Numerical approximation of a feedback-control data assimilation algorithm: uniform in time error estimates

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(Joint work with E. S. Titi)

Outline

- General idea of Data Assimilation (DA).
- Feedback-control algorithm.
- Numerical approximation – Postprocessing Galerkin.
- Summary.
- Remarks/Future work.

Question: How to make a weather forecast?

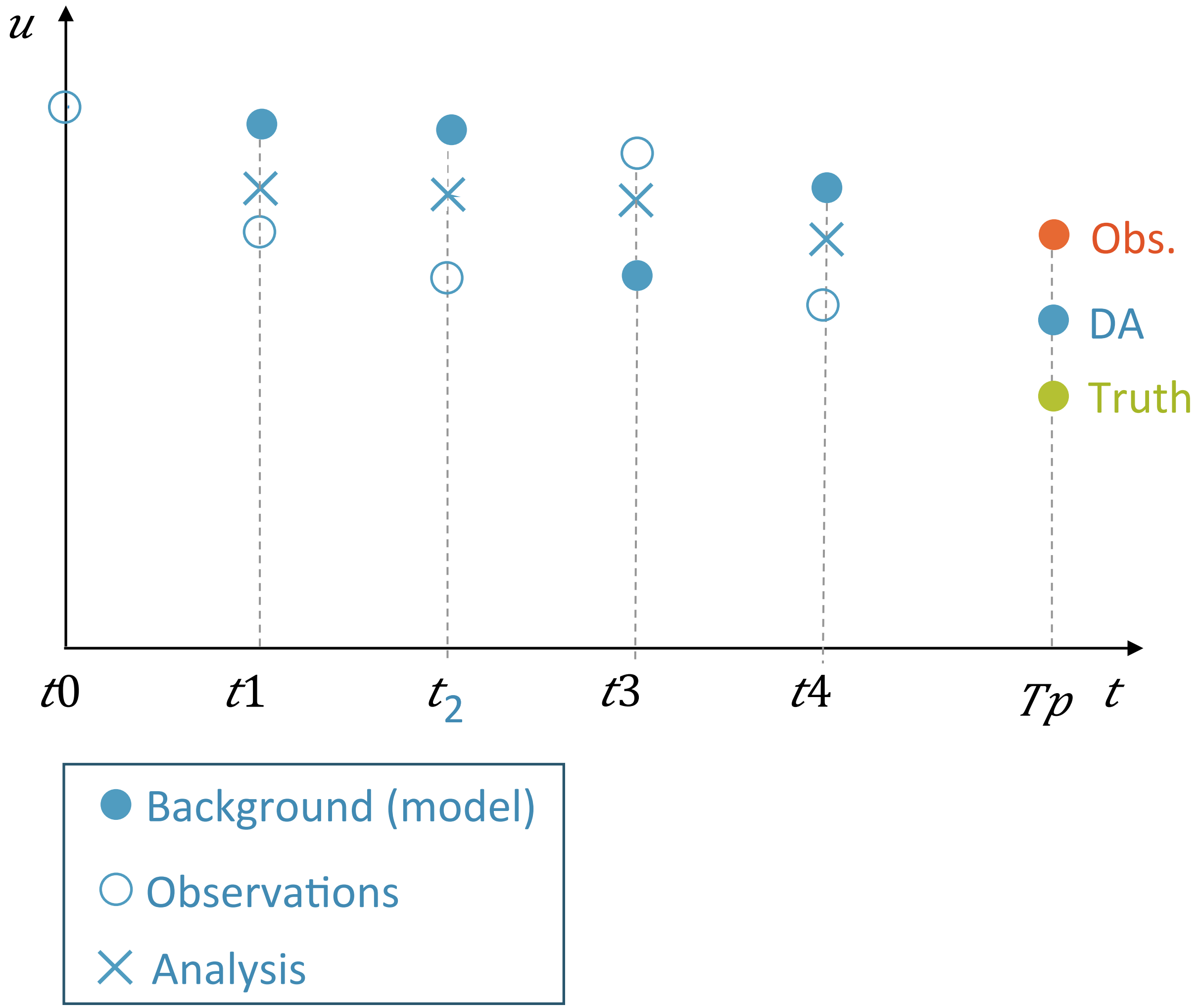
You will need...

