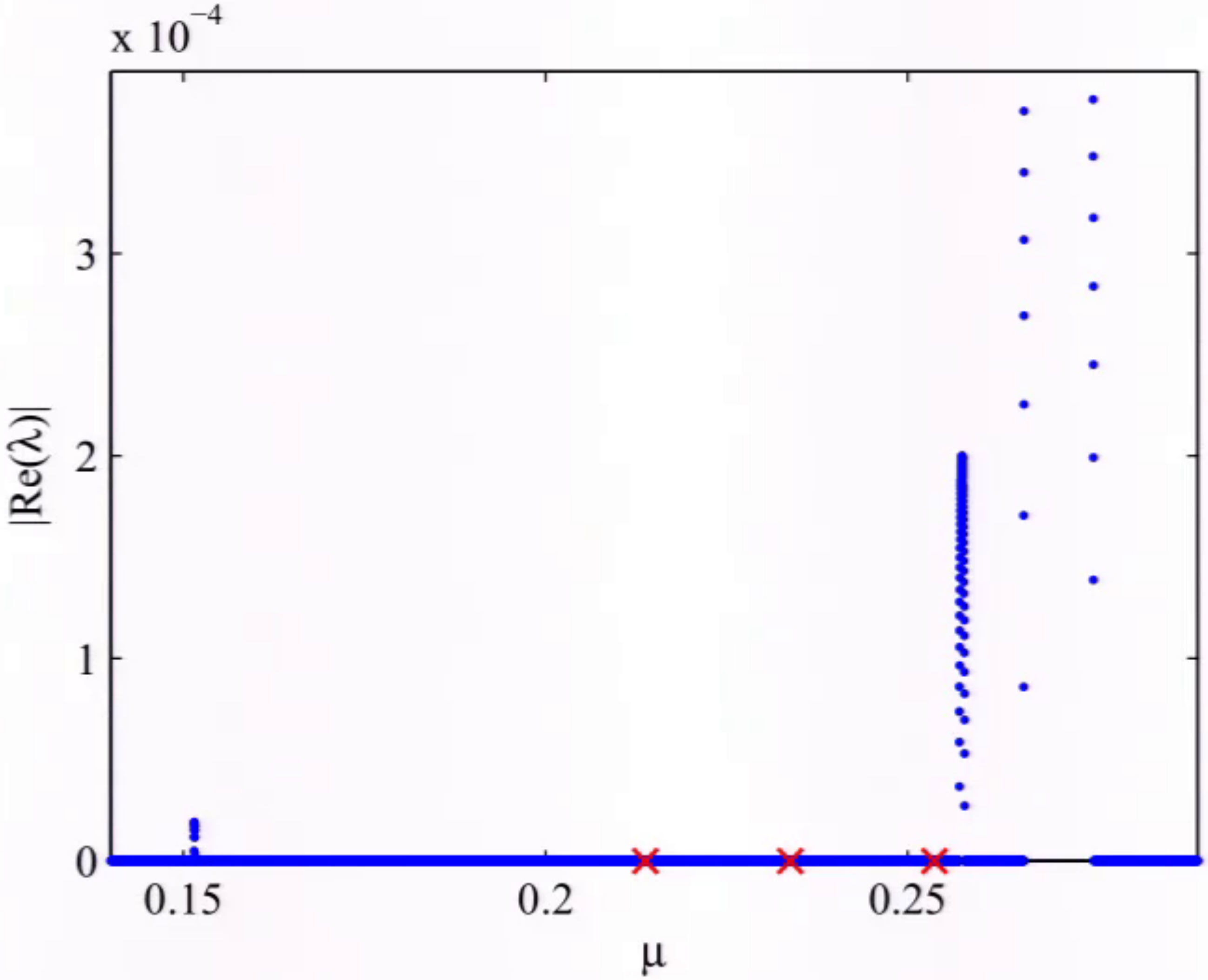




KG spectrum as a function of Bloch-wave parameter μ with wave amplitude ~ 0.11:



In addition to the unstable spectra emanating from the origin, there are three bubbles of unstable spectra.

Big question

Can an upper bound be placed on the total number of bubbles of unstable spectra for (at least) small waves? If so, under what conditions?

Miller/Marangell showed that superliminal waves have an *infinite* number of instability bubbles if the wave corresponds to an infinite-gap potential of a corresponding Hill's equation.

Under reasonable assumptions the spectrum of the polynomial operator,

$$\mathcal{P}_2(\lambda) = \mathcal{A}_0 + \lambda\mathcal{A}_1 + \lambda^2\mathcal{I},$$

satisfies:

1. the polynomial eigenvalues have the Hamiltonian spectral symmetry $\{\lambda, -\overline{\lambda}\}$
2. point spectra only
3. each polynomial eigenvalue has finite multiplicity
4. the only possible accumulation point is infinity.

For our purposes, the coefficients are smooth functions of the Bloch-wave parameter, μ.

