Optimizing the Kelvin force in a moving target subdomain

Harbir Antil

Department of Mathematical Sciences
George Mason University
hantil@gmu.edu

July 15, 2016

SIAM Annual Meeting
Boston, July 15, 2016

Collaborators: R.H. Nochetto (UMD) and P. Venegas (Chile)

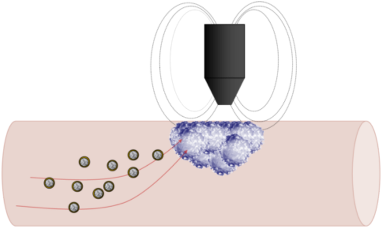
Funding: National Science Foundation (NSF) DMS–1521590

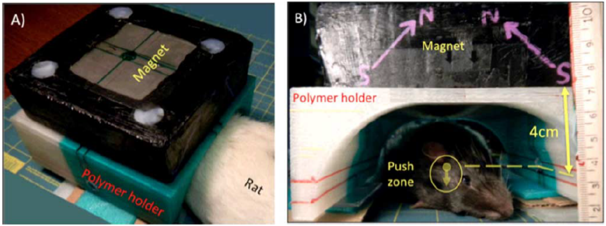
Motivation

The magnetic field exerts a force on magnetic materials such as magnetic nanoparticles (MNPs). MNPs under the action of external magnetic field are used in:

- medical sciences:
  - as contrast agents to enhance the contrast in MRI
  - as carriers for targeted drug delivery, for instance, to treat cancer cells, tumors ($< 0.1\%$ is taken by tumor cells)
  - in gene therapy
  - in magnetized stem-cells
- magnetic tweezers
- lab-on-a-chip systems that include magnetic particles or fluids
- magnetofection a transfection method

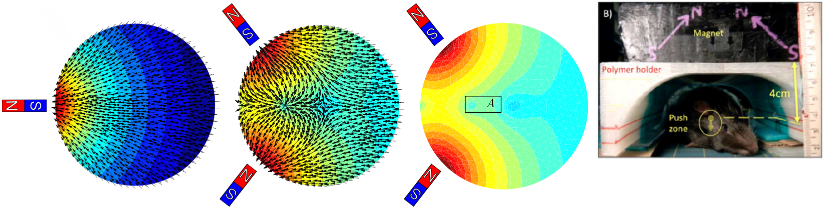
Magnetic drug targeting (MDT)





Experimental setup ([Shapiro et al., 2013])

What are engineers interested in?



- Pull in or attract particles (left): there are already human trials on this.
- **Key difficulty:** To push or to control particles (center and right). In region A the force is pointing outward, allowing us to push particles.
- The success of the aforementioned applications highly depend on the accurate control of the magnetic force.
- **Goal:** how to approximate a desired magnetic force $\mathbf{f}$ by a fixed configuration of magnetic field sources.
- **Approach:**

$$\min_{\mathbf{F}} \int_0^T \|\mathbf{F} - \mathbf{f}\|^2_{\mathrm{L}^2(D)} \, dt \quad \text{for } T > 0 \text{ and } D \subset \mathbb{R}^d, d = 2, 3.$$

How does magnetic field manipulate MNPs?

- **Magnetic force:** Magnetic field gradient is required to exert a force and such a force is given by [Rosensweig '97]:

$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{H}.$$

- **A simplification:** After some simplifications (weakly diamagnetic medium, no current sources) and using $\operatorname{curl} \mathbf{H} = 0$, the force on a single MNP

$$\mathbf{F} = \frac{V_m \Delta\chi}{2} \nabla |\mathbf{H}|^2$$

  V_m: is the volume of the particle
  $\Delta\chi = \chi_p - \chi_m$: effective susceptibility.

- **Fundamental difficulty:** magnetic field intensity $\mathbf{H}$ is not parallel to the magnetic force $\mathbf{F}$.

