Snowbird May 22, 2019

Why period-doubling cascades exist

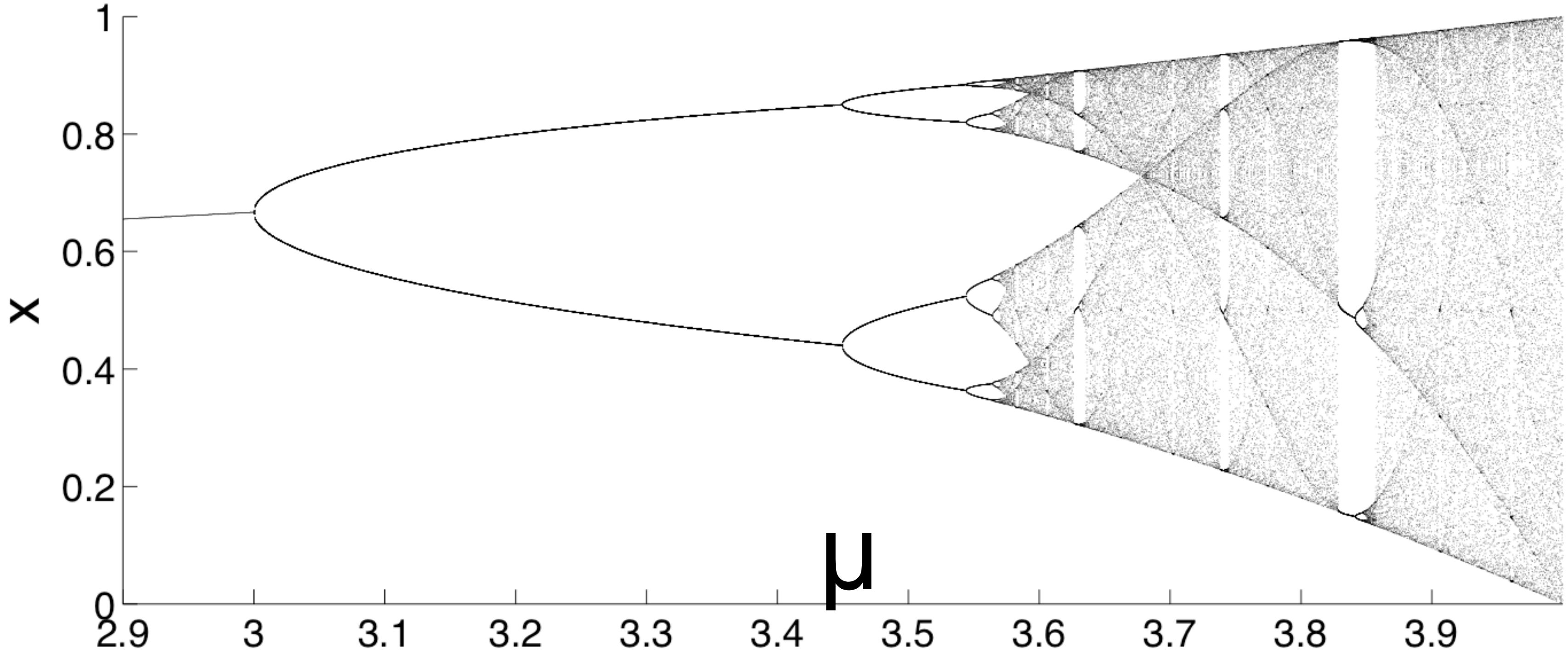
Preliminary joint work with R. B. Kellogg and T.-Y. Li 1976
S.N. Chow and J. Mallet-Paret 1978
J. Mallet-Paret 1982 (Snakes, orbit index

Joint work on cascades with Kathy Alligood 1983, 1985
Evelyn Sander 2011, 2013…

This talk uses the term even for non flip hyperbolic orbits
See also my 2013 lecture at "Canada North Bay Summer Top…"

A Cascade –Logistic map: $F(x) := \mu x(1-x)$

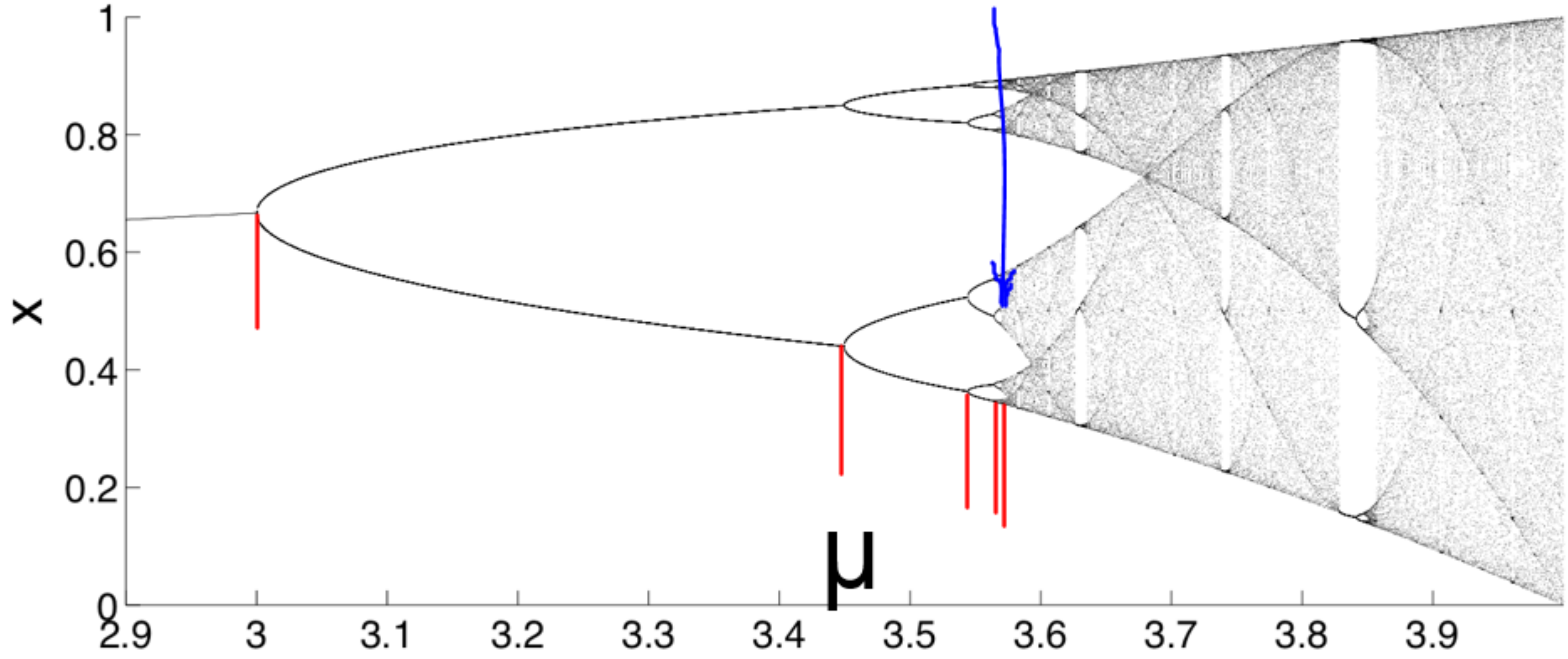
In a series of papers in 1958–1963, Finish mathematician Pekka Myrberg (1952 to 1962 the chancellor of the University of Helsinki - Finland) was the first to discover that as a parameter is varied in the quadratic map, periodic orbits of periods k, 2k, 4k, 8k, … occur for a variety of k values.



Mitchell Feignbaum’s “universal number” 4.6692...

- the quotient of successive distances between bifurcation events tends to 4.6692... And the vertical rescaling is about -2.502

Rescale μ= 3.569946…,x=0.5



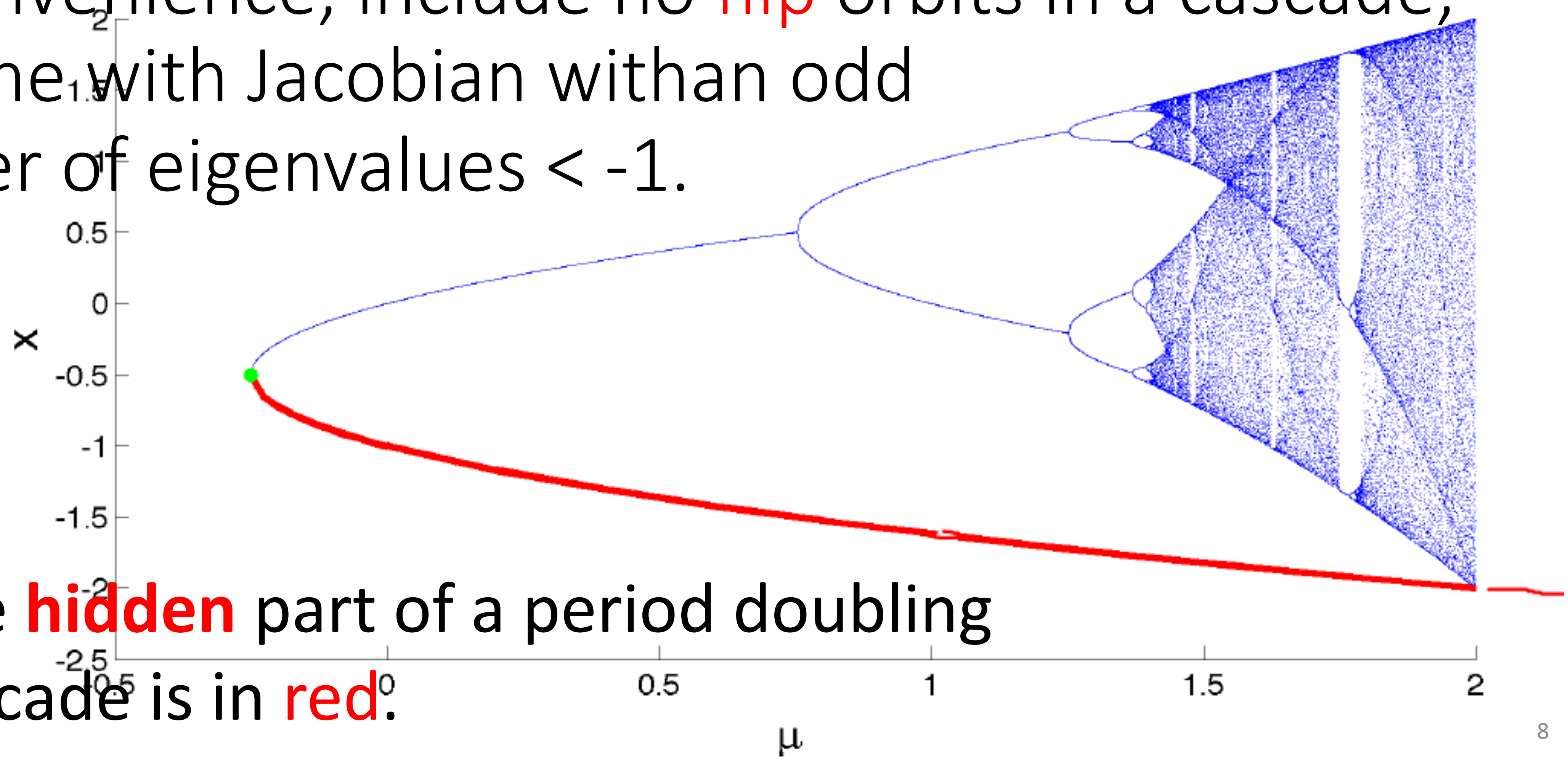
Universal Rescaling of Feigenbaum

- The brilliant asymptotic rescaling ideas of Feigenbaum have been verified using powerful techniques from complex analysis, thanks to the combined effort of such mathematicians as Douady, Hubbard, Sullivan, and McMullen.
- In practice a wide variety of cascades have the same rescaling eigenvalues that Feigenbaum identified.