We are first concerned with determining the number of (potentially) unstable polynomial eigenvalues in terms of the coefficient operators. Set

1. k_{r}: total number of real positive polynomial eigenvalues
2. k_{c}: total number of complex polynomial eigenvalues with positive real part and nonzero imaginary part.

The negative Krein index of a purely imaginary polynomial eigenvalue, λ_0, with associated eigenspace, $\mathbb{E}_{\lambda_0}$, is

$$k_{\mathrm{i}}^{-}(\lambda_0) = \mathrm{n}\left(-\lambda_0 \mathcal{P}_2'(\lambda_0)|_{\mathbb{E}_{\lambda_0}}\right).$$

The total negative Krein index, k_{i}^{-}, is the sum of $k_{\mathrm{i}}^{-}(\lambda_0)$ for each polynomial eigenvalue $\lambda_0 \in \mathrm{i}\mathbb{R}$.

The Hamiltonian-Krein index is $K_{\mathrm{Ham}} = k_{\mathrm{r}} + k_{\mathrm{c}} + k_{\mathrm{i}}^{-}$.

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For $\mathcal{P}_2(\lambda) = \mathcal{A}_0 + \lambda\mathcal{A}_1 + \lambda^2\mathcal{I}$ with $\mathcal{A}_0$ invertible,

$$K_{\mathrm{Ham}} = \mathrm{n}\left(\mathcal{A}_0\right)$$

(Pelinovsky et al., Kapitula et al., Grillakis et al., others).

Under suitable (quite generic) assumptions, an underlying wave is orbitally stable if $K_{\mathrm{Ham}} = 0$ (Grillakis et al.).

We wish to graphically locate those purely imaginary polynomial eigenvalues with negative Krein signature. The Krein matrix for a polynomial operator,

$$\mathcal{P}_2(\lambda) = \mathcal{A}_0 + \lambda\mathcal{A}_1 + \lambda^2\mathcal{I},$$

for $\lambda = \mathrm{i}z$ is:

$$\boldsymbol{K}_S(z) = -z\mathcal{P}_2(\mathrm{i}z)|_S \cdots$$
$$\cdots + z\mathcal{P}_2(\mathrm{i}z)P_{S^\perp}(P_{S^\perp}\mathcal{P}_2(\mathrm{i}z)P_{S^\perp})^{-1}P_{S^\perp}\mathcal{P}_2(\mathrm{i}z)|_S.$$

Here S is the finite-dimensional negative subspace for $\mathcal{A}_0$ (or an approximation). For the problems at hand,

$$\dim[S] = K_{\mathrm{Ham}}.$$

Polynomial eigenvalues are those values for which the Krein matrix is singular.

The Krein eigenvalues, $r_j(z)$, for $j = 1, \ldots, \dim[S]$, are the eigenvalues of the Krein matrix. The Krein eigenvalues satisfy:

- if $\lambda = \mathrm{i}z_0$ is an eigenvalue, then for some j, $r_j(z_0) = 0$ with

$$r_j'(z_0) \begin{cases} < 0, & k_{\mathrm{i}}^{-}(\mathrm{i}z_0) = 1 \\ > 0, & k_{\mathrm{i}}^{-}(\mathrm{i}z_0) = 0 \end{cases}$$

- $r_j'(z) > 0$ near poles (eigenvalues of $P_{S^\perp}\mathcal{P}_2(\mathrm{i}z)P_{S^\perp}$)
- each pole of the Krein matrix is a removable singularity for all but one of the Krein eigenvalues.