Minimization problem and control action

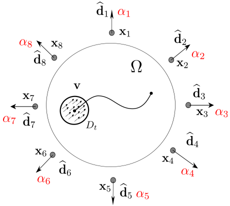
- $\Omega \subset \mathbb{R}^d$, $d = 2, 3$ open, bounded
  $D_t \subset \Omega$: time dependent subdomain.
- **Maxwell's equations:** We consider magnetic sources outside Ω then

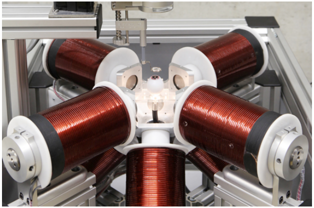
$$\operatorname{curl} \mathbf{H} = \mathbf{0}, \quad \operatorname{div} \mathbf{H} = 0 \quad \text{in } \Omega.$$

- **Dipole approximation:**

$$\mathbf{H}(\boldsymbol{x}, t) = \sum_{i=1}^{n_p} \alpha_i(t) \left(d \frac{(\boldsymbol{x} - \boldsymbol{x}_i)(\boldsymbol{x} - \boldsymbol{x}_i)^\top}{|\boldsymbol{x} - \boldsymbol{x}_i|^2} - \mathbb{I} \right) \frac{\widehat{\mathbf{d}}_i}{|\boldsymbol{x} - \boldsymbol{x}_i|^d} = \sum_{i=1}^{n_p} \alpha_i(t) \mathbf{H}_i(\boldsymbol{x})$$

  $\boldsymbol{x}_i \in \mathbb{R}^d \setminus \overline{\Omega}$: dipole positions
  $\widehat{\mathbf{d}}_i \in \mathbb{R}^d$: field direction





OctoMag. IRIS - Institute (Zürich)

Problem 1: Fixed final time

- **Minimization problem:**

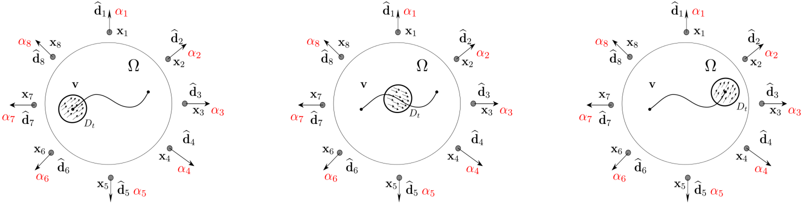
$$\min_{\boldsymbol{\alpha}\in\mathcal{H}_{ad}} \mathcal{J}(\boldsymbol{\alpha}), \quad \text{with } \frac{1}{2}\int_0^T \|\nabla|\mathbf{H}(\boldsymbol{\alpha})|^2 - \mathbf{f}\|^2_{\mathrm{L}^2(D_t)} dt + \frac{\lambda}{2}\int_0^T |d_t\boldsymbol{\alpha}|^2 dt,$$

with $\boldsymbol{\alpha}(t) := (\alpha_1(t), \ldots, \alpha_{n_p}(t))^\top$ in

$$\mathcal{H}_{ad} := \left\{\boldsymbol{\alpha} \in [\mathrm{H}^1(0,T)]^{n_p} : \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \quad \text{and} \quad \boldsymbol{\alpha}_* \le \boldsymbol{\alpha}(t) \le \boldsymbol{\alpha}^*, \ \forall t \in [0,T]\right\}.$$

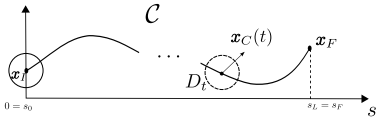
- **Reformulation:**

$$\min_{\boldsymbol{\alpha}\in\mathcal{H}_{ad}} \mathcal{J}(\boldsymbol{\alpha}), \quad \text{with } \mathcal{J}(\boldsymbol{\alpha}) = \frac{1}{2}\int_0^T \left(\sum_{i=1}^d \|\boldsymbol{\alpha}^\top \mathbf{P}_i\boldsymbol{\alpha} - \mathbf{f}_i\|^2_{\mathrm{L}^2(D_t)} + \lambda|d_t\boldsymbol{\alpha}|^2\right) dt.$$



Problem 2: Minimizing the final time

- **Unknown $\mathbf{f}$.** Since the final time T_F is an unknown, thus $\mathbf{f}$ is not meaningful quantity. We treat $\mathbf{f}$ as an unknown.
- Let $\boldsymbol{\rho} \in C^1[0, s_F]$ be a parameterization of $\mathcal{C}$ with respect to arc length.



