Dynamics Days 2018

Denver, Colorado, January 4-6 2018

Preliminary list of invited speakers:

Elizabeth Bradley
Alain Goriely
Mark Hoefer
Peko Hosoi
James Hudspeth
William Irvine
Chris Jones
Panos Kevrekidis
Nathan Kutz
Laura Miller
Kandice Tanner
Jean Luc Thiffeault
Cris Moore
John Bush

Wave Breaking:

Its Parametrization and Its Role in Dissipation

JUAN M. RESTREPO

If you are expecting this, run to Superior B

Spring 2017

Synchronization of interacting quantum dipoles

Juan G. Restrepo

Department of Applied Mathematics
University of Colorado at Boulder

Collaborators

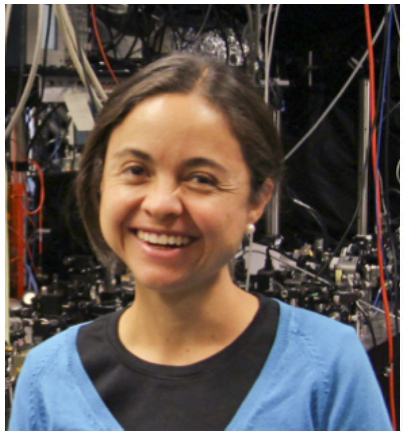
JILA, CU


B. Zhu


J. Schachenmayer


M. Xu


M. Holland


Ana M. Rey

U. San. Chile


Felipe Herrera

Applied Math, CU


Felix Jimenez

Outline

Interacting quantum dipoles

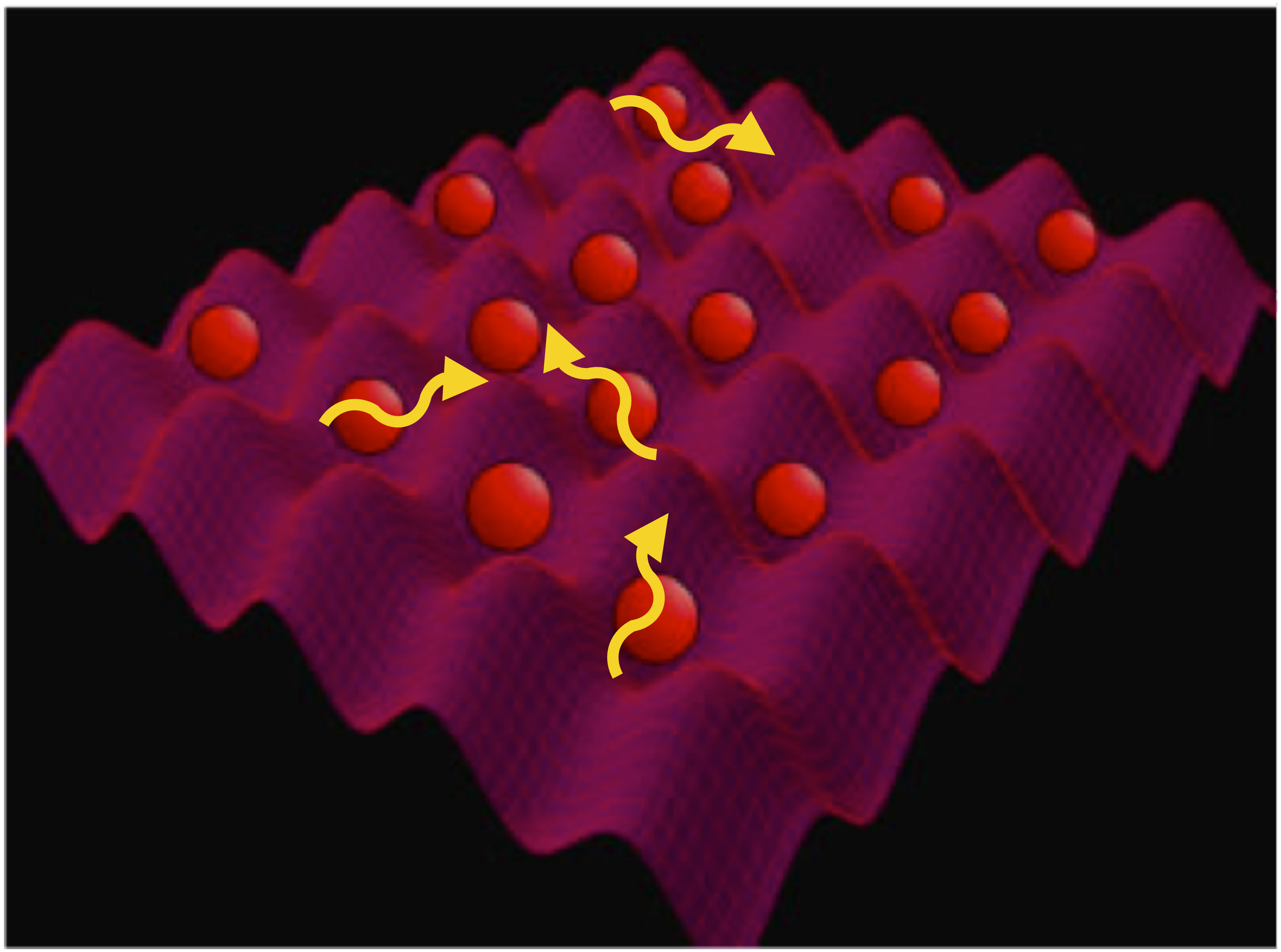
Mean field description

Challenges

Obligatory network

System: interacting quantum dipoles

Atoms or molecules held in an optical lattice