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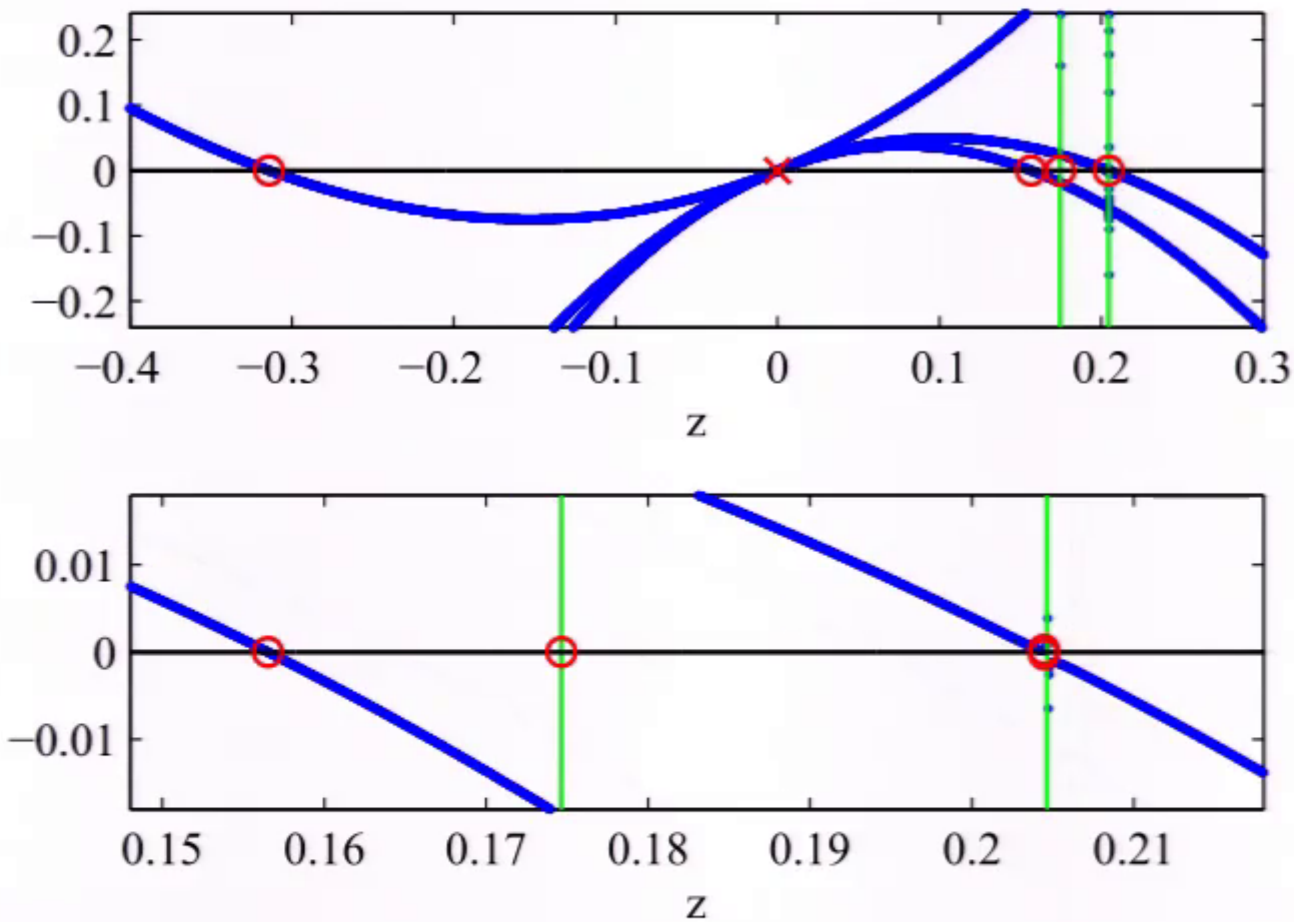
Krein eigenvalue for small waves (~ 0.05) to KG:



$\mu = 0.2435$

Conclusion

The number of instability bubbles for small waves is bounded above by the number of (Krein eigenvalue) zero/pole collisions for the unperturbed problem.

The Hamiltonian-Krein index:

1. determines the size of the Krein matrix
2. gives an upper bound for the number of unstable polynomial eigenvalues for a *fixed* value of μ.

However, the index does not necessarily bound the total number of instability bubbles.

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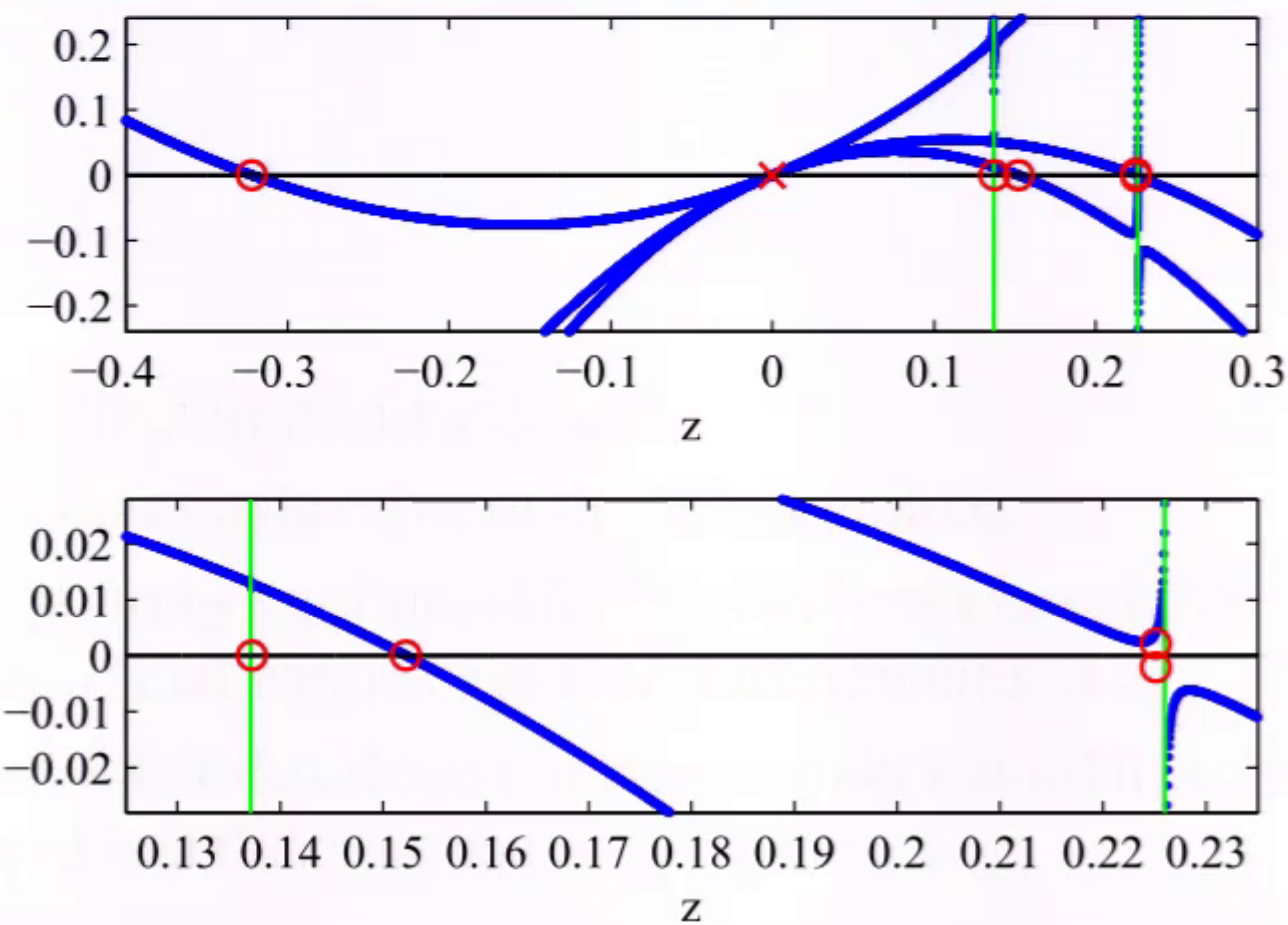