- Assume that the barycenter $\boldsymbol{x}_C(t)$ of D_t moves along curve $\mathcal{C}$ at speed $\theta(t) > 0$ with initial position $\boldsymbol{x}_I$ and final $\boldsymbol{x}_F$ such that

$$\int_0^{T_F} \theta(\tau) d\tau = s_F.$$

- We define the map $\sigma(\cdot) : [0, T_F] \to [0, s_F]$ as

$$s = \sigma(t) = \int_0^t \theta(\tau) d\tau.$$

- Whence $\boldsymbol{x}_C(\cdot) = \boldsymbol{\rho} \circ \sigma(\cdot)$. Also $d_t \boldsymbol{x}_C(t) = \theta(t) d_t \boldsymbol{\rho}(\sigma(t))$.

Problem 2: Minimizing the final time

$$\min_{(\boldsymbol{\alpha},\theta)\in\mathcal{H}_{ad}\times\mathcal{V}_{ad}} \mathcal{F}(\boldsymbol{\alpha},\theta)$$

with

$$\mathcal{F}(\boldsymbol{\alpha},\theta) := \int_0^{s_F} \left(\frac{1}{2\theta(s)} \sum_{i=1}^{d} \|\boldsymbol{\alpha}(s)^\top \mathbf{P}_i \boldsymbol{\alpha}(s) - \boldsymbol{\rho}_i'(s)\theta(s)\|^2_{\mathrm{L}^2(D_s)} \right.$$
$$\left. + \frac{\beta}{\theta(s)} + \frac{\lambda}{2}|d_s\boldsymbol{\alpha}(s)|^2 + \frac{\eta}{2}|d_s\theta(s)|^2 \right) ds$$

and

$$\mathcal{U}_{ad} := \left\{ \boldsymbol{\alpha} \in \mathrm{H}^1(0,s_F) : \boldsymbol{\alpha}(0) = \boldsymbol{\alpha}_0 \quad \text{and} \quad \boldsymbol{\alpha}_* \le \boldsymbol{\alpha}(s) \le \boldsymbol{\alpha}^*,\ \forall s \in [0,s_F] \right\},$$
$$\mathcal{V}_{ad} := \left\{ \theta \in \mathrm{H}^1(0,s_F) : \theta(0) = \theta_0 \quad \text{and} \quad 0 < \theta_* \le \theta(s) \le \theta^*,\ \forall s \in [0,s_F] \right\}.$$

Well-posedness of continuous problem

- **Existence of solution.** If $\mathbf{f} \in [\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))]^d$ then using direct method of calculus of variations, there exist a solution to Problem 1. Same holds for Problem 2 if $\boldsymbol{\rho}$ is C^1.
- **First order necessary optimality conditions.**
  - If $\bar{\boldsymbol{\alpha}} \in \mathcal{U}_{ad}$ solves Problem 1 then

$$\mathcal{J}'(\bar{\boldsymbol{\alpha}})(\boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}) \geq 0 \qquad \forall \boldsymbol{\alpha} \in \mathcal{H}_{ad}.$$