- A theoretical model:

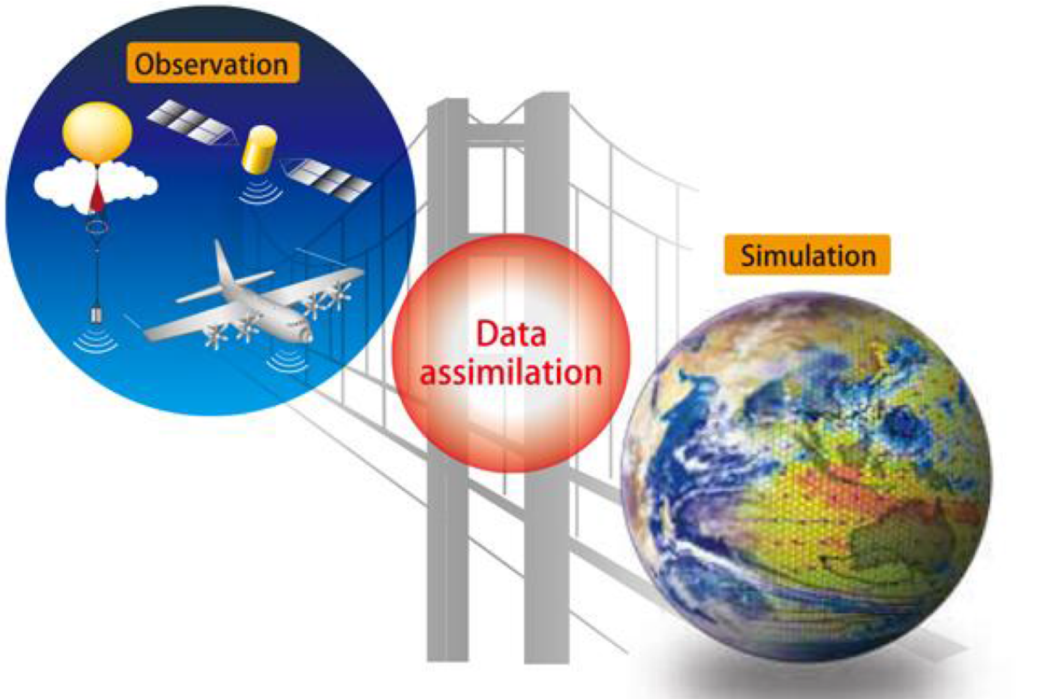
$$du/dt = F(t, u(t))$$

u: unknown variable representing the state of the atmosphere (velocity field, temperature, pressure, ...).

- Observational measurements.



- ***Data Assimilation*** combines the theoretical model with information from observations in order to obtain a good approximation of the state of the physical system at a certain future time.



Source: http://daconf15.umd.edu/

- Numerous applications: meteorology, oceanography, oil industry, neuroscience, etc.

- Several approaches:
  - Nudging.
  - Kalman Filter (KF).
  - Ensemble Kalman Filter (EnKF).
  - Local Ensemble Transform Kalman Filter (LETKF).
  - 3DVAR.
  - 4DVAR.

Feedback-control (nudging) approach
(Azouani-Olson-Titi, ‘14)

- Combine model and measurements by adding a feedback-control term to the equations.

approximate model = original model + feedback control term

⇒ Approximate solution v; arbitrary $v(0)=v0$

Feedback control term forces coarse spatial scales of v toward those of u

original (forecast) model

⇒ Reference solution u; missing $u(0)=u0$

Background idea

- Long-time behavior of solutions to dissipative evolution equations is determined by only a ***finite*** number of degrees of freedom.
  - Fourier modes, 2D-NSE (Foias-Prodi, '67):

Let P_N be the projection operator onto the first N Fourier modes.
$\exists\, N \gg 1$ s.t. if $\mathbf{u}_1,\ \mathbf{u}_2$ are two solutions of 2D-NSE with

$$\|P_N \mathbf{u}_1 - P_N \mathbf{u}_2\|_{L^2} \to 0, \quad t \to \infty$$

then

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{L^2} \to 0, \quad t \to \infty.$$

  - Spatial nodes, 2D-NSE (Foias-Temam, '84).
  - Finite volume elements, 2D-NSE (Foias-Titi, '91; Jones-Titi, '92).
  - Other dissipative evolution eqs. (Cockburn-Jones-Titi, '97).