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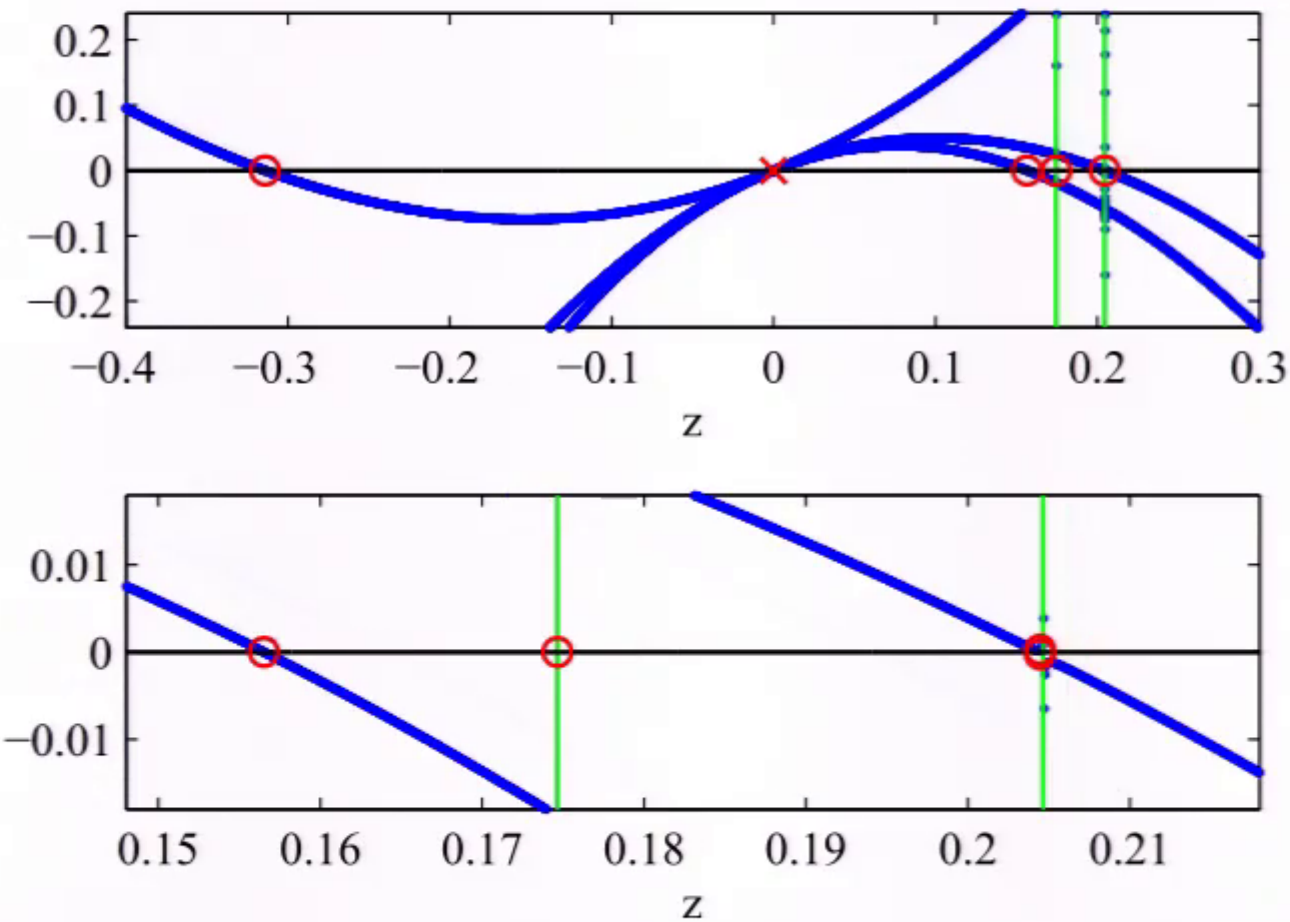