  - If $(\bar{\boldsymbol{\alpha}}, \bar{\theta}) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$ solves Problem 2 then

$$\nabla \mathcal{F}(\bar{\boldsymbol{\alpha}}, \bar{\theta})(\delta\boldsymbol{\alpha}, \delta\theta) \geq 0 \qquad \forall (\boldsymbol{\alpha}, \theta) \in \mathcal{U}_{ad} \times \mathcal{V}_{ad}$$

  where $\delta\boldsymbol{\alpha} = \boldsymbol{\alpha} - \bar{\boldsymbol{\alpha}}$, $\delta\theta = \theta - \bar{\theta}$.
- **Second order sufficient condition.** Under the assumption

$$\mathcal{J}''(\bar{\boldsymbol{\alpha}})(\delta\boldsymbol{\alpha})^2 \geq \omega |\delta\boldsymbol{\alpha}|^2_{\mathrm{H}^1(0,T)} \qquad \forall \delta\boldsymbol{\alpha} \in \mathcal{A}(\bar{\boldsymbol{\alpha}})$$

  where $\mathcal{A}(\boldsymbol{\alpha}) := \left\{ \mathbf{h} \in \mathrm{H}^1_0(0,T) : \boldsymbol{\alpha} + \zeta\mathbf{h} \in \mathcal{H}_{ad}, \zeta > 0 \right\}$ there exists a local unique solution to Problem 1. Same applies to Problem 2.

Problem 1: Reference domain

- **Reference domain.** We define a reference domain $\widehat{D} \subset \mathbb{R}^d$ and a map $\mathbf{X} : [0,T] \times \overline{\widehat{D}} \to \overline{\Omega}$, such that for all $t \in [0,T]$

$$\begin{aligned} \mathbf{X}(t,\cdot) : & \overline{\widehat{D}} \to \overline{D}_t \\ & \widehat{\boldsymbol{x}} \to \mathbf{X}(t,\widehat{\boldsymbol{x}}) = \boldsymbol{\psi}(t) + \psi(t)\widehat{\boldsymbol{x}}, \end{aligned}$$

  $\boldsymbol{\psi} : [0,T] \to \mathbb{R}^d$, $\psi : [0,T] \to (0,+\infty)$, and $\boldsymbol{\psi}, \psi \in \mathrm{H}^1(0,T)$. Moreover, $\widehat{D} = D_0$.

- **Reference domain cost.** Then, we rewrite $\mathcal{J}$ as

$$\mathcal{J}(\boldsymbol{\alpha}) = \frac{1}{2}\sum_{i=1}^{d}\int_0^T \|\boldsymbol{\alpha}^\top \widehat{\mathbf{P}}_i\boldsymbol{\alpha} - \widehat{\mathbf{f}}_i\|^2_{\mathrm{L}^2(\widehat{D})} + \frac{\lambda}{2}\int_0^T |d_t\boldsymbol{\alpha}|^2 = \mathcal{J}^1(\boldsymbol{\alpha}) + \mathcal{J}^2(\boldsymbol{\alpha})$$

  with $\widehat{\mathbf{P}}_i(t,\widehat{\boldsymbol{x}}) := \mathbf{P}_i(\mathbf{X}(t,\widehat{\boldsymbol{x}}))\psi(t)^{d/2}$ and $\widehat{\mathbf{f}}_i(t,\widehat{\boldsymbol{x}}) = \mathrm{v}_i(t,\mathbf{X}(t,\widehat{\boldsymbol{x}}))\psi(t)^{d/2}$, for $i = 1,\ldots,d$.

Problem 1: Time discretization

- Let us fix $N \in \mathbb{N}$ and let $\tau := T/N$ be the time step. Now, for $n = 1, \ldots, N$, we define $t^n := n\tau$, $\widehat{\mathbf{P}}_i^n = \widehat{\mathbf{P}}_i(t^n)$ and $\widehat{\mathbf{f}}_i^n$ to be

$$\widehat{\mathbf{f}}_i^n(\cdot) = \frac{1}{\tau}\int_{t^{n-1}}^{t^n} \widehat{\mathbf{f}}_i(t,\cdot)dt, \qquad i = 1, \ldots, d,$$

  which in turn allows us to incorporate a general $\mathbf{f}$.