Example

- Consider the forecast (theoretical) model given by the ***2D incompressible Navier-Stokes equations***:

$$\frac{\partial \mathbf{u}}{\partial t} - \nu \Delta \mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{u} + \nabla p = \mathbf{f}, \quad \nabla \cdot \mathbf{u} = 0 \qquad \text{(2D-NSE)}$$

$\mathbf{u}$: velocity field

ν : kinematic viscosity

p : pressure

$\mathbf{f}$: density of volume forces

- Assume:
  - No model error.
  - Continuous in time and error-free measurements.

Approximate model

controls small scales

controls large scales

$$\frac{\partial \mathbf{v}}{\partial t} \overbrace{- \nu \Delta \mathbf{v}}^{\text{controls small scales}} + (\mathbf{v} \cdot \nabla)\mathbf{v} + \nabla \pi = \mathbf{f} \overbrace{- \beta [I_h(\mathbf{v}) - I_h(\mathbf{u})]}^{\text{controls large scales}}, \quad \nabla \cdot \mathbf{v} = 0.$$

ν, $\mathbf{f}$: same as for the 2D-NSE

π : modified pressure

h : resolution of spatial mesh

β : relaxation parameter

I_h : linear interpolant operator in space

- Denote $\mathbf{w} = \mathbf{v} - \mathbf{u}$.

$$\frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + [(\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u} - (\mathbf{w}\cdot\nabla)\mathbf{w}] + \nabla(\pi - p) = -\beta I_h(\mathbf{w})$$
$$= -\beta[I_h(\mathbf{w}) - \mathbf{w}] - \beta\mathbf{w}$$

$$\Rightarrow \frac{\partial \mathbf{w}}{\partial t} - \nu\Delta\mathbf{w} + \beta\mathbf{w} + \nabla(\pi - p) = [(\mathbf{u}\cdot\nabla)\mathbf{w} + (\mathbf{w}\cdot\nabla)\mathbf{u} - (\mathbf{w}\cdot\nabla)\mathbf{w}] - \beta[I_h(\mathbf{w}) - \mathbf{w}]$$

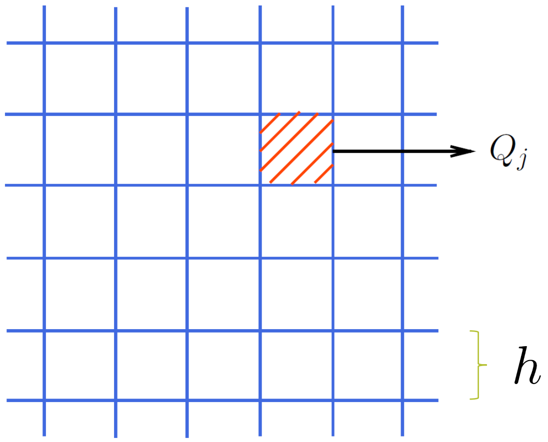
- Assume

$$\|I_h(\varphi) - \varphi\|_{L^2} \le c_0 h \|\nabla\varphi\|_{L^2} \quad \forall\varphi \in (H^1)^2.$$

Ex.:

- Low modes projector: $I_h(\varphi) = P_N\varphi,\ N \in \mathbb{N}.$

- Finite volume elements: $\Omega = \bigcup_{j=1}^{N} Q_j$.



$$I_h(\varphi) = \sum_{j=1}^{N} \overline{\varphi_j} \chi_{Q_j}, \text{ where } \overline{\varphi_j} = \frac{1}{|Q_j|} \int_{Q_j} \varphi \mathrm{d}x.$$

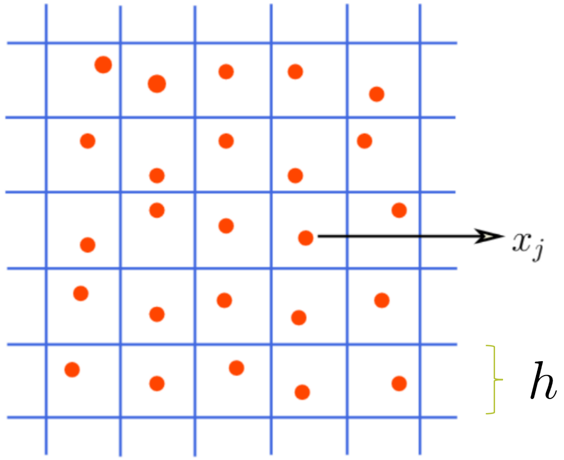
OR:

$$\|I_h(\varphi) - \varphi\|_{L^2} \le c_0 h \|\varphi\|_{H^1} + c_1 h^2 \|\varphi\|_{H^2} \quad \forall \varphi \in (H^2)^2.$$

Ex.:

- Nodal values: $x_j \in Q_j$, $j = 1, \ldots, N$.

$$I_h(\varphi) = \sum_{j=1}^{N} \varphi(x_j) \chi_{Q_j}.$$



Theorem (Azouani-Olson-Titi, '14)

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\|\mathbf{v}(t) - \mathbf{u}(t)\| \leq O(\mathrm{e}^{-\beta t})$.

