Does fast migration imply well-mixing of the population dynamics?

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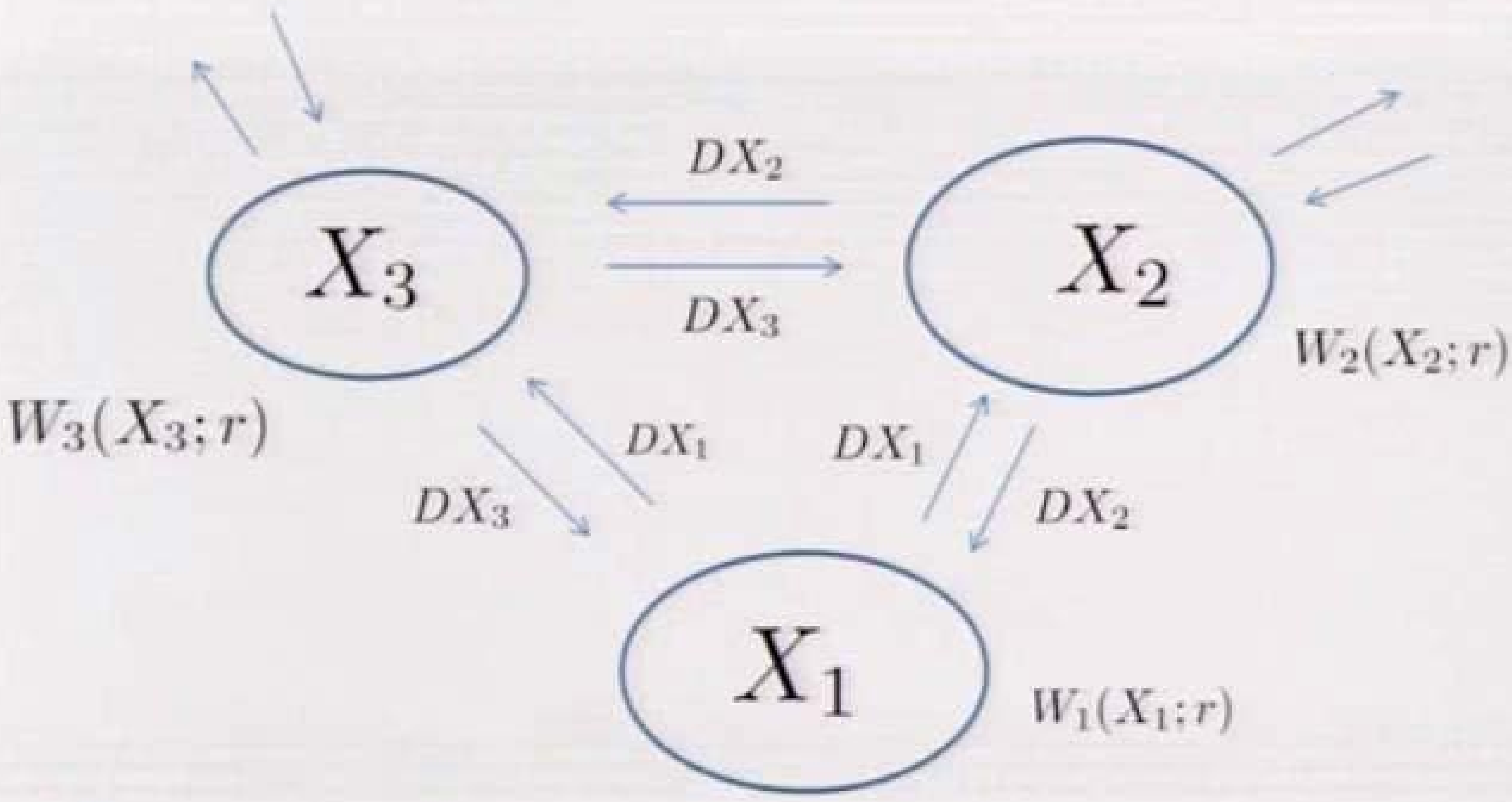
Well-mixed population dynamics