- **Time discrete problem:** given the initial condition $\boldsymbol{\alpha}_0 =: \bar{\boldsymbol{\alpha}}_\tau(0)$, find $\bar{\boldsymbol{\alpha}}_\tau \subset \mathcal{H}_{ad}^\tau$ solving

$$\bar{\boldsymbol{\alpha}}_\tau = \operatorname*{argmin}_{\boldsymbol{\alpha}_\tau \in \mathcal{H}_{ad}^\tau} \mathcal{J}_\tau(\boldsymbol{\alpha}_\tau), \quad \mathcal{J}_\tau(\boldsymbol{\alpha}_\tau) = \mathcal{J}_\tau^1(\boldsymbol{\alpha}_\tau) + \mathcal{J}_\tau^2(\boldsymbol{\alpha}_\tau),$$

  where

$$\begin{aligned} &\mathcal{J}_\tau^1(\boldsymbol{\alpha}_\tau) + \mathcal{J}_\tau^2(\boldsymbol{\alpha}_\tau) \\ &\quad = \tau \sum_{n=1}^{N} \frac{1}{2} \sum_{i=1}^{d} \|(\boldsymbol{\alpha}_\tau^n)^\top \widehat{\mathbf{P}}_i^n \boldsymbol{\alpha}_\tau^n - \widehat{\mathbf{f}}_i^n\|_{\mathrm{L}^2(\widehat{D})}^2 + \tau \sum_{n=1}^{N} \frac{\lambda}{2\tau^2} |\boldsymbol{\alpha}_\tau^n - \boldsymbol{\alpha}_\tau^{n-1}|^2. \end{aligned}$$

  and

$$\mathcal{H}_{ad}^\tau := \{\boldsymbol{\alpha}_\tau \in \mathrm{H}^1(0,T) : \boldsymbol{\alpha}_\tau|_{[t^{n-1},t^n]} \in \mathbb{P}^1,\ n = 1, \ldots, N\} \cap \mathcal{H}_{ad}.$$

Convergence of scheme

Theorem. The family of minimizers $\{\bar{\boldsymbol{\alpha}}_\tau\}_{\tau>0}$ to the discrete problem is

- uniformly bounded in $\mathrm{H}^1(0,T)$.
- it contains a subsequence that converges weakly to $\bar{\boldsymbol{\alpha}}$ in $\mathrm{H}^1(0,T)$.
- $\lim_{\tau\to 0} \mathcal{J}_\tau(\bar{\boldsymbol{\alpha}}_\tau) = \mathcal{J}(\bar{\boldsymbol{\alpha}})$.

The proof is motivated by Γ convergence.

Strong convergence to local minimizers

- **Auxiliary problem:** For a fixed $\varepsilon > 0$, we construct a family $\{\boldsymbol{\alpha}_\tau^\varepsilon\}_{\tau>0}$ upon solving the minimization problem

$$\boldsymbol{\alpha}_\tau^\varepsilon = \operatorname*{argmin}_{\boldsymbol{\alpha}_\tau \in \mathcal{H}_{ad}^{\tau,\varepsilon}} \mathcal{J}_\tau(\boldsymbol{\alpha}_\tau),$$

  where $\mathcal{H}_{ad}^{\tau,\varepsilon} = \left\{ \boldsymbol{\alpha}_\tau \in \mathcal{H}_{ad}^\tau : \|\Pi_\tau \bar{\boldsymbol{\alpha}} - \boldsymbol{\alpha}_\tau\|_{\mathrm{L}^2(0,T)} \le \varepsilon \right\}$.
- Next using the second order sufficient condition we show that $\{\boldsymbol{\alpha}_\tau^\varepsilon\}_{\tau>0}$ forms a local solution to our discrete problem.
- We conclude by showing that $\|\boldsymbol{\alpha}_\tau^\varepsilon - \bar{\boldsymbol{\alpha}}\|_{\mathrm{H}^1(0,T)} \to 0$ as $\tau \to 0$.

This approach is inspired by Casas and Troeltzsch '02.