- See for example http://www.math.harvard.edu/library/sternberg/slides/1180904.pdf

What does Feigenbaum universality say about the existence of cascades?

Def. A (period-doubling) cascade is a connected path of periodic orbits along which the period goes to infinity.
For convenience, include no flip orbits in a cascade, i.e. none with Jacobian withan odd number of eigenvalues < -1.



The hidden part of a period doubling cascade is in red.

Cascades for orbits from period 2 to period 6

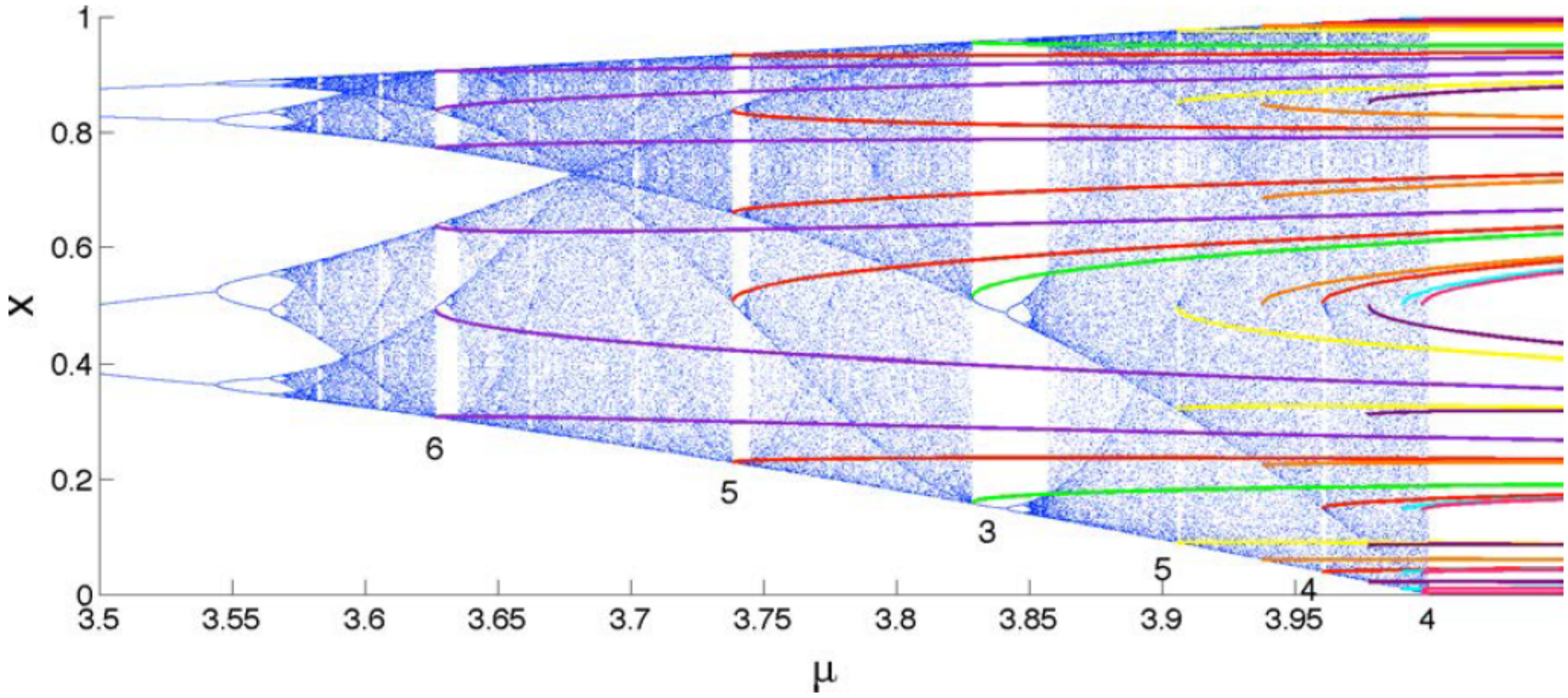
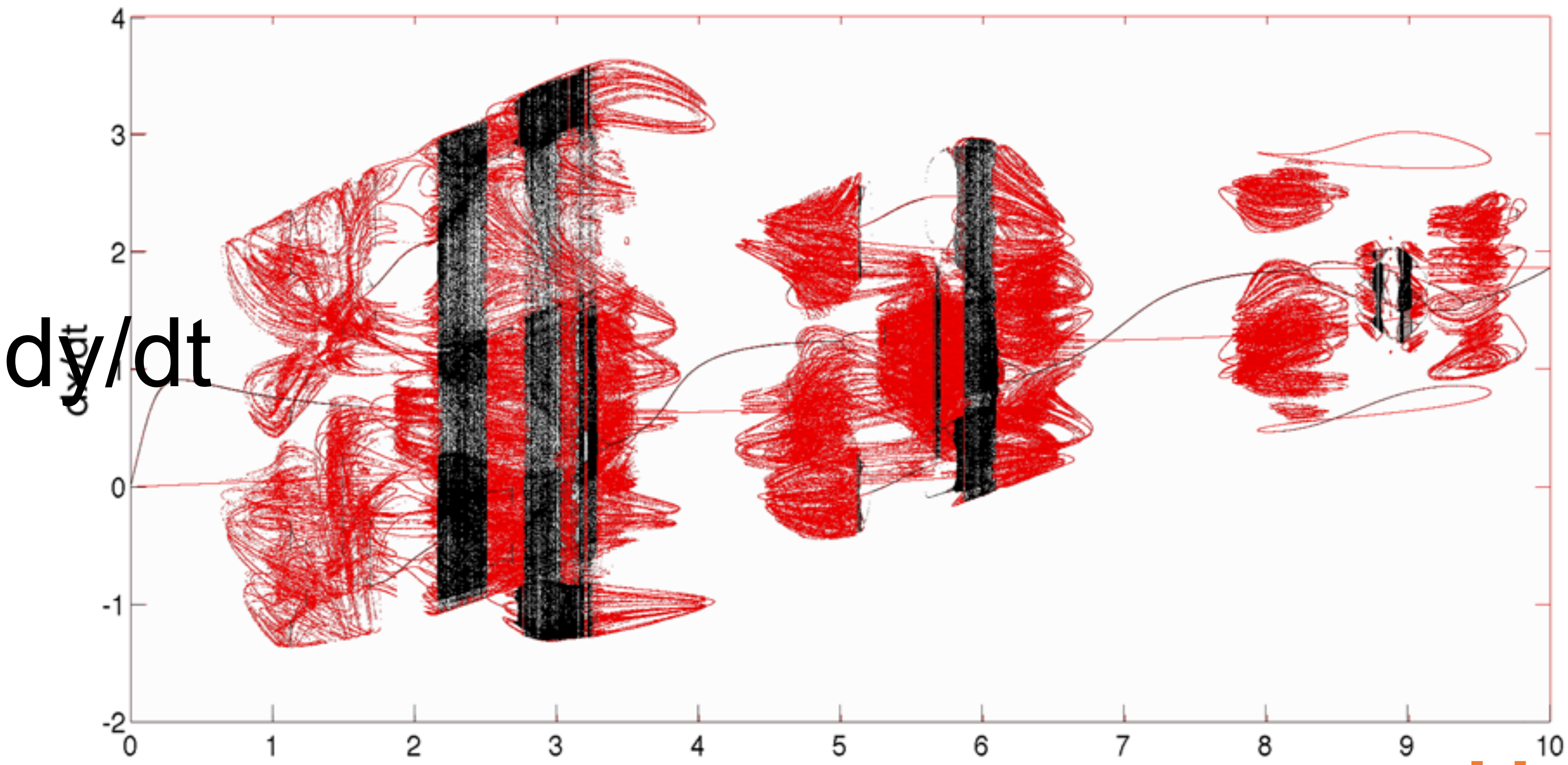


Fig. 4. *Cascades for* $F(\mu, x) = \mu x(1 - x)$. The logistic map has infinitely many cascades of attracting periodic orbits, and all cascades start at the stable orbit of a saddle-node bifurcation. The unstable orbits form what we call the **stems** of the cascades (shown in color). Each stem continues to exist for all large μ values. By our terminology, this means that all the cascades shown are solitary (on any parameter interval $[\mu_1, \mu_2]$, for $\mu_1 = 3.5$ and any $\mu_2 > 4$) since the stem does not connect its cascade to a second cascade. The stems are shown here up to period six. Different colors are used for different periods.

Forced damped pendulum
Unstable orbits in red
Then attractors are plotted in black



Time 2π map for

$$y'' + 0.2\, y' + \sin(y) = \mu \cos t$$

- Here we investigate only families $F(\mu,x)$ on R^n with μ in R that are generic: i.e. all periodic orbit bifurcations are generic.
- We assume there is one parameter μ_0 for which there are finitely many periodic points and another μ_1 for which there is chaos.

"Chaos" here means there are infinitely many regular saddles (having its unstable eigenvalue > +1). Of course there can also be infinitely many flip saddles (having its unstable eigenvalue < -1).

For maps $F(\mu,x)$ on R^n with μ in R^1.

Our cascade **theorem**:

- Assume at one parameter value μ_0 there are only finitely many periodic orbits, and at another μ_1 there is chaos. Then between those two parameter values (μ_0, μ_1) there must be infinitely many cascades, with the following assumptions:

1) F is generic: i.e. all periodic orbit bifurcations are generic.

2) The set of periodic orbits in (μ_0, μ_1) is bounded.

3) At μ_1, all periodic orbits have the same unstable dimension (including infinitely many unstable regular (non-flip) orbits. (Finitely many exceptional orbits are allowed.)

This lecture describes WHY cascades exist, how many exist, and the large scale connectivity.

- To motivate the method of proof, we now show a constructive proof of the Brouwer fixed point theorem.
- This 1976 result by RB Kellogg -   TY Li   - Yorke

was the first constructive or implementable proof.

It shows the spirit of the proof that cascades exist.