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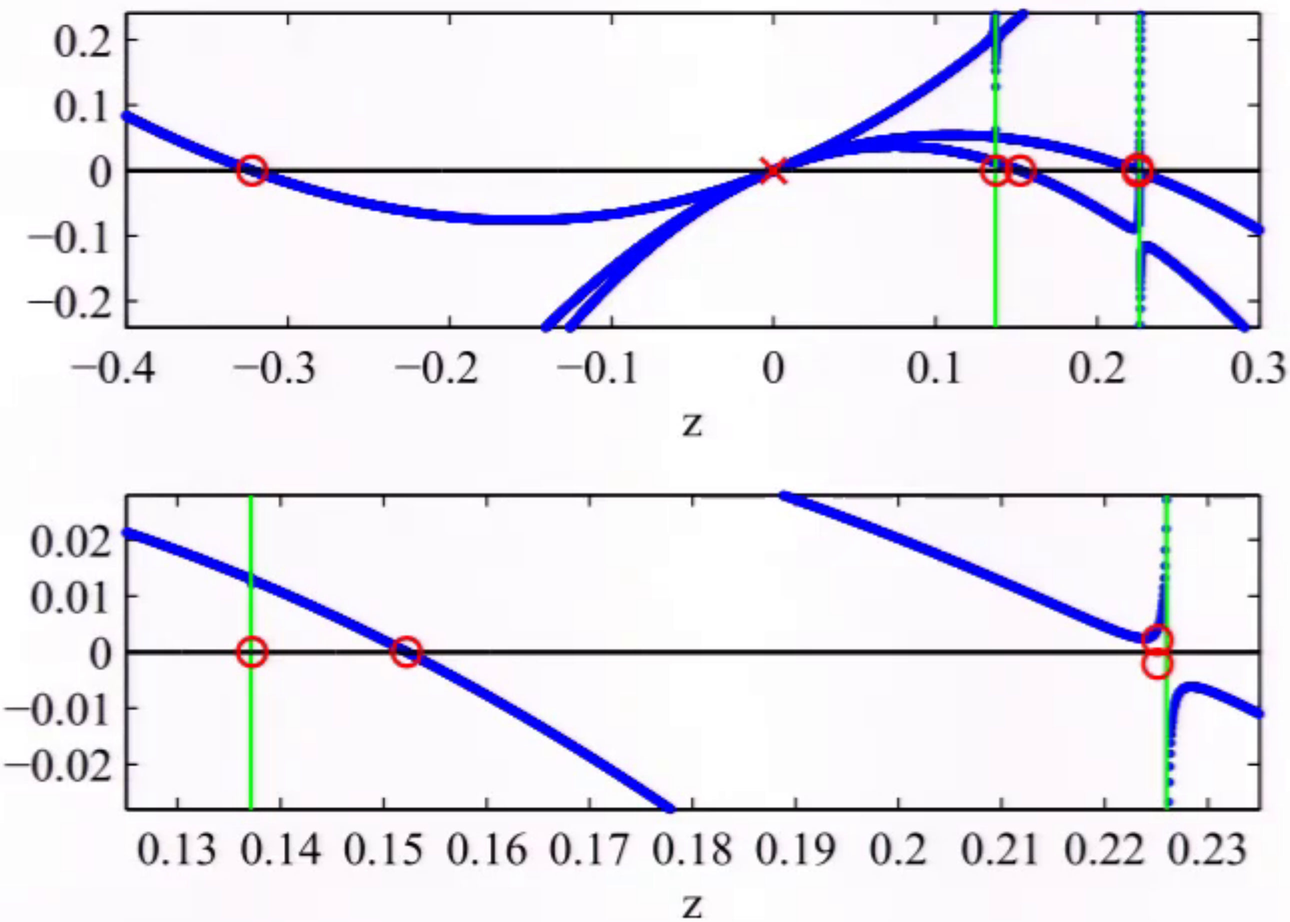