Problem 2: Discretization

- We consider the following discrete problem: given an initial condition $(\boldsymbol{\alpha}_0, \theta_0) =: (\boldsymbol{\alpha}_\kappa(0), \theta_\kappa(0))$ find a solution $(\boldsymbol{\alpha}_\kappa, \theta_\kappa) \in \mathcal{U}_{ad}^\kappa \times \mathcal{V}_{ad}^\kappa$ to

$$\min_{(\boldsymbol{\alpha}_\kappa, \theta_\kappa) \in \mathcal{U}_{ad}^\kappa \times \mathcal{V}_{ad}^\kappa} \mathcal{F}_\kappa(\boldsymbol{\alpha}_\kappa, \theta_\kappa) := \mathcal{F}_\kappa^1(\boldsymbol{\alpha}_\kappa, \theta_\kappa) + \mathcal{F}_\kappa^2(\theta_\kappa) + \mathcal{F}_\kappa^3(\boldsymbol{\alpha}_\kappa) + \mathcal{F}_\kappa^4(\theta_\kappa)$$

where

$$\mathcal{F}_\kappa^1(\boldsymbol{\alpha}_\kappa, \theta_\kappa) = \sum_{m=1}^{M} \frac{\kappa}{2\theta_\kappa^m} \sum_{i=1}^{d} \|(\boldsymbol{\alpha}_\kappa^m)^\top \widetilde{\mathbf{P}}_i^m \boldsymbol{\alpha}_\kappa^m - \boldsymbol{\rho}_i'(s^m)\theta_\kappa^m\|_{\mathrm{L}^2(\widehat{D})}^2$$

$$\mathcal{F}_\kappa^2(\theta_\kappa) = \sum_{m=1}^{M} \frac{\beta\kappa}{\theta_\kappa^m}, \quad \mathcal{F}_\kappa^3(\boldsymbol{\alpha}_\kappa) = \kappa \sum_{m=1}^{M} \frac{\lambda}{2\kappa^2} |\boldsymbol{\alpha}_\kappa^m - \boldsymbol{\alpha}_\kappa^{m-1}|^2,$$

$$\mathcal{F}_\kappa^4(\theta_\kappa) = \kappa \sum_{m=1}^{M} \frac{\eta}{2\kappa^2} |\theta_\kappa^m - \theta_\kappa^{m-1}|^2.$$

- and admissible sets

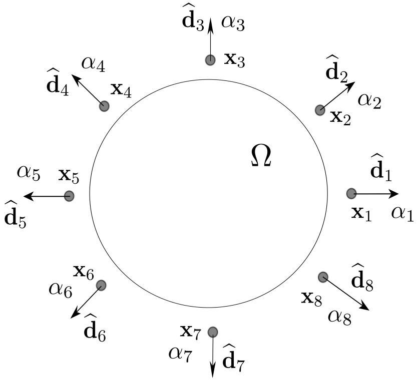
$$\mathcal{U}_{ad}^\kappa := \left\{ \boldsymbol{\alpha}_\kappa \in \mathrm{H}^1(0, s_F) : \boldsymbol{\alpha}_\kappa|_{[s^{m-1}, s^m]} \in \mathbb{P}^1, m = 1, \ldots, M \right\} \cap \mathcal{U}_{ad},$$

$$\mathcal{V}_{ad}^\kappa := \left\{ \theta_\kappa \in \mathrm{H}^1(0, s_F) : \theta_\kappa|_{[s^{m-1}, s^m]} \in \mathbb{P}^1, m = 1, \ldots, M \right\} \cap \mathcal{V}_{ad}.$$

- We again show the weak convergence using Γ-convergence.