- With a later version by Chow, Mallet-Paret, Y   Math. of Comp. 32 (1978), 887-899.
- Later extensions by J Alexander-Yorke

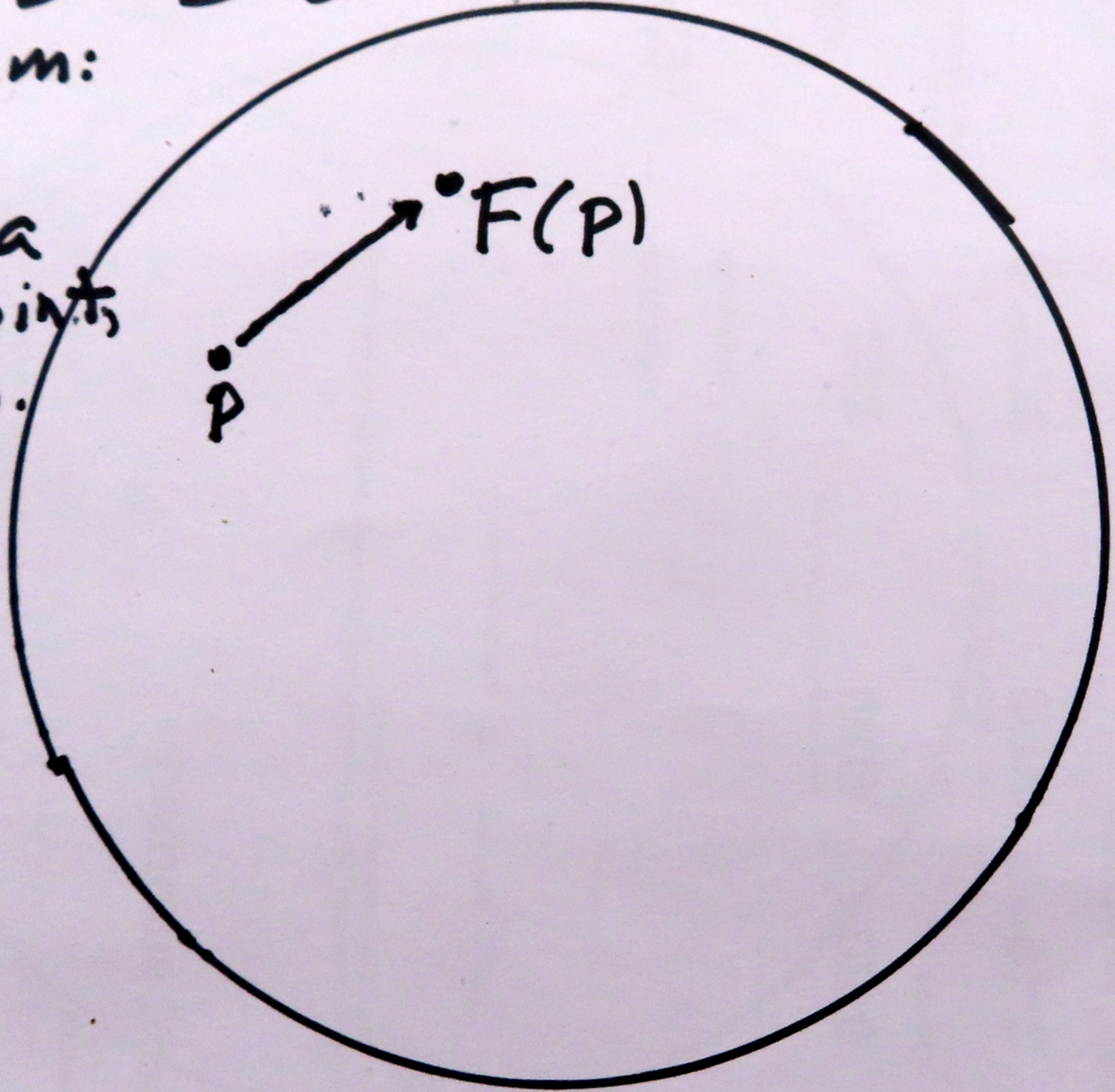
- Let B be a smooth convex ball in R^n. And F: R^n -> R^n

Let B be a smooth ball in $\mathbb{R}^n$.

Let $F: B \to B$ be smooth.

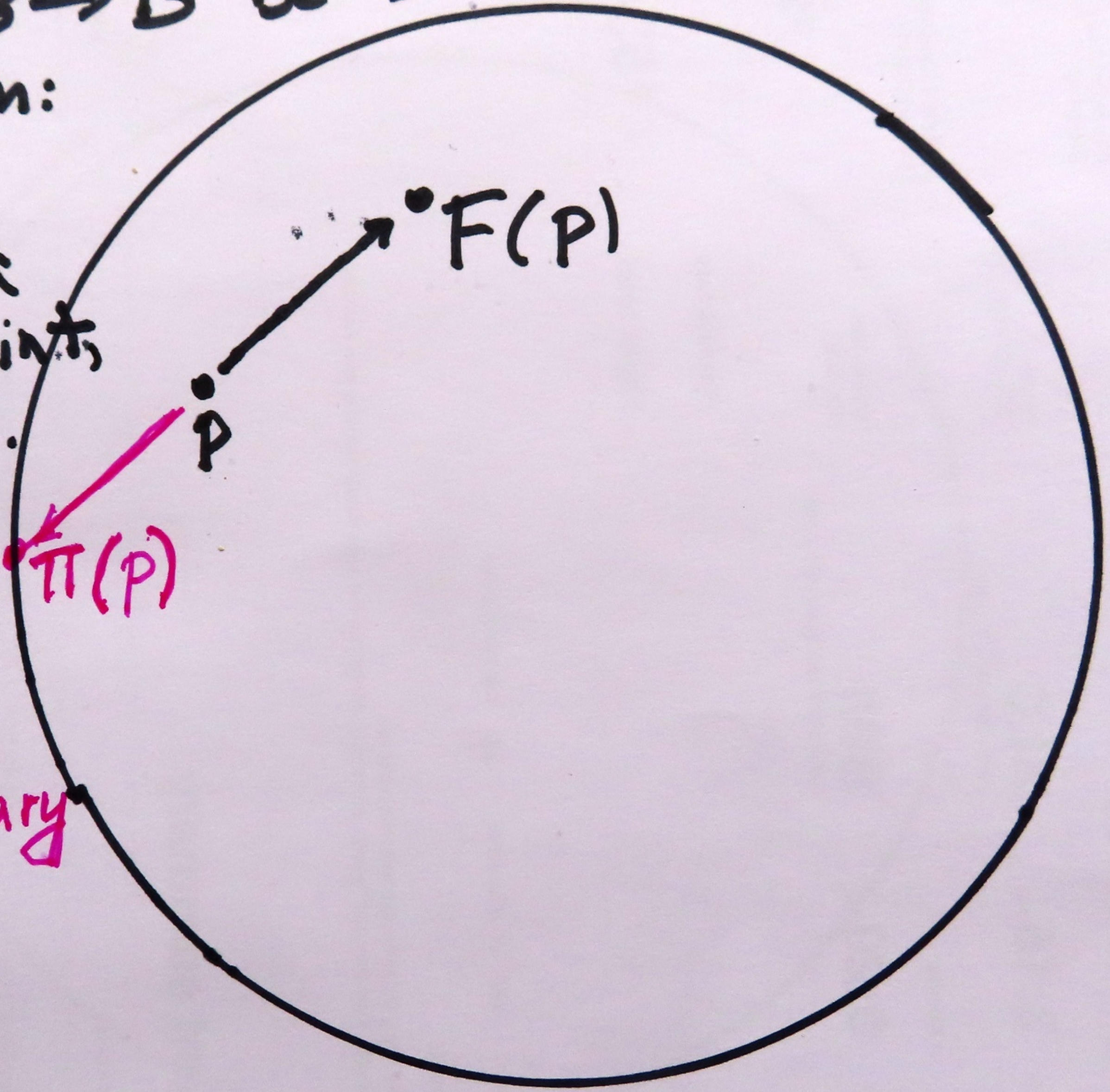
Theorem:

There exists a fixed point, $F(p_0) = p_0$.



Let B be a smooth ball in $\mathbb{R}^n$.
Let $F: B \to B$ be smooth.

Theorem:
There exists a fixed point, $F(p_0) = p_0$.



DEFINE "RETRACT" π
MAP ONTO
THE Boundary

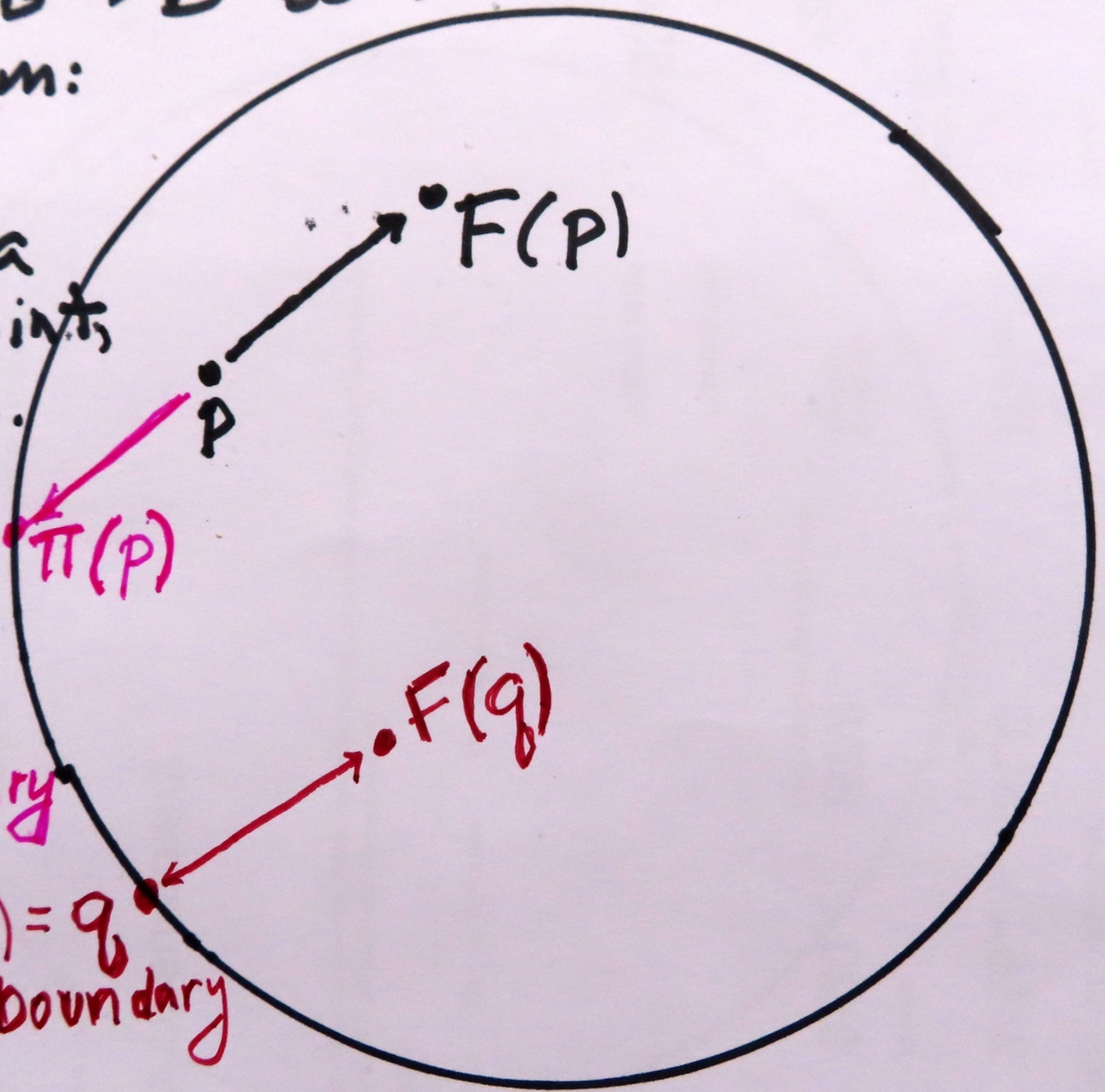
Let B be a smooth ball in $\mathbb{R}^n$.