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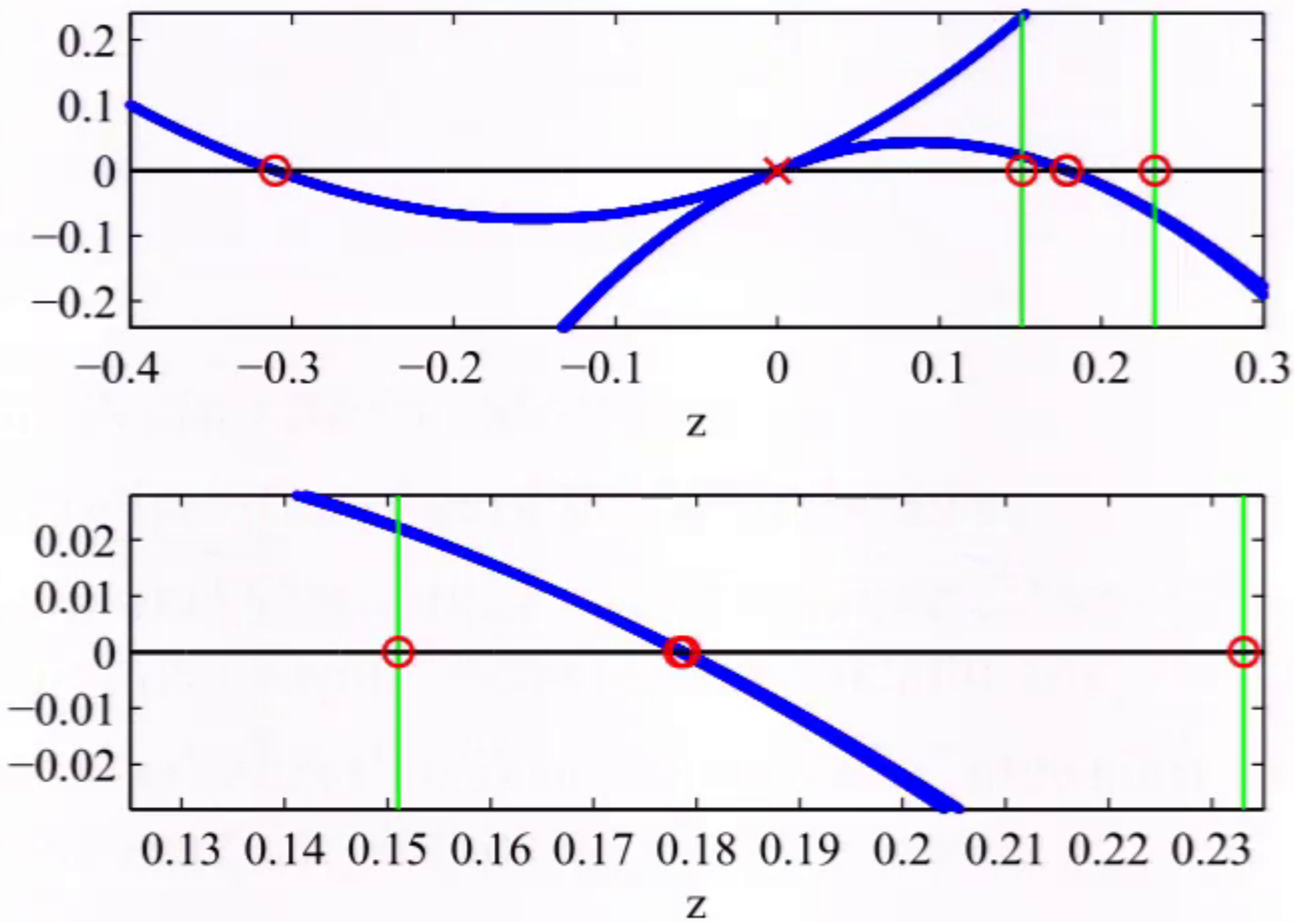