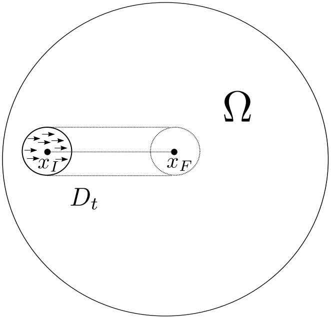
Numerical examples

- $\Omega = B_1(0,0) \subset \mathbb{R}^2$, $\widehat{D} = B_{0.2}(-0.75, 0)$
- $\boldsymbol{x}_{k+1} = 1.2(\cos(k\pi/4), \sin(k\pi/4))$
- $\widehat{\mathbf{d}}_{k+1} = (\cos(k\pi/4), \sin(k\pi/4)), k = 0, \dots, 7$ ($n_p = 8$)

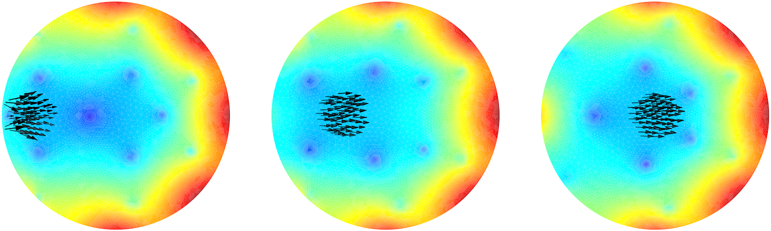


Problem 1: approximate $\mathbf{f}_1(\boldsymbol{x}, t) = (1, 0)^\top$

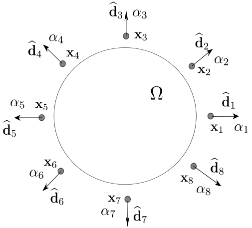
- $T = 1$, $\lambda = 10^{-5}$.
- $\boldsymbol{\alpha}^* = (2, \ldots, 2) \in \mathbb{R}^8$ and $\boldsymbol{\alpha}_* = (-2, \ldots, -2) \in \mathbb{R}^8$.

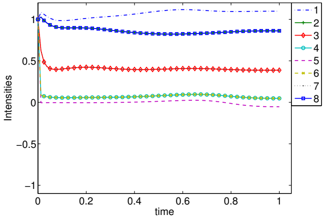


Problem 1: approximate $\mathbf{f}_1(\boldsymbol{x}, t) = (1, 0)^\top$



Magnetic force at $t = 0.0125, 0.5$, and 1.





Dipoles on the left (dipoles 4, 5 and 6) have small intensities at initial times. This is expected because $D_{1,t}$ is close to the boundary of Ω, where $\mathbf{H}$ is large, thus it is difficult for dipoles 4, 5 and 6 to "push" in the $\mathbf{f}_1$ direction.

Problem 1: approximate $\mathbf{f}_1(\boldsymbol{x}, t) = (1, 0)^\top$

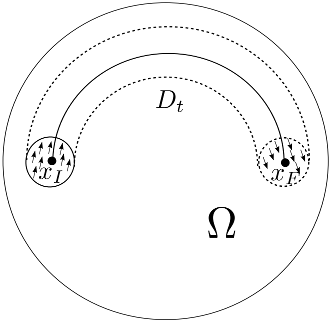
Loading movie ...

Play/Pause

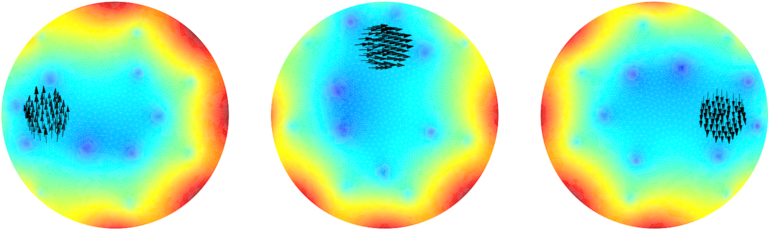
Loading movie ...

Play/Pause

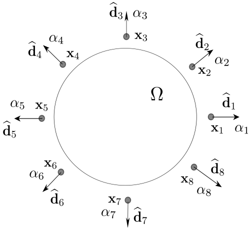
Problem 1: approximate $\mathbf{f}_2(\boldsymbol{x}, t) = (\sin(\pi(1-t)), -\cos(\pi(1-t))^\top$

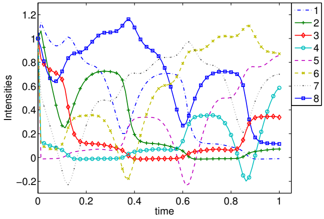


Problem 1: approximate $\mathbf{f}_2(\boldsymbol{x},t) = (\sin(\pi(1-t)), -\cos(\pi(1-t)))^\top$



Magnetic force at $t = 0.015, 0.375$ and 0.75.