Let $F: B \to B$ be smooth.

Theorem:

There exists a fixed point, $F(p_0) = p_0$.

DEFINE "RETRACT" π MAP ONTO THE Boundary



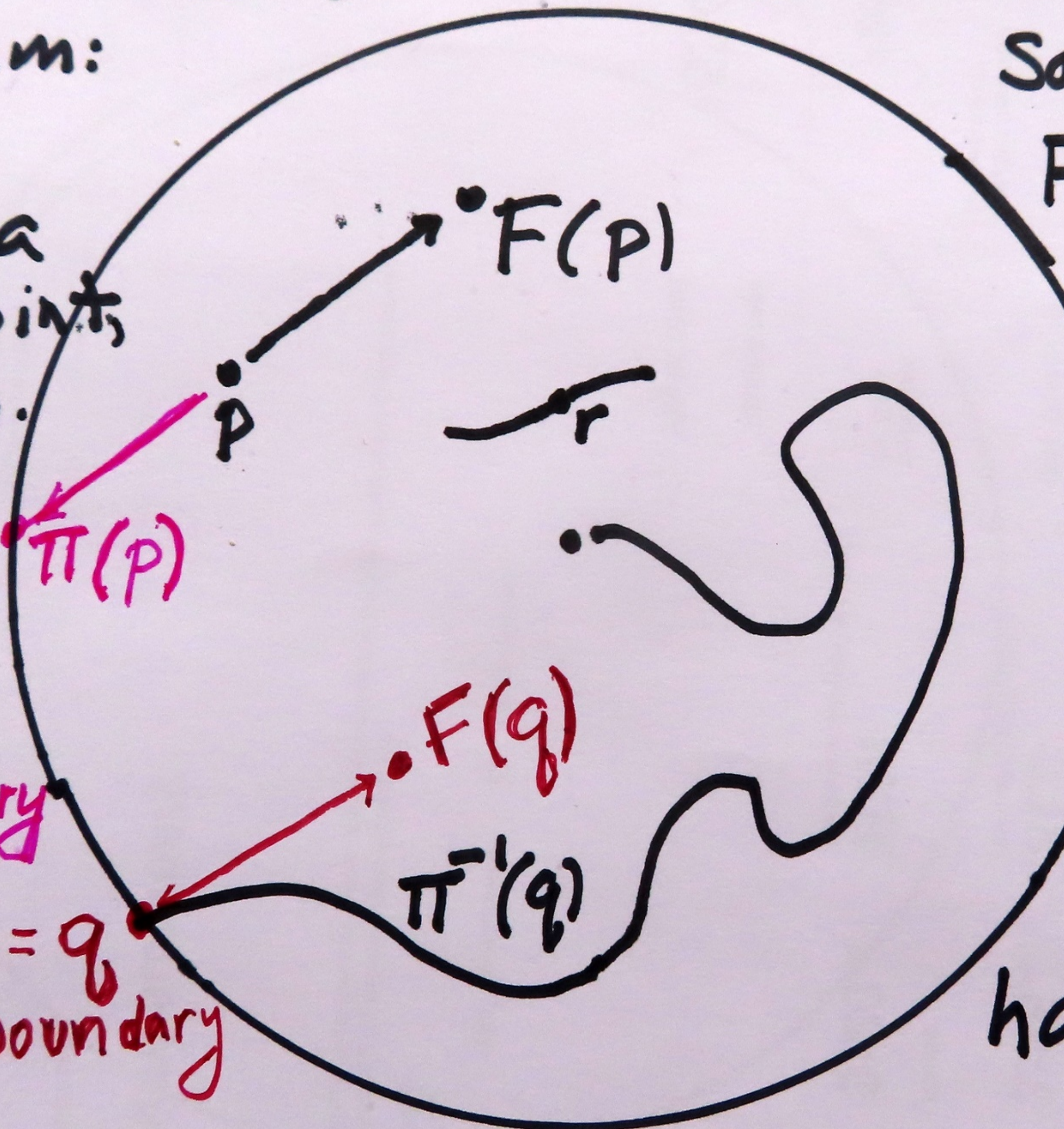
Let B be a smooth ball in $\mathbb{R}^n$.

Let $F: B \to B$ be smooth.

Theorem: There exists a fixed point, $F(p_0) = p_0$.

Sard's Thm (1943)

For almost every $q \in$ boundary, q is a regular value: If $\pi(r) = q$ then $D\pi(r)$ has full rank.

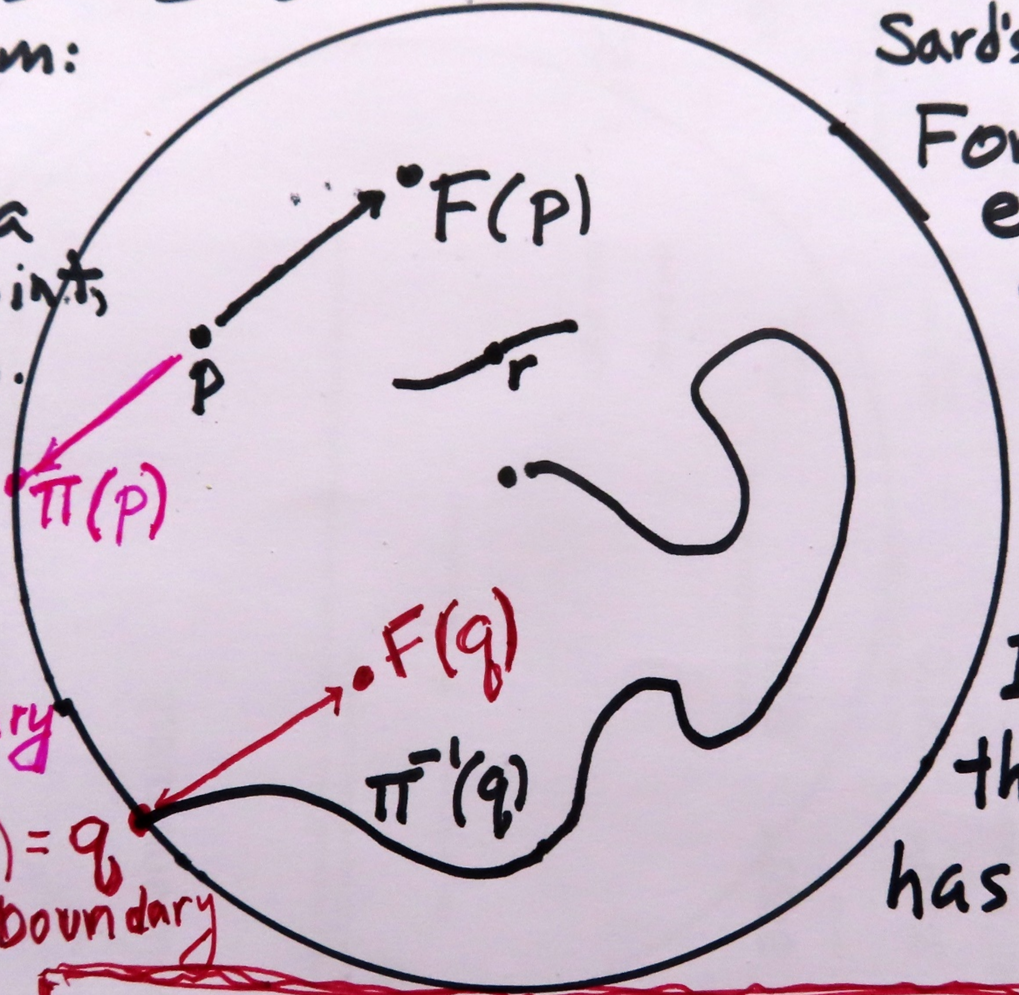


DEFINE "RETRACT" π MAP ONTO THE Boundary

$\pi(q) = q$ on the boundary

Let B be a smooth ball in $\mathbb{R}^n$.
Let $F: B \to B$ be smooth.

Theorem: There exists a fixed point, $F(p_0) = p_0$.



DEFINE "RETRACT" π MAP ONTO THE Boundary

$\pi(q) = q$ on the boundary

Sard's Thm (1943)
For almost every $q \in$ boundary, q is a regular value: If $\pi(r) = q$ then $D\pi(r)$ has full rank.

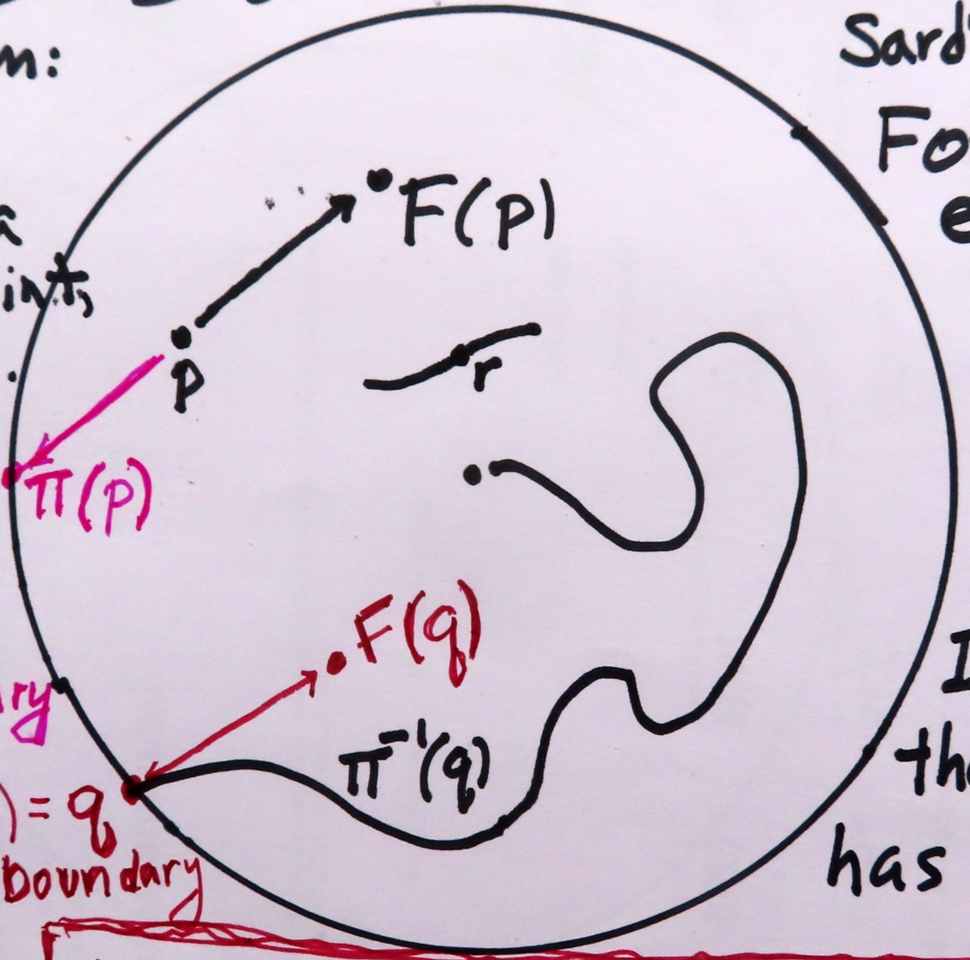
WHERE Does the $\pi^{-1}(q)$ path from q End?