Some related works

- Other models: 3D NS-alpha (Albanez-Nussenzveig Lopes-Titi, '16), 3D Brinkman-Forchheimer-extended Darcy (Markowich-Titi-Trabelsi, '16), 2D-SQG (Jolly-Martinez-Titi, '17).
- Using observations of less components:
  - 2D Bénard, only velocity (Farhat-Jolly-Titi, '15).
  - 2D-NSE, one velocity component (Farhat-Lunasin-Titi, '16) .
  - 3D planetary geostrophic model, only temperature (Farhat-Lunasin-Titi, '16).
  - 2D Bénard, only horizontal velocity component (Farhat-Lunasin-Titi, '17).
  - 3D Bénard in porous media, only temperature (Farhat-Lunasin-Titi, '17).
  - 3D Leray-alpha, only two components of velocity (Farhat-Lunasin-Titi, 17).

Some related works (cont'd)

- Higher order convergence, Gevrey class and L^∞ (Biswas-Martinez, '17).
- Measurements with stochastic errors (Blomker-Law-Stuart-Zygalakis, '13; Bessaih-Olson-Titi, '15).
- Time-averaged meas.: 2D-SQG (Jolly-Olson-Titi-Martinez), Lorenz (Blocher-Olson-Martinez).
- Discrete in time meas. with syst. errors, 2D-NSE (Foias-M-Titi, '16).
- Numerical computations:
  - 2D-NSE (Gesho-Olson-Titi, '16).
  - 2D Bénard (Altaf-Titi-Gebrael-Knio-Zhao-McCabe-Hoteit, '16).
- Numerical approximation by PPGM, 2D-NSE (M-Titi).

Numerical Approximation

- In practice, numerical models can only compute ***finite-dimensional*** approximations.
- ***Goal:*** Obtain an analytical estimate of the error between a numerical approximation of $\mathbf{v}$ and the (full) reference solution $\mathbf{u}$.
- For simplicity, assume: continuous in time and error-free measurements.
- Setting:
  - Phase space of 2D-NSE: $H = \{\mathbf{u} \in (L^2)^2 \,|\, \nabla \cdot \mathbf{u} = 0 \;+\; b.c.\}.$
  - Apply projector $P_\sigma : (L^2)^2 \to H$ to the feedback-control equation:

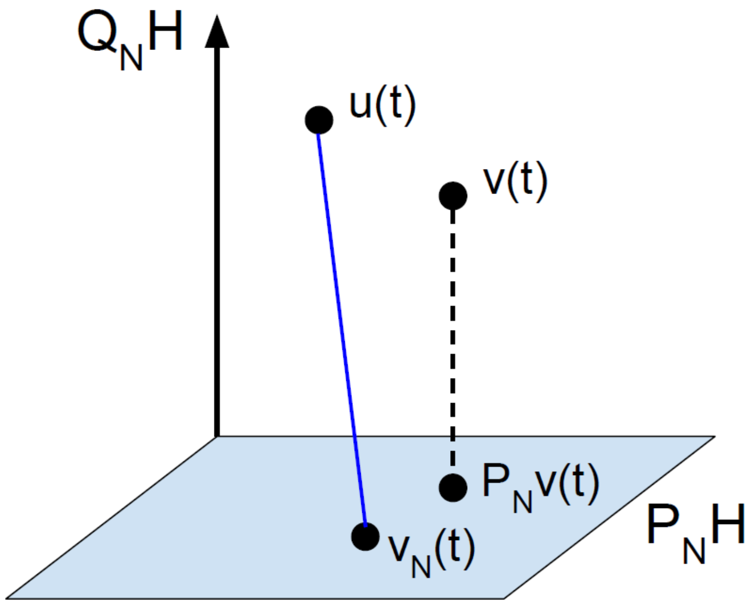
$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} + \nu A\mathbf{v} + B(\mathbf{v},\mathbf{v}) = \mathbf{f} - \beta P_\sigma I_h(\mathbf{v} - \mathbf{u}),$$