Population size: X

Birth-death processes: $X \overset{W(X;r)}{\rightarrow} X + r$

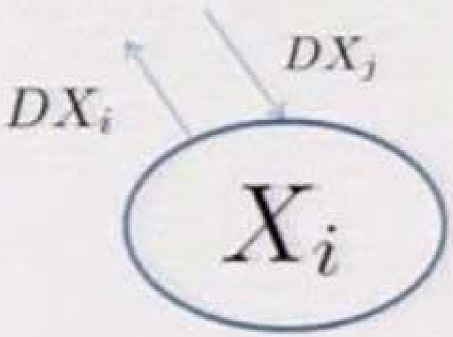
$$\dot{P}(X,t) = \sum W(X - r; r)P(X - r, t) - W(X;r)P(X,t)$$

Qualitative statement of the problem



Does fast migration lead to well-mixing of the total population?

Strong well-mixing condition



$$W_i(X_i;r) = K_i w(x_i;r)$$
$$x_i = K_i^{-1} X_i.$$

(Local dynamics are identical up to carrying capacity)

Strong well-mixing condition for the total population $X = \sum_i X_i$

$$D \to \infty$$

$$\dot{P}(X,t) = \sum W(X-r;r)P(X-r,t) - W(X;r)P(X,t)$$

$$W(X;r) = \tilde{K} w(x;r) \quad x = \tilde{K}^{-1} X$$

Mean-field evolution vs. rare events

$K \gg 1$

Mean-field evolution

$$\dot{X}_i = \sum_r r W_i(X_i; r) + D \sum_{j \in I_i} (X_j - X_i)$$

$$= \sum_r r K_i w(K_i^{-1} X_i; r) + D \sum_{j \in I_i} (X_j - X_i)$$

$$\dot{X} = \sum_r r \tilde{K} w(\tilde{K}^{-1} X; r) \quad D \to \infty$$

$$\sum_{r,i} r K_i w((N K_i)^{-1} X; r) = \sum_r r \tilde{K} w(\tilde{K}^{-1} X; r)$$

Dynamics of rare events: WKB solution of master equation.

Master equation:

$$\dot{P}(\mathbf{X},t) = \sum_{\mathbf{r}} [W(\mathbf{X}-\mathbf{r},\mathbf{r})P(\mathbf{X}-\mathbf{r},t) - W(\mathbf{X},\mathbf{r})P(\mathbf{X},t)]$$

Assumptions

$W(\mathbf{X},\mathbf{r}) \sim K \gg 1$

$|\mathbf{r}| \sim 1$

$P(\mathbf{X},t) = \exp[-K s(\mathbf{x},t)]$

$\mathbf{x} = \mathbf{X}/K$

WKB Ansatz

Hamilton-Jacobi equation:

$$\partial_t s = -H(\mathbf{x}, \partial_{\mathbf{x}} s)$$

Semiclassical Hamiltonian:

$$H(\mathbf{x},\mathbf{p}) = \sum_{\mathbf{r}} w(\mathbf{x},\mathbf{r})(e^{\mathbf{pr}} - 1) \qquad w(\mathbf{x},\mathbf{r}) = W(\mathbf{X},\mathbf{r})/K$$

Kubo, Matsua, & Kitahara (1973); Wentzell (1975), Hu Gang (1987), Dykman, Mori, Ross, & Hunt (1994);

Rare events dynamics of the total population

$$K_i = \kappa_i K, \quad K \gg 1$$

$$H_{total}(Q, P) = \sum_r \sum_{i=1}^{N} \kappa_i w\left(\frac{Q}{N\kappa_i}; r\right)\left(e^{Pr} - 1\right)$$

$$Q = K^{-1} \sum_{i=1}^{N} X_i$$

$$\sum_{i=1}^{N} \kappa_i w\left(\frac{Q}{N\kappa_i}; r\right) = \bar{\kappa} w\left(\bar{\kappa}^{-1} Q; r\right)$$

Weak well-mixing condition

$$\sum_{i=1}^{N} \kappa_i w\left(\frac{Q}{N\kappa_i}; r\right) = \bar{\kappa} w\left(\bar{\kappa}^{-1} Q; r\right)$$

$$w(x; r) = \sum_{n=0}^{\infty} a_n^r x^n$$

$$a_n^r \left[\bar{\kappa}^{1-n} - N^{-n} \sum_{i=1}^{N} \kappa_i^{1-n}\right] = 0, \quad \forall r, n.$$

$$\begin{cases} \kappa_i = \kappa_j \rightarrow \bar{\kappa} = N \\ \kappa_i \neq \kappa_j \rightarrow w(x; r) = a_1^r x + a_{n^*}^r x^{n^*} \quad \forall r \end{cases}$$