Let B be a smooth ball in $\mathbb{R}^n$.
Let $F: B \to B$ be smooth.

Theorem: There exists a fixed point, $F(p_0) = p_0$.

Sard's Thm (1943)
For almost every $q \in$ boundary, q is a regular value: If $\pi(r) = q$ then $D\pi(r)$ has full rank.

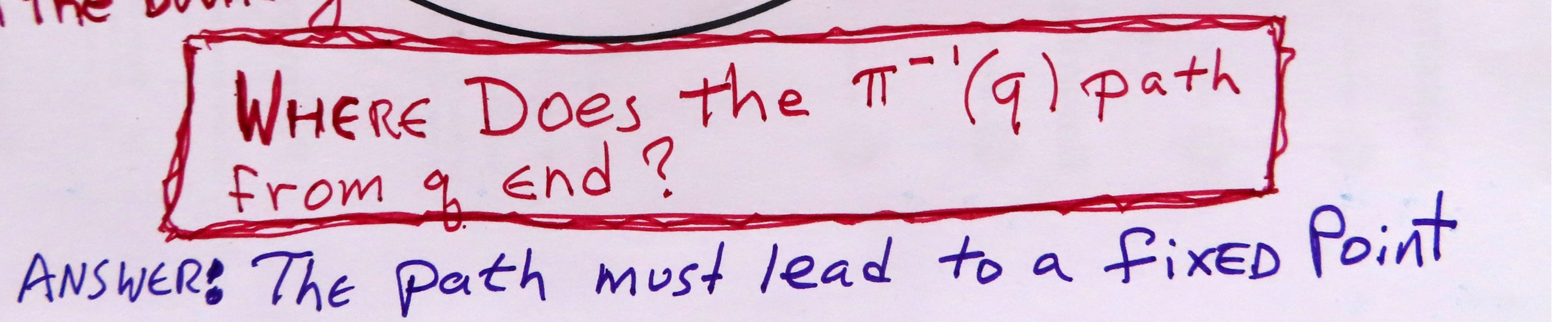
DEFINE "RETRACT" π MAP ONTO THE Boundary



π(q) = q on the boundary

WHERE DOES the $\pi^{-1}(q)$ path from q End?

ANSWER: The path must lead to a fixed Point



Thus a fixed point exists.

This path-following method is “essentially” constructive.

Follow the path from almost any q to find a fixed point.

Now apply the methods of that Brouwer Fixed Point Proof to:

Period Doubling Cascades

Mallet-Paret's orbit index: Orbit index assigns values to hyperbolic periodic orbits:

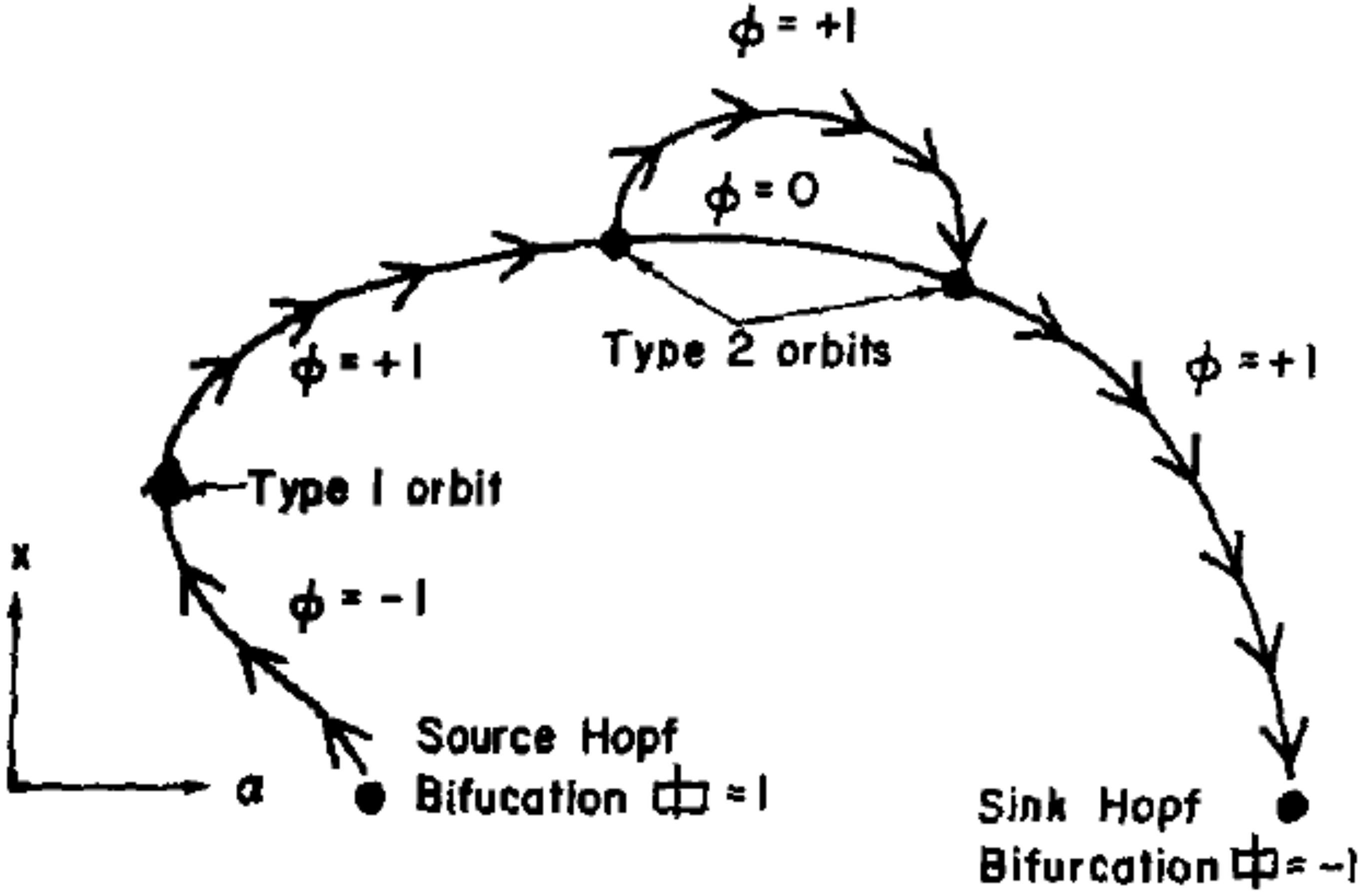
+1 (attractors)   use orientation >>>

-1 ("regular" saddles)   <<<

0 ("flip" saddles) are not followed

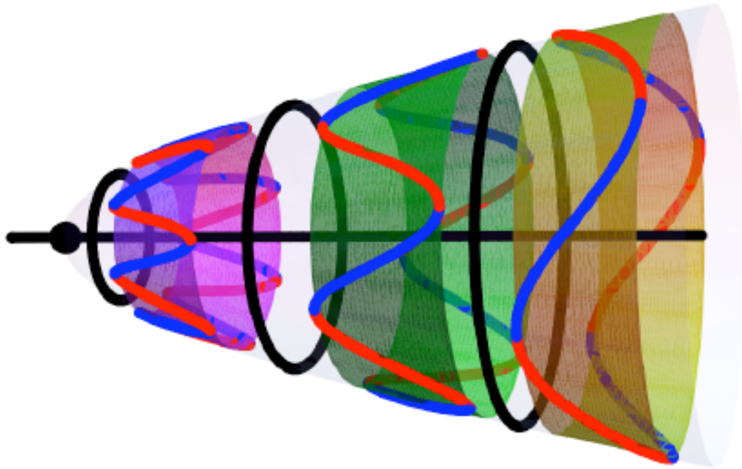
- Attracting periodic orbits are on paths with orientation >>>
- Regular saddles are on paths with orientation <<<
- Flip saddles are not on paths.

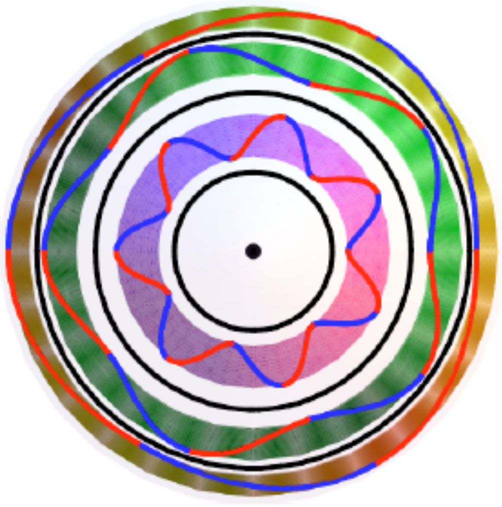
Periodic Orbit Index (Mallet-Paret & Y 1982)
Our periodic orbit index is a bifurcation invariant.

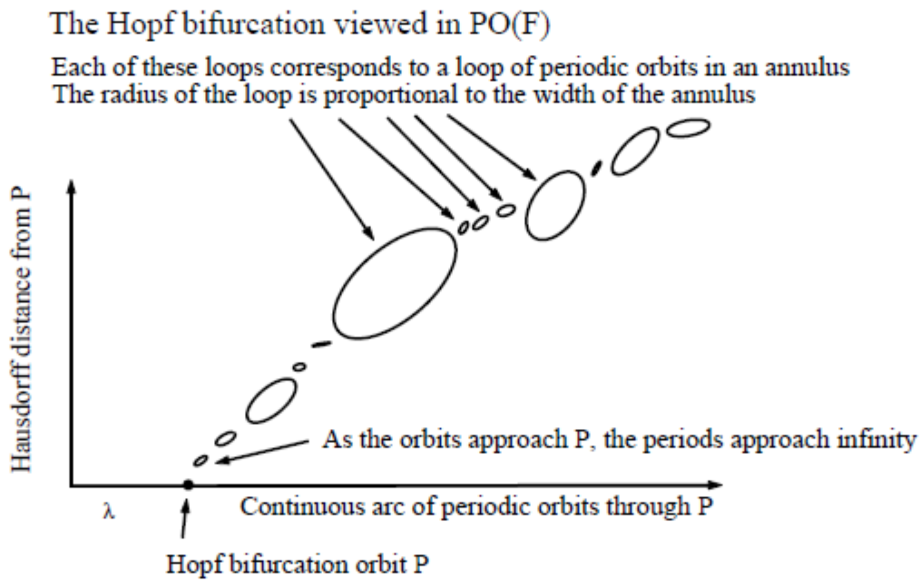


A snake of periodic orbits: hypothetical example illustrating terminology.

Generic Map-Hopf bifurcations can be added to the menu without loss.
The path simply does not see them.
Numerical orbit tracking ignores them.