  - Eigenvectors of $A = P_\sigma(-\Delta)$: $\{\mathbf{w}_j\}_j$, with eigenvalues $\{\lambda_j\}_j$.
  - Finite-dimensional space: $\mathrm{span}\{\mathbf{w}_1, \ldots, \mathbf{w}_N\} = P_N H.$

Galerkin spectral method

Find $\mathbf{v}_N \in P_N H$ satisfying

$$\frac{\mathrm{d}\mathbf{v}_N}{\mathrm{d}t} + \nu A \mathbf{v}_N + P_N B(\mathbf{v}_N, \mathbf{v}_N) = P_N \mathbf{f} - \beta P_N P_\sigma I_h(\mathbf{v}_N - \mathbf{u}).$$



Notation: $Q_N = I - P_N$.

Theorem (M.-Titi)

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\exists\, \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N}{\lambda_{N+1}}.$$

Thus, $\exists\, T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|\mathbf{v}_N(t) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}}, \quad \forall t \geq T,$$

where

$$L_N = \left[1 + log\left(\frac{\lambda_N}{\lambda_1}\right)\right]^{1/2}.$$

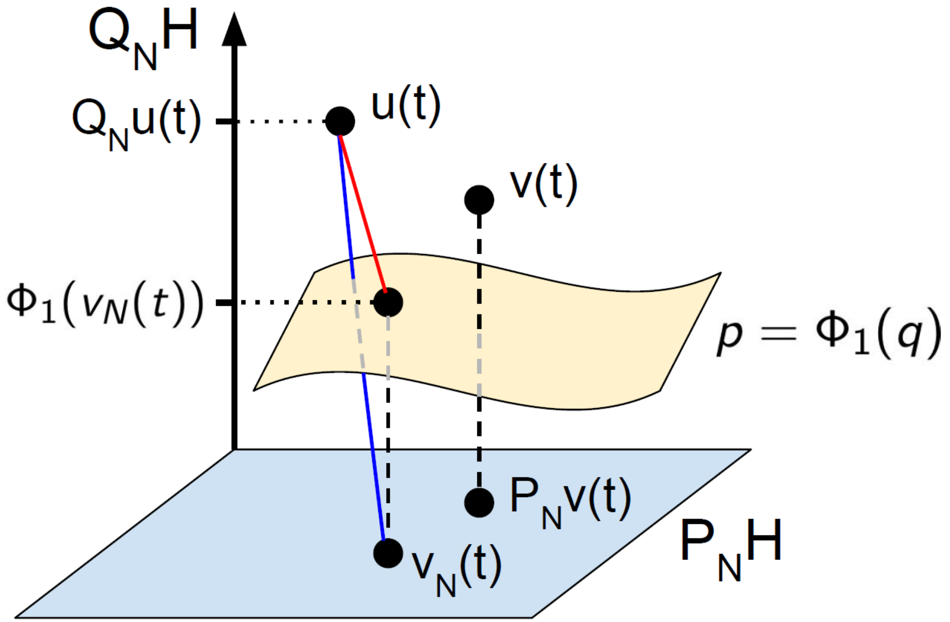
A Postprocessing of the Galerkin method

('García-Archilla'-Novo-Titi, '98)

- Idea: Add to the Galerkin approximation of $\mathbf{v}$ a suitable approximation of $\mathbf{q}$:

$$\mathbf{q} \approx \Phi_1(\mathbf{p}) = (\nu A)^{-1} Q_N[\mathbf{f} - B(\mathbf{p}, \mathbf{p})]$$

(Approximate inertial manifold, Foias-Manley-Temam, '88)



Notation: $\mathbf{p} = P_N \mathbf{u}$, $\mathbf{q} = Q_N \mathbf{u}$

$(\mathbf{u} = \mathbf{p} + \mathbf{q})$

Postprocessing Galerkin Algorithm

For obtaining an approximation of $\mathbf{v}$, and thus $\mathbf{u}$, at a certain time $T > t_0$:

1. Integrate the Galerkin system over $[t_0, T]$ to obtain $\mathbf{v}_N(T)$.
2. Obtain $\mathbf{q}_N$ satisfying $\nu A \mathbf{q}_N = Q_N[\mathbf{f} - B(\mathbf{v}_N(T), \mathbf{v}_N(T))]$.
3. Compute $\mathbf{v}_N(T) + \mathbf{q}_N$.