Examples. Logistic growth

$$\dot{X}_i = X_i(1 - X_i/K_i)$$

$$W(X_i; r) = aX_i + bX_i^2/K_i$$

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$$\bar{K} = N\left(\frac{1}{N}\sum_{i=1}^{N}\frac{1}{K_i}\right)^{-1}$$

Examples. Allee effect

$$A \to 0 \qquad 2A \to 3A \qquad 3A \to 2A$$

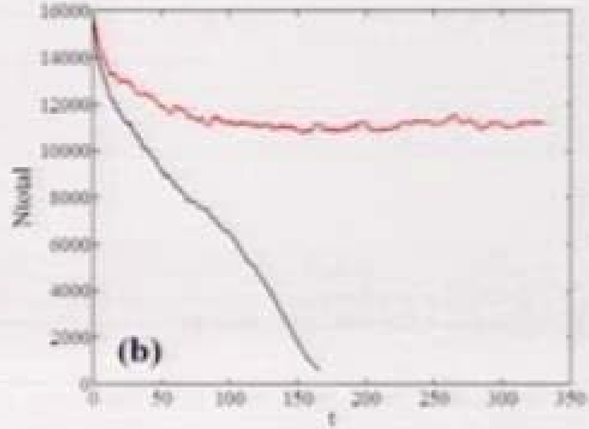
$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\mu X_i \qquad (\lambda/2K_i)X_i^2 \qquad (\sigma/6K_i)X_i^3$$

$$\lambda^2 = \mu\sigma$$

$$\lambda^2 > \sigma\mu$$

$$\lambda^2 < \sigma\mu$$



Examples. Allee effect

Local birth-death processes: $A \rightarrow 0$ $\quad 2A \rightarrow 3A$ $\quad 3A \rightarrow 2A$

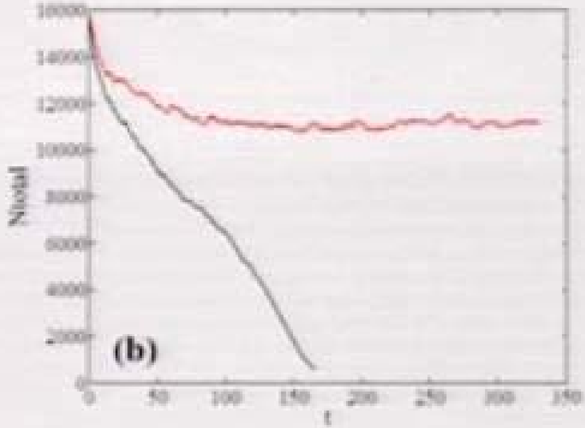
Local transition rates: μX_i $\quad (\lambda/2K_i)X_i^2$ $\quad (\sigma/6K_i)X_i^3$

No well-mixing!

Saddle-node bifurcation: $\lambda^2 = \mu\sigma$

Bistability: $\lambda^2 > \sigma\mu$

Monostability (population extinction): $\lambda^2 < \sigma\mu$



Mean-field evolution vs. rare events again

$\kappa_i \approx \kappa_j \quad \forall i, j$

$K_i = K\kappa_i, \; K \gg 1$

$|\kappa_i - 1| \sim \epsilon \ll 1$

$\epsilon \ll 1/K$

Summary

- Formulated strong/weak well-mixing condition
- Derived equations of motion for large fluctuations (rare events) of the total population for fast migration case
- Derived necessary & sufficient condition for weak well-mixing by fast migration
- Considered common examples where the condition is met" logistic growth", etc., and is not met: Allee effect.
- Mean-field dynamics is much more robust with respect to the well-mixing assumption than rare events

Fast Migration and Emergent Population Dynamics, M. Khasin, E. Khain, and L. M. Sander Phys. Rev. Lett. 109, 248102 (2012)