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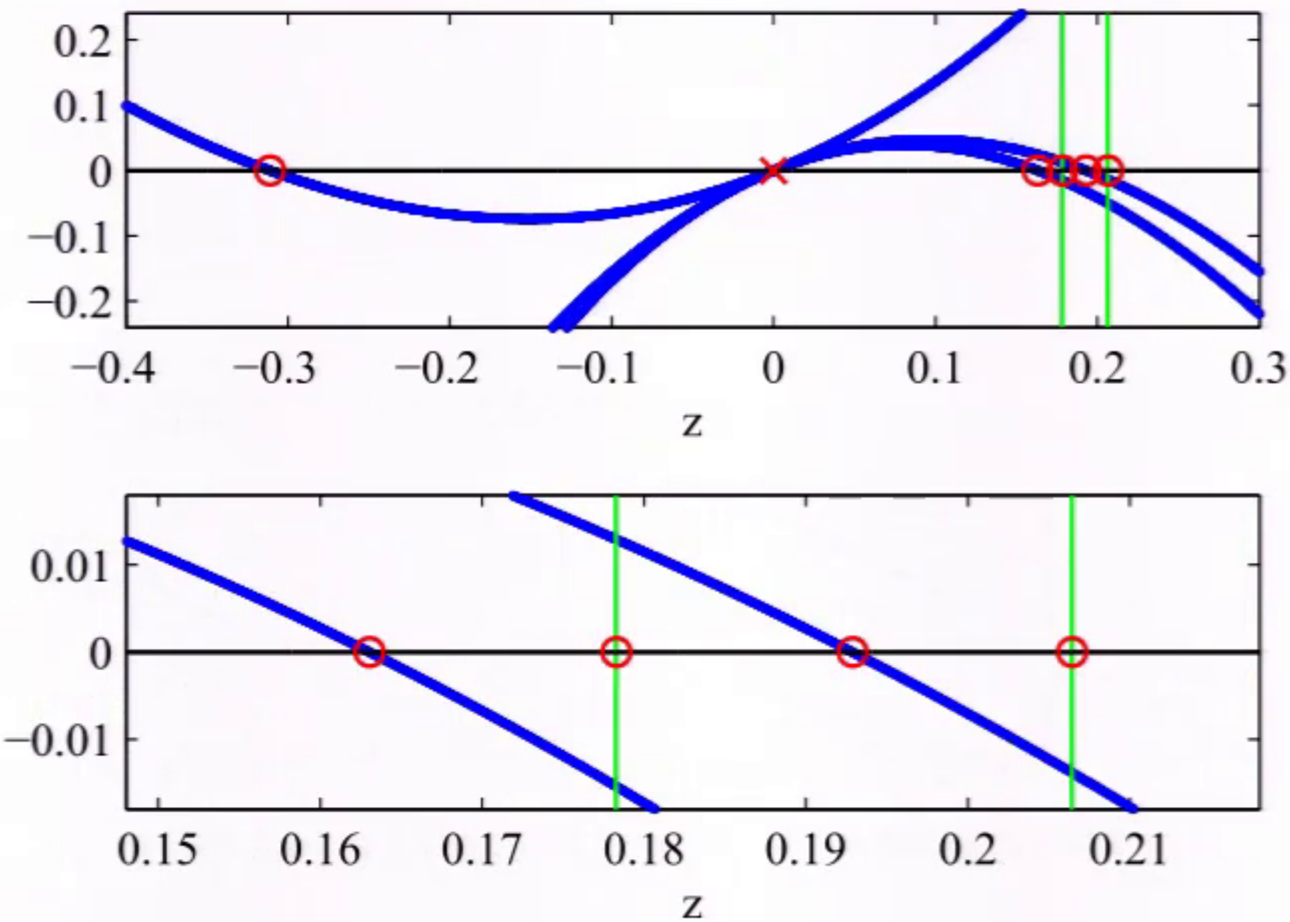
Krein eigenvalue for unperturbed KG ($K_{\mathrm{Ham}} = 3$):



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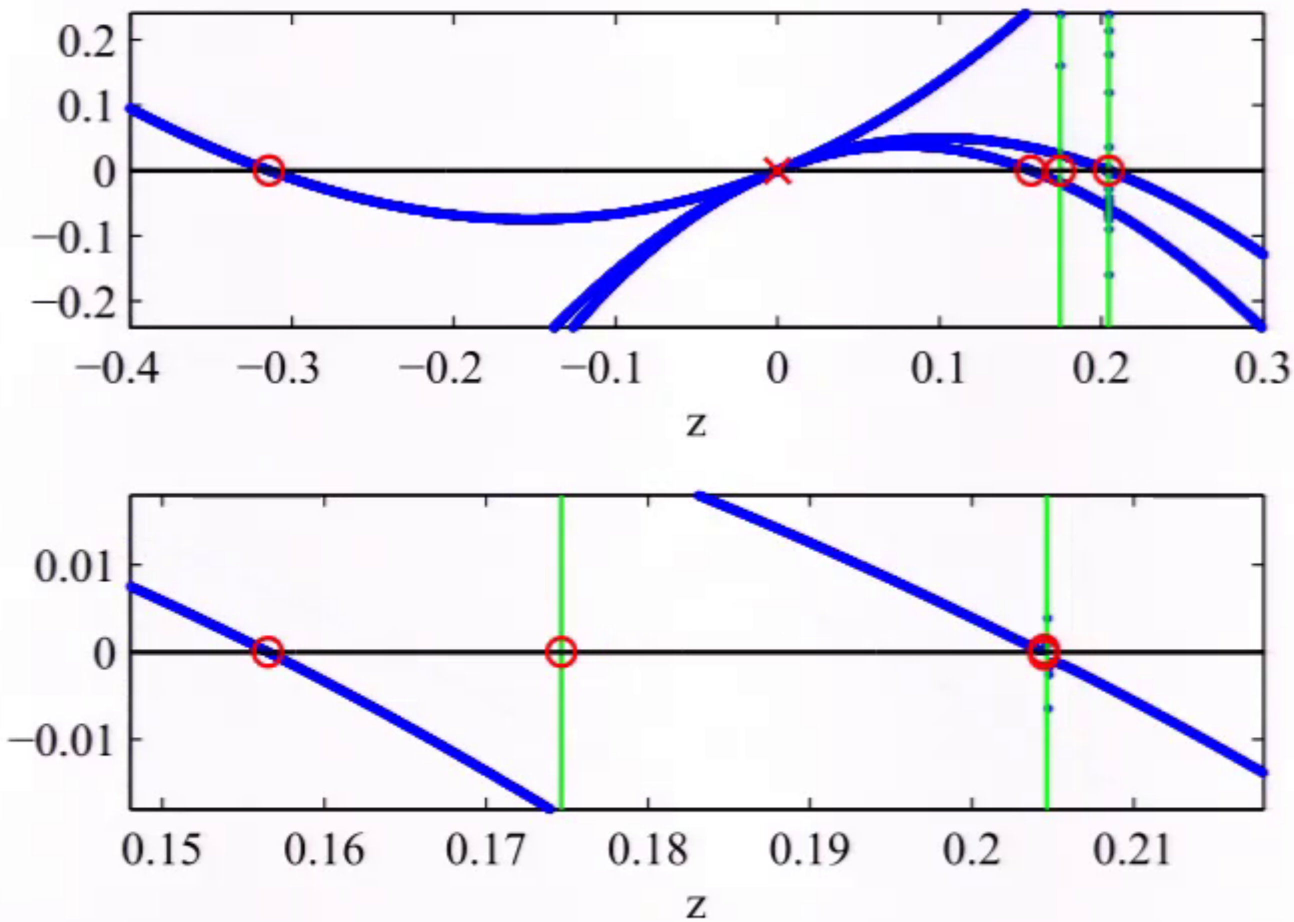