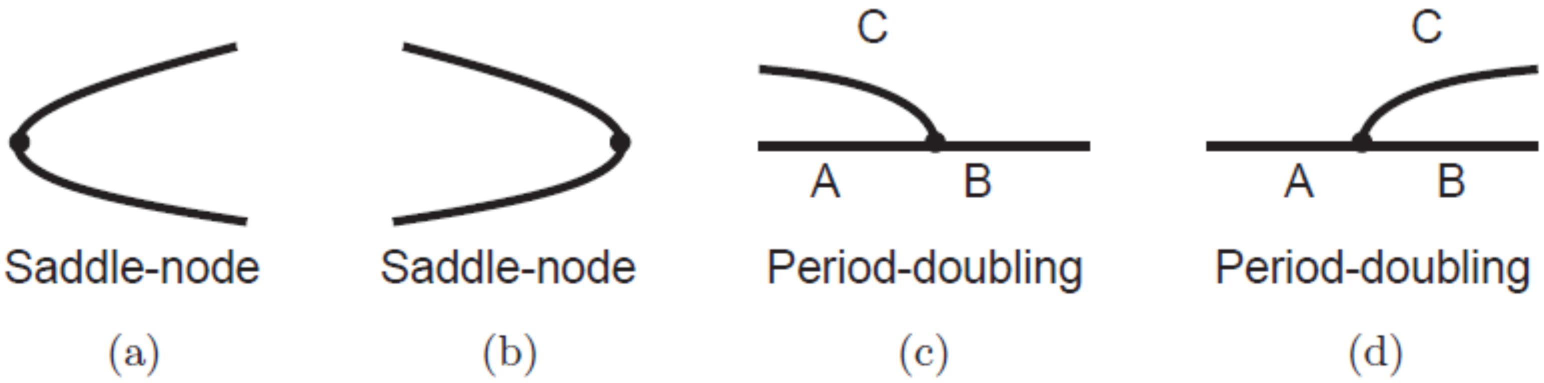


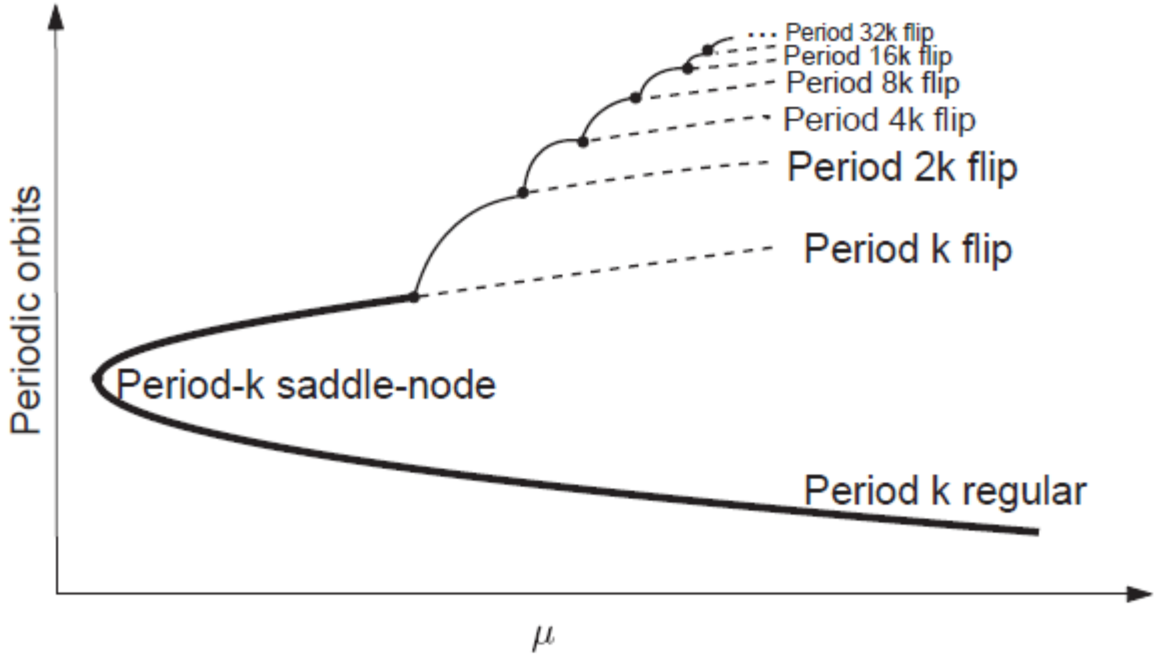


These loops are all disconnected.

Regular component is locally a one-manifold.

Generic bifurcations with 1 eigenvalue crossing the unit circle – path representation- each point is a periodic orbit.





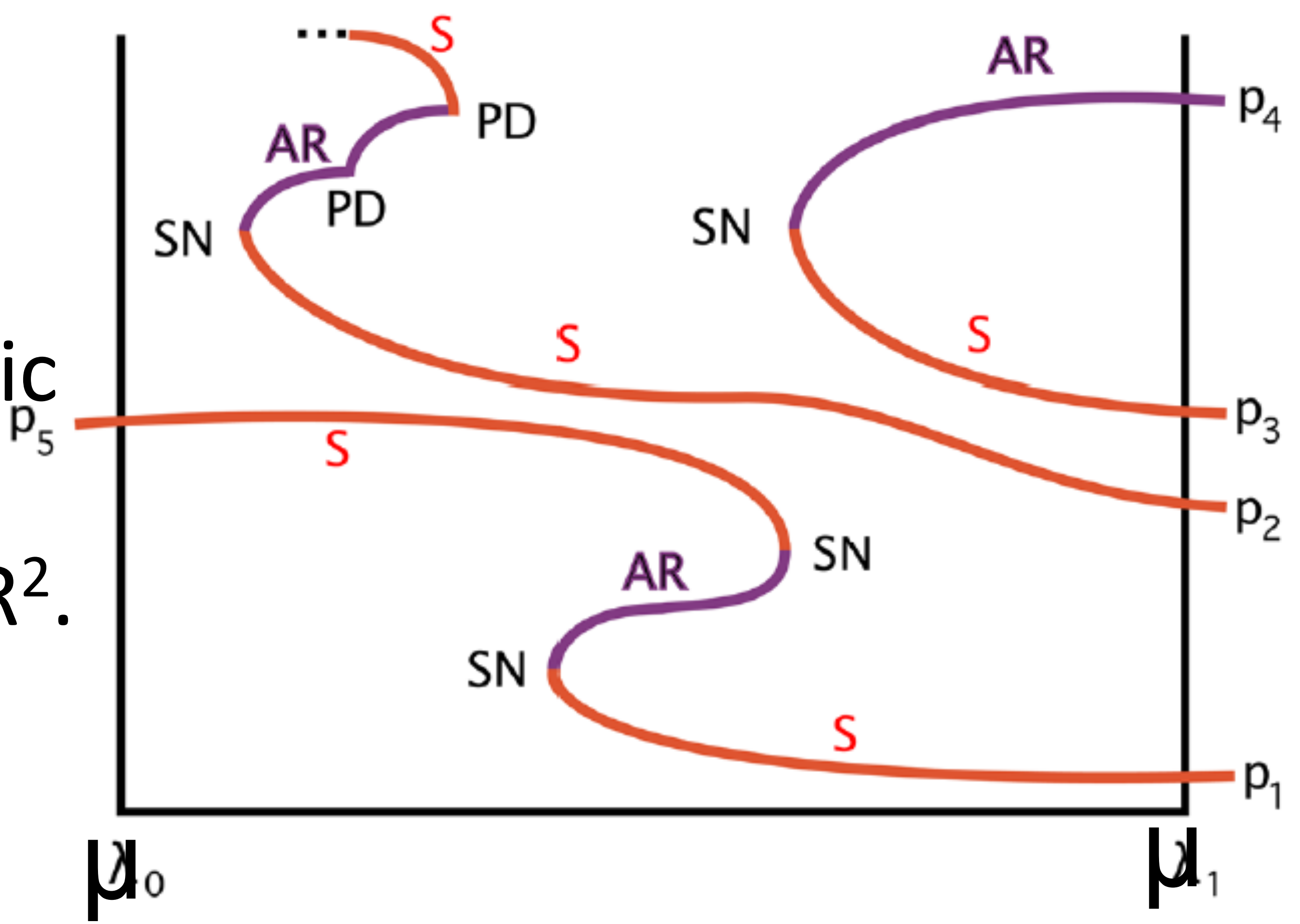
S= even saddle

AR= attractor or even repellor (has 0 or 2 e-vals <-1)

PD = period-doubling point.

Assume F: $R^{2+1} \rightarrow R^2$

Assume the set of periodic points is bounded in R^2.



We return to the topic of period doubling cascades, where we apply similar ideas.

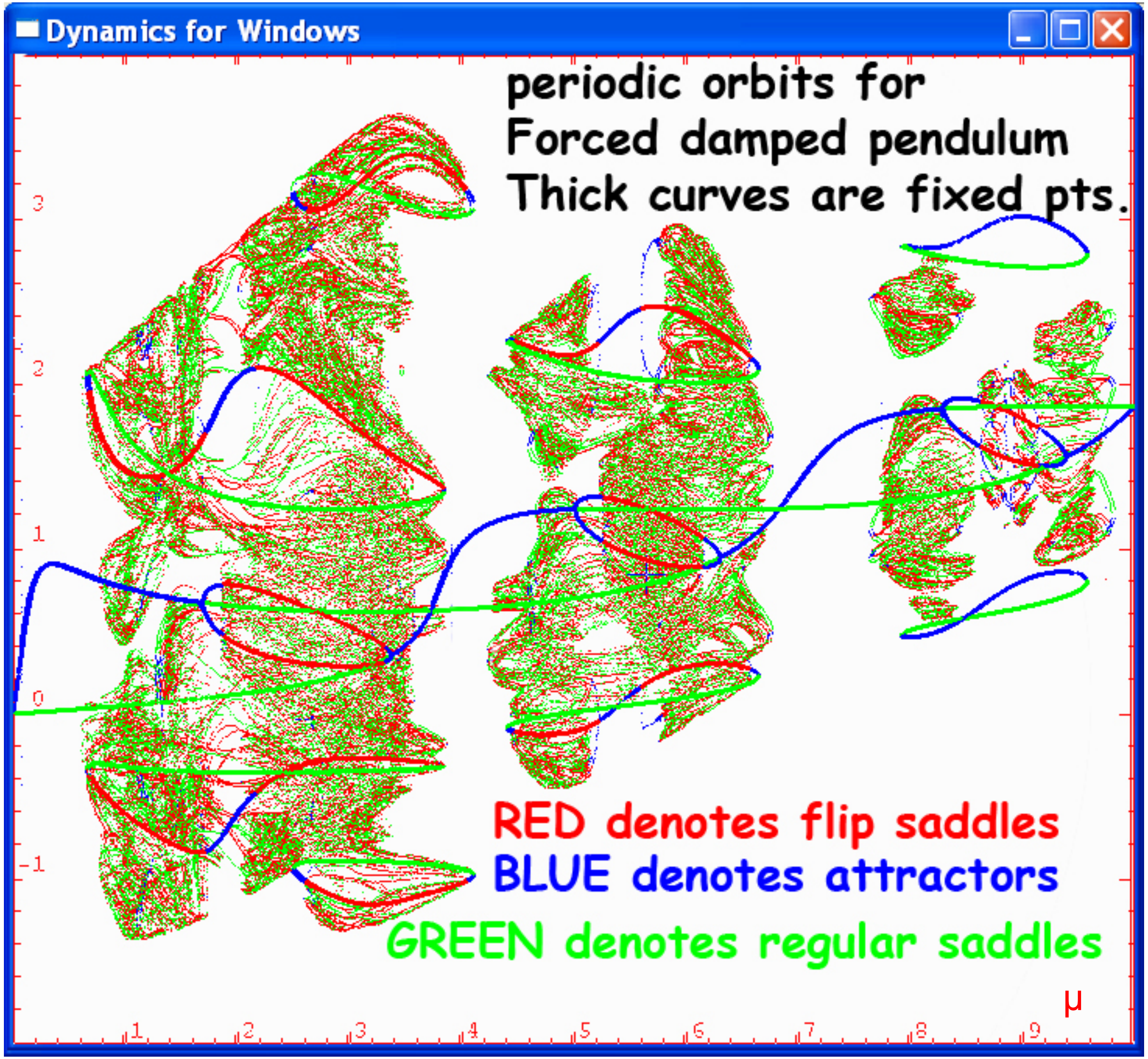
K. T. Alligood, J. Mallet-Paret and J. A. Yorke, Families of periodic orbits: Local continuability does not imply global continuability, J. Differential Geom. 16 (1981), 483-492.