Problem 1: approximate $\mathbf{f}_2(\boldsymbol{x}, t) = (\sin(\pi(1-t)), -\cos(\pi(1-t))^\top$

Loading movie ...

Loading movie ...

Play/Pause

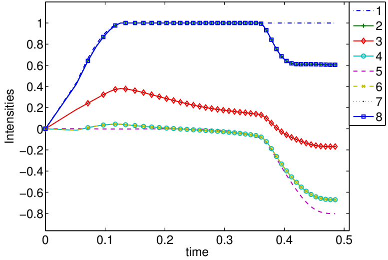
Play/Pause

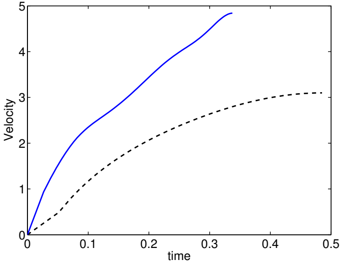
Problem 2

- Let the curve $\mathcal{C}$ is parameterized by

$$\boldsymbol{\rho}(s) = \boldsymbol{x}_I + s\frac{(\boldsymbol{x}_F - \boldsymbol{x}_I)}{\|\boldsymbol{x}_F - \boldsymbol{x}_I\|}, \qquad s \in [0, 0.75]$$

  with $\boldsymbol{x}_I = (0, -0.75)$ and $\boldsymbol{x}_F = (0, 0)$.
- $(\boldsymbol{\alpha}_*, \boldsymbol{\alpha}^*, \theta_*, \theta^*) = (-1, 1, 10^{-10}, 10)$.
- $\beta = 10^{-1}$, $\lambda = 10^{-6}$, and $\eta = 10^{-4}$).





Application: concentration transport

- **MNPs**. We assume a concentration of magnetic nanoparticles confined in a domain $\Omega \subset \mathbb{R}^d, d = 2, 3$.
- **Drug concentration** is evolved using

$$\frac{\partial c}{\partial t} + \operatorname{div}\big(-A\nabla c + c\mathbf{u} + \gamma_1 c f(\mathbf{H})\big) = 0 \quad \text{in } \Omega \times (0,T)$$
$$c = 0 \quad \text{on } \partial\Omega \times (0,T) \qquad c(x,0) = c_0 \quad \text{in } \Omega$$
$$\operatorname{curl}\mathbf{H} = \mathbf{0} \quad \text{in } \Omega \qquad \operatorname{div}(\mathbf{H}) = 0 \quad \text{in } \Omega$$

  where $A = 10^{-3}$ is a diffusion coefficient matrix, $\mathbf{u}$ is a fixed velocity vector and f is the *Kelvin force*.
- **Goal.** Move c_0 from one subdomain to another (desired location) using the magnetic force f while minimizing the spreading.

Application: concentration transport

We solve the parabolic problem with magnetic force given by Problem 1 with $\mathbf{f}_1$.

Loading movie ...

Play/Pause

Loading movie ...

Play/Pause

Conclusions

- We have approximated a vector field by using the Kelvin force. In particular, we study two problems:
  - Fixed final time
  - Unknown final time
- We prove the existence of solution and using second order sufficient conditions we show the local uniqueness.
- Motivated by Γ-convergence we show the H^1-weak convergence of the time-discrete problems.
- In presence of second order sufficient condition, a H^1-strong local convergence result is proved for Problem 1.
- As an application, we study the control of magnetic nanoparticles as those used in magnetic drug delivery. The optimized Kelvin force is used to transport the drug to a desired location.