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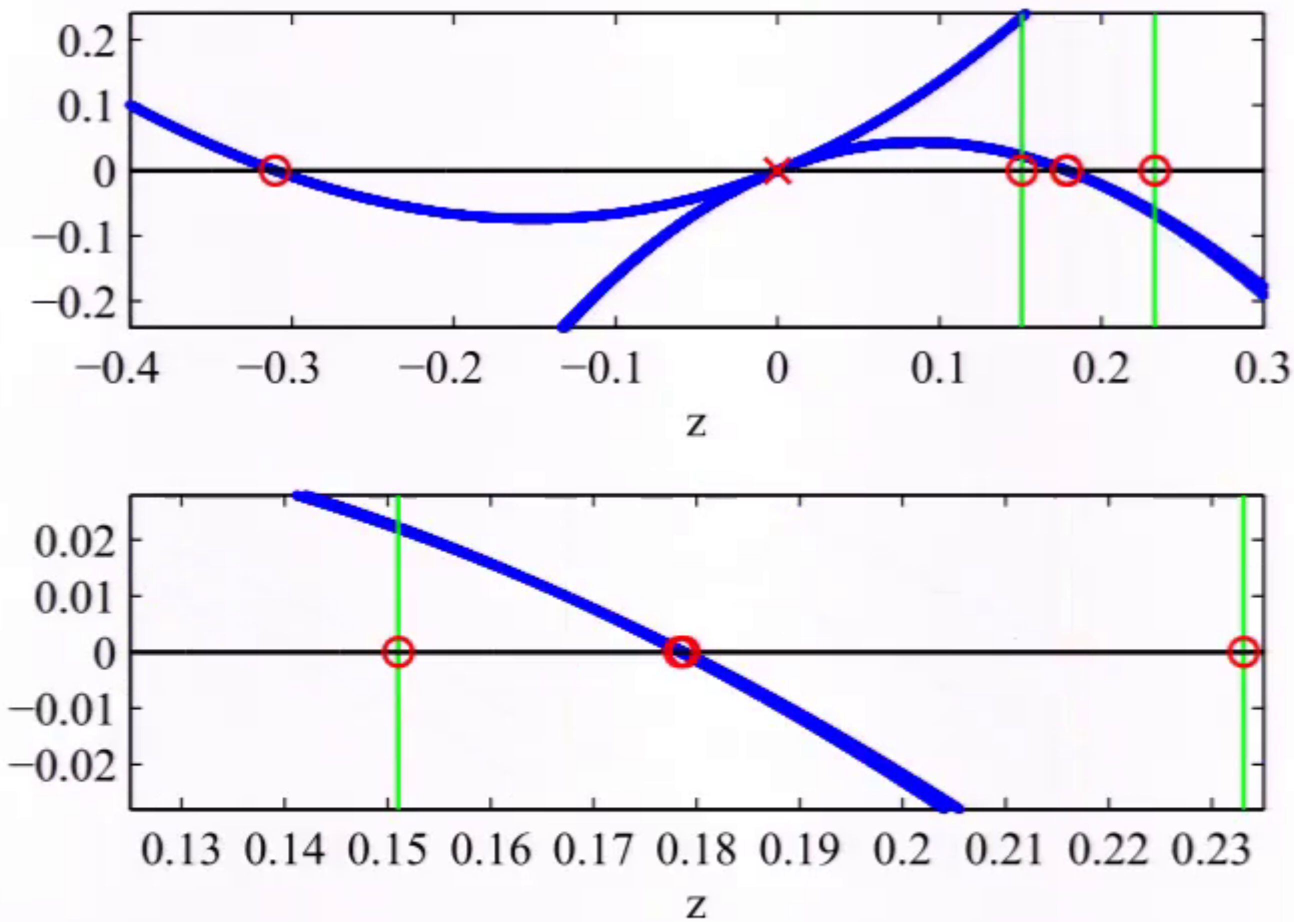

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