- Information on the high modes (fine spatial scales) is only used at the final time T! This is one of the reasons for the efficiency of the Postprocessing Galerkin method (compared to, e.g., the Nonlinear Galerkin method).

Particular case: $I_h = P_K,\ K \in \mathbb{N}$

Theorem (M.-Titi)

If $\beta \gg \nu\lambda_1^2$ and $\lambda_K \gtrsim \beta/\nu$, then $\exists\ \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t)) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N^4}{\lambda_{N+1}^{3/2}}.$$

Thus, $\exists\ T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N^4}{\lambda_{N+1}^{3/2}}, \quad \forall t \geq T.$$

General case

- Assume $I_h : (L^2)^2 \to (L^2)^2$ is a linear operator satisfying:
  - $\exists\, c_0 > 0$ s.t.
  $$\|\varphi - I_h(\varphi)\|_{L^2} \le c_0 h \|\varphi\|_{H^1}, \quad \forall \varphi \in H^1(\Omega)^2.$$
  - $\exists\, c_{-1} > 0$ s.t.
  $$\|\varphi - I_h(\varphi)\|_{H^{-1}} \le c_{-1} h \|\varphi\|_{L^2}, \quad \forall \varphi \in L^2(\Omega)^2.$$
  - $\exists\, \tilde{c}_0 > 0$ s.t.
  $$\|I_h(\mathbf{q})\|_{L^2} \le \widetilde{c_0} \frac{|\Omega|^{3/4}}{h^2 \lambda_{N+1}^{1/4}} \|\mathbf{q}\|_{L^2}, \quad \forall \mathbf{q} \in Q_N H.$$
- Examples: low modes projector; finite volume elements.

Theorem (M.-Titi)

If $\beta \gg \nu\lambda_1^2$ and $h \lesssim \nu^{1/2}/\beta^{1/2}$, then $\exists\ \theta = \theta(\beta) \in [0,1)$ and $C = C(\nu, \lambda_1, |\mathbf{f}|_{L^2})$ s.t., for N sufficiently large,

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq c\theta^{(t-t_0)\nu\lambda_1 - 1}\|\mathbf{v}_N(t_0) - \mathbf{p}(t_0)\|_{L^2} + C\frac{L_N}{\lambda_{N+1}^{5/4}}.$$

Thus, $\exists\ T = T(\nu, \lambda_1, |\mathbf{f}|_{L^2}, N)$ s.t.

$$\|(\mathbf{v}_N(t) + \Phi_1(\mathbf{v}_N(t))) - \mathbf{u}(t)\|_{L^2} \leq C\frac{L_N}{\lambda_{N+1}^{5/4}}, \quad \forall t \geq T.$$

Comparison

- Error using the Galerkin method (both types of I_h):

$$\|\mathbf{v}_N - \mathbf{u}\|_{L^2} \leq O(L_N \lambda_{N+1}^{-1}).$$

- Error using the Postprocessing Galerkin method:
  - Case $I_h = P_K$:

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N^4 \lambda_{N+1}^{-3/2}).$$

  - General class of I_h :

$$\|(\mathbf{v}_N + \Phi_1(\mathbf{v}_N)) - \mathbf{u}\|_{L^2} = O(L_N \lambda_{N+1}^{-5/4}).$$

Summary

- Original feedback-control data assimilation algorithm (Azouani-Olson-Titi, ’14): continuous in time and error-free measurements.
- Numerical approximations of v, and thus u (M.-Titi):
  - Postprocessing Galerkin method has a better convergence rate than the Galerkin method, with respect to the numerical resolution.
  - Error estimates are ***uniform in time*** – feedback-control term stabilizes the large scales of the difference v - u, resulting in a globally asymptotically stable system.

Remarks/Future work

- Theoretical condition on the spatial resolution of the measurements, h, is far from being valid for real flows.
  - Numerical simulations done in, e.g. [Gesho-Olson-Titi, ‘16] and [Altaf et al., ‘16] show that a much less restrictive condition on h is sufficient for exponential convergence.
- Other types of numerical methods (e.g., finite volume elements) need to be considered for approximating v. This may yield better convergence rates with respect to the numerical resolution.

Thank you!