- J. Mallet-Paret and J. A. Yorke, Snakes: Oriented families of periodic orbits, their sources, sinks, and continuation, J. Differential Equations 43 (1982), 419-450.
- J. A. Yorke and K. T. Alligood, Cascades of period doubling bifurcations: A prerequisite for horseshoes, Bull. Amer. Math. Soc. 9J. Announcement.
- J. A. Yorke and K. T. Alligood, Period doubling cascades of attractors: A prerequisite for horseshoes, Comm. Math. Phys. 101 (1985), 305-321.
- Evelyn Sander and J. A. Yorke, Period-doubling cascades for large perturbations of Henon families, J Fixed Point Theory and Applications, 6(1): 153-163, 2009
- Evelyn Sander and J. A. Yorke, Period-doubling cascades galore, Ergodic Theory and Dynamical Systems, 31 (2011), 1249-1267.
- Evelyn Sander and J. A. Yorke, A Period-Doubling Cascade Precedes Chaos for Planar Maps, Chaos **23**, 033113 (2013).

We return to the topic of period doubling cascades, where we apply similar ideas.

K. T. Alligood, J. Mallet-Paret and J. A. Yorke, Families of periodic orbits: Local continuability does not imply global continuability, J. Differential Geom. 16 (1981), 483-492.

- J. Mallet-Paret and J. A. Yorke, Snakes: Oriented families of periodic orbits, their sources, sinks, and continuation, J. Differential Equations 43 (1982), 419-450.
- J. A. Yorke and K. T. Alligood, Cascades of period doubling bifurcations: A prerequisite for horseshoes, Bull. Amer. Math. Soc. 9J. Announcement.
- J. A. Yorke and K. T. Alligood, Period doubling cascades of attractors: A prerequisite for horseshoes, Comm. Math. Phys. 101 (1985), 305-321.
- Evelyn Sander and J. A. Yorke, Period-doubling cascades for large perturbations of Henon families, J Fixed Point Theory and Applications, 6(1): 153-163, 2009
- Evelyn Sander and J. A. Yorke, Period-doubling cascades galore, Ergodic Theory and Dynamical Systems, 31 (2011), 1249-1267.
- Evelyn Sander and J. A. Yorke, A Period-Doubling Cascade Precedes Chaos for Planar Maps, Chaos **23**, 033113 (2013).

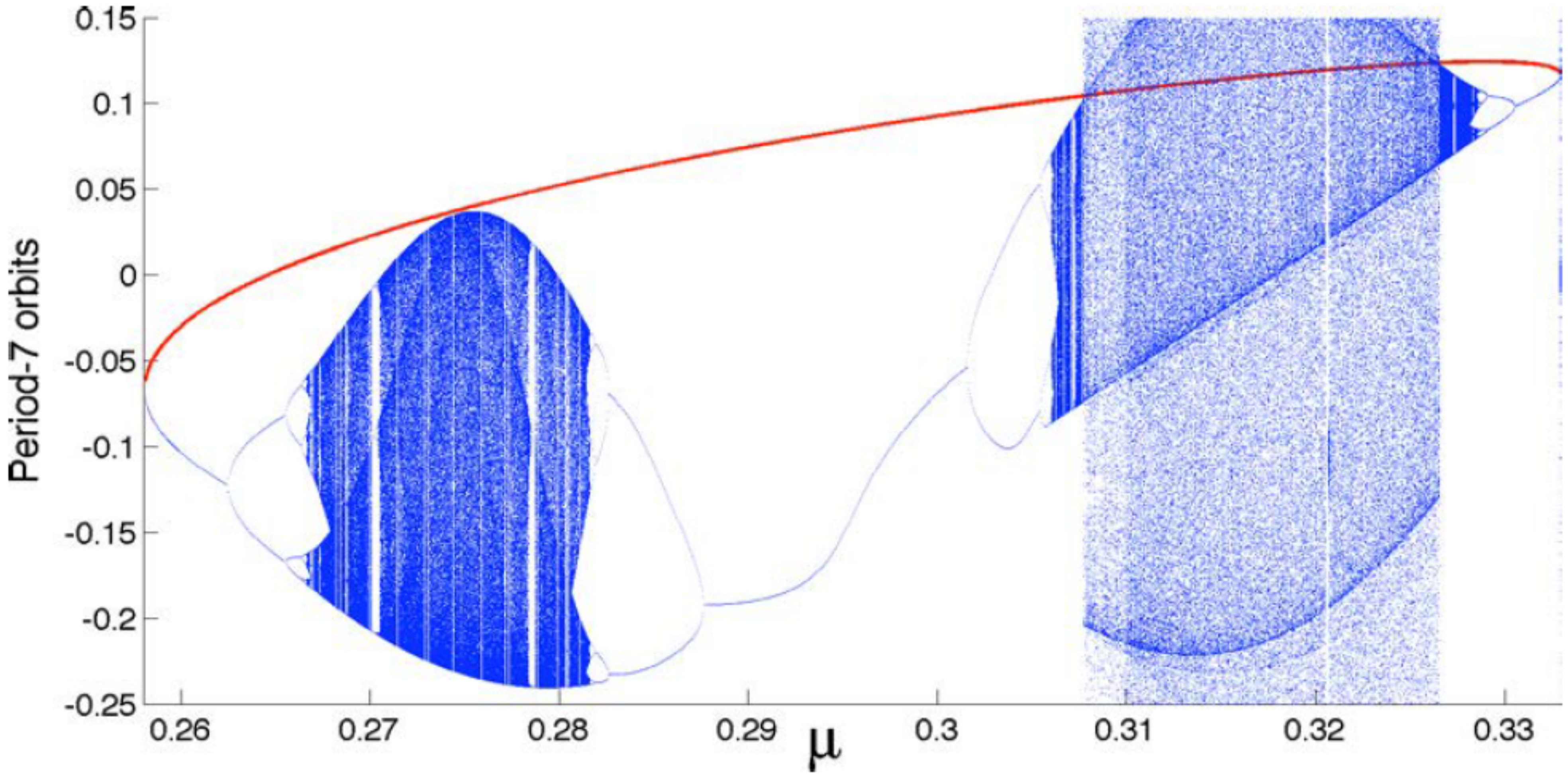

Dynamics for Windows
periodic orbits for
Forced damped pendulum
Thick curves are fixed pts.
RED denotes flip saddles
BLUE denotes attractors
GREEN denotes regular saddles
μ
3
2
1
0
-1
1
2
3
4
5
6
7
8
9

2 kinds of cascades

- There are two kinds of cascades:

  "paired" and

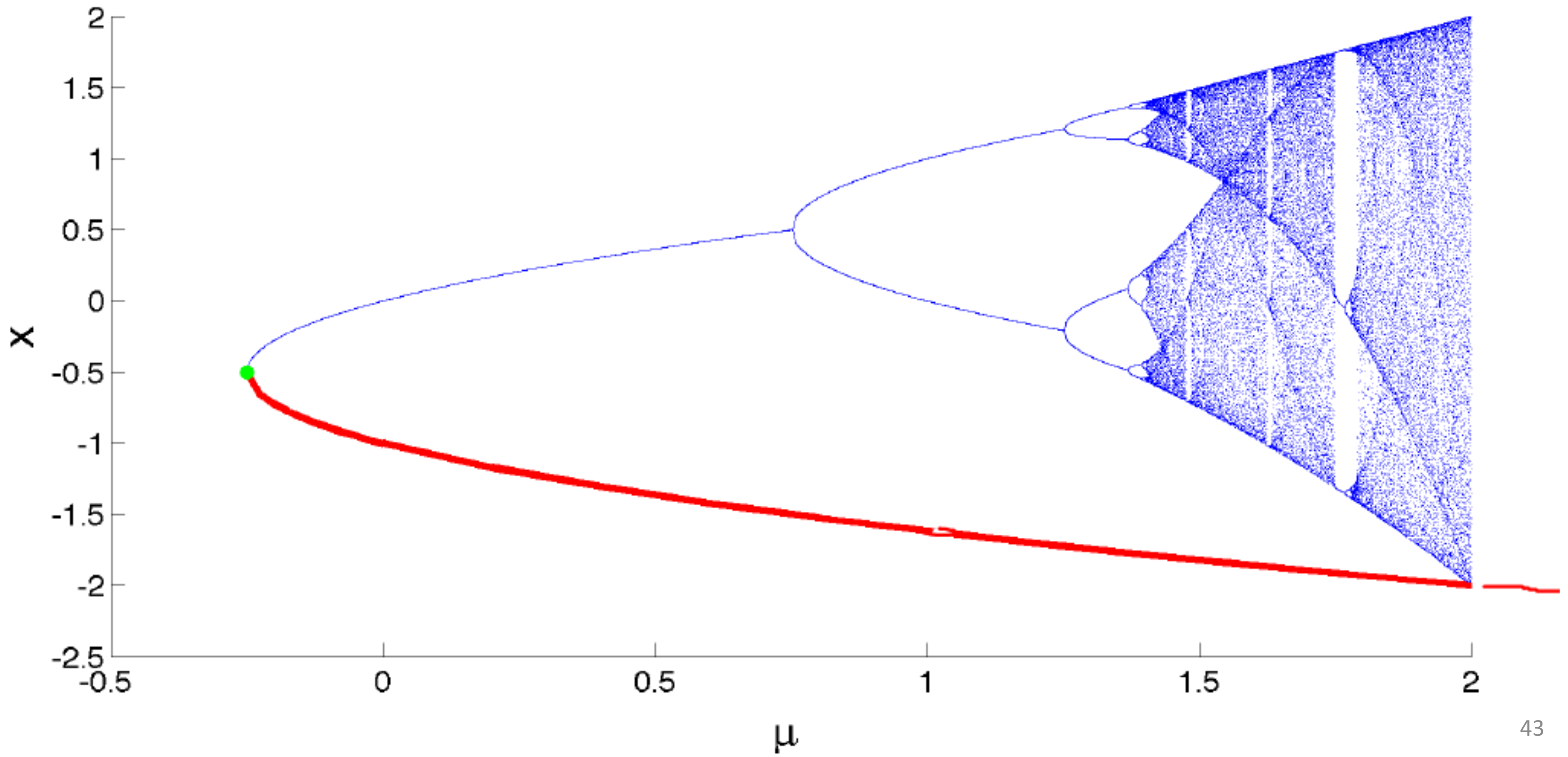
  "solitary" (as in the quadratic map).



- Figures from Sander Yorke 2012 IJBC paper

Fig. 5. *Paired cascades in the Hénon map* $(u, v) \mapsto (1.25 - u^2 + \mu v, u)$. The top bifurcation diagram shows a set of four period-7 cascades. The bottom bifurcation diagram shows the detail of the top part. Only one point of each of the period-7 orbits of the Hénon map are shown so that it is clearer how the two pairs connect to each other. The leftmost and rightmost cascades form a pair that is connected by a path of unstable regular periodic orbits (shown in red). Likewise, the two middle cascades form a pair. It is connected by a path of attracting period-seven orbits (blue). Paired cascades are not robust to moderate changes in the map.

The hidden part of a period doubling cascade



A more complex cascade pattern: black for attractors, red for periodic orbits

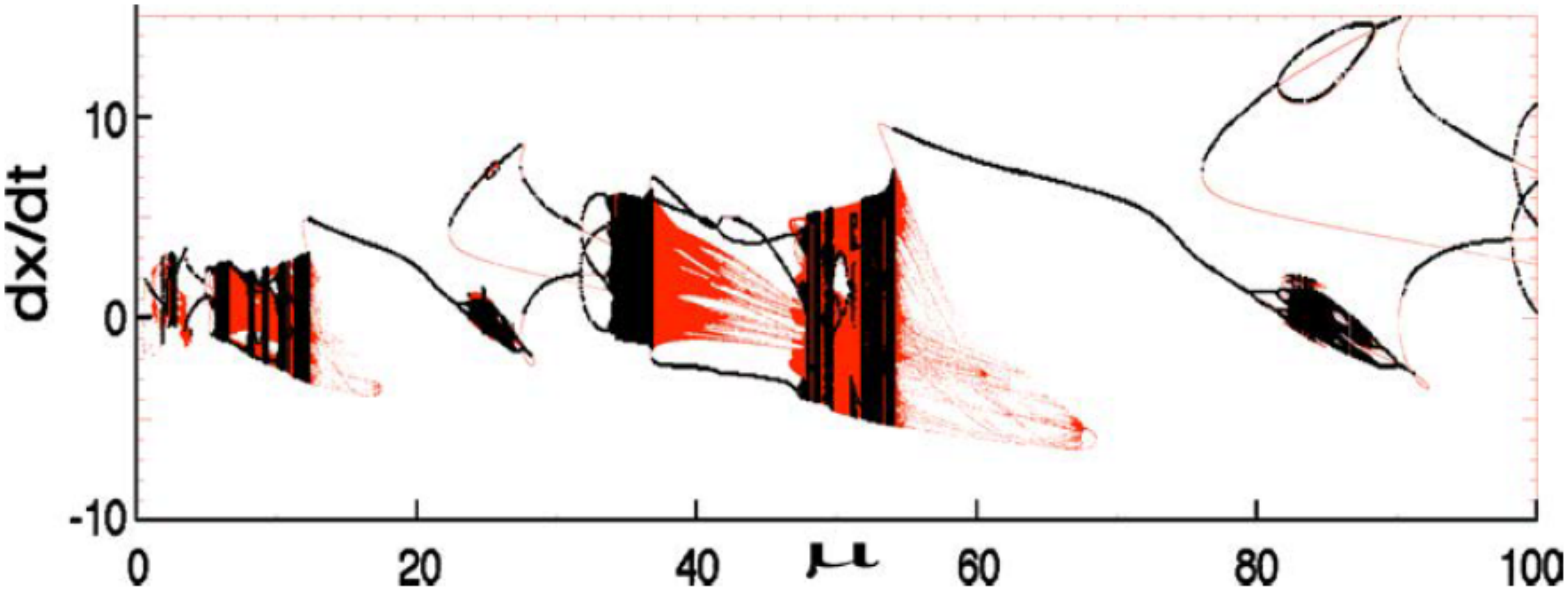


Fig. 2. *Cascades in the double-well Duffing equation.* The attracting sets (in black) and periodic orbits up to period ten (in red) for the time-2π map of the double-well Duffing equation: $x''(t) + 0.3x'(t) - x(t) + (x(t))^2 + (x(t))^3 = \mu \sin t$. Numerical studies show regions of chaos interspersed with regions without chaos, as in the Off–On–Off Chaos Theorem (Theorem 5).

Forced damped pendulum

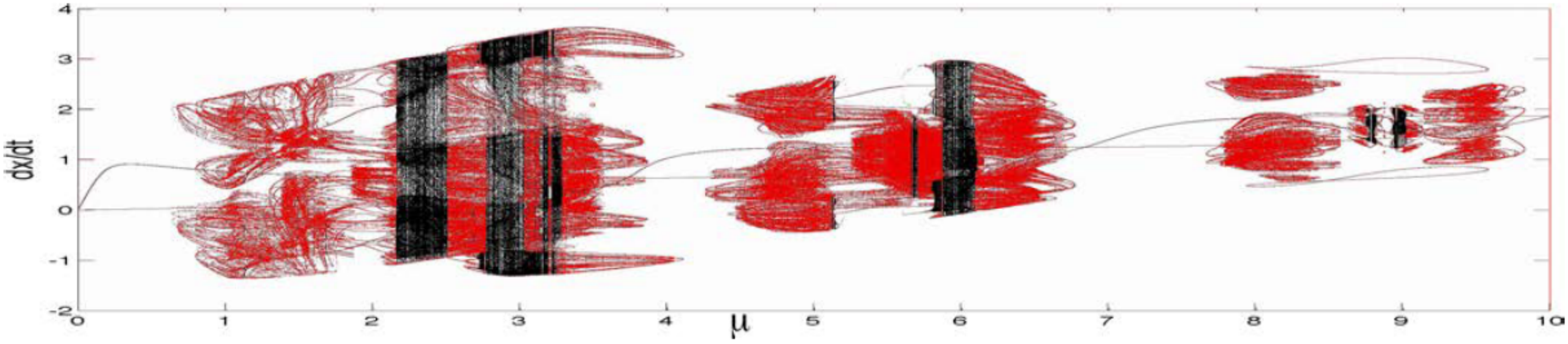


Fig. 3. *The forced-damped pendulum.* For this figure, periodic points with periods up to ten were plotted in red for the time-2π map of the forced-damped pendulum equation: $x''(t) + 0.2x'(t) + \sin(x(t)) = \mu\cos(t)$, indicating the general areas with chaotic dynamics for this map. Then the attracting sets were plotted in black, hiding some periodic points. Parameter ranges with and without chaos are interspersed.

Cascades for each orbit up to period 6

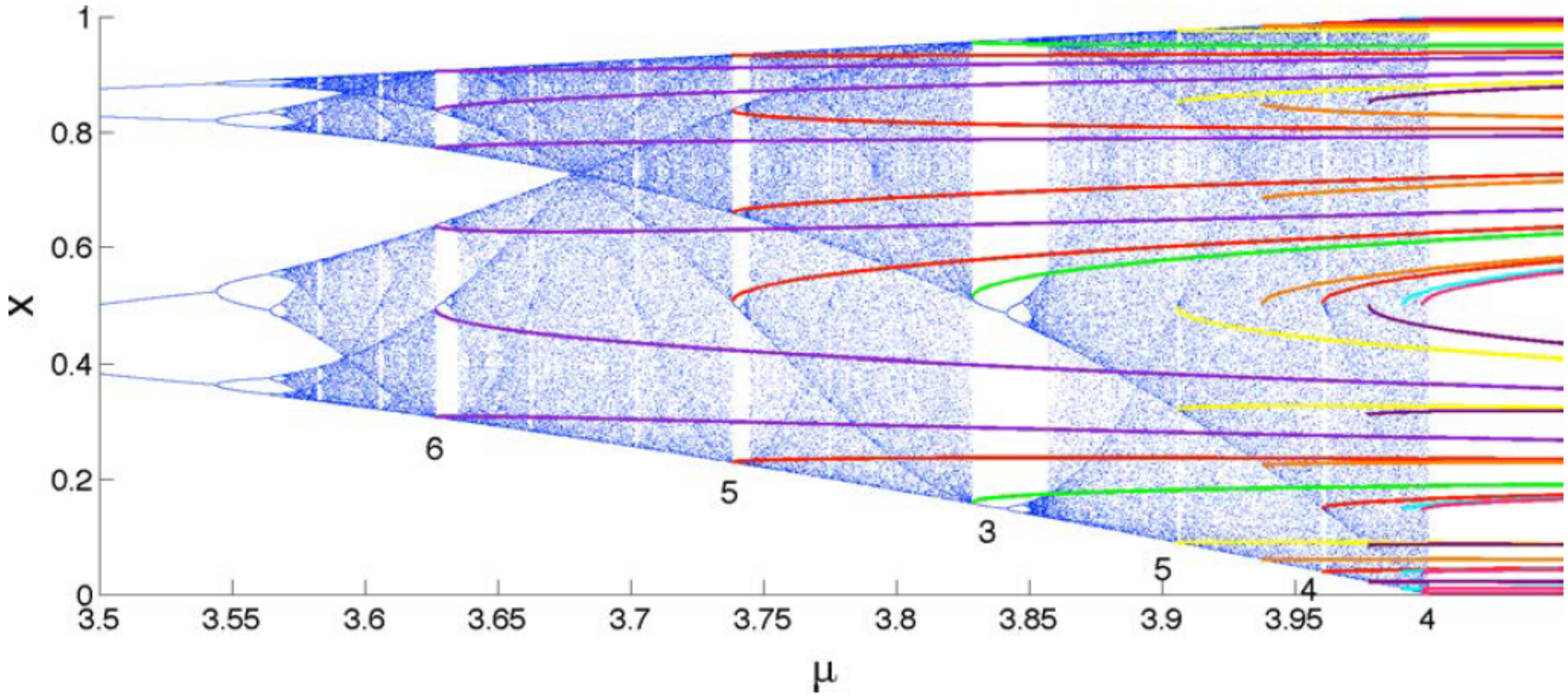


Fig. 4. *Cascades for* $F(\mu, x) = \mu x(1 - x)$. The logistic map has infinitely many cascades of attracting periodic orbits, and all cascades start at the stable orbit of a saddle-node bifurcation. The unstable orbits form what we call the **stems** of the cascades (shown in color). Each stem continues to exist for all large μ values. By our terminology, this means that all the cascades shown are solitary (on any parameter interval $[\mu_1, \mu_2]$, for $\mu_1 = 3.5$ and any $\mu_2 > 4$) since the stem does not connect its cascade to a second cascade. The stems are shown here up to period six. Different colors are used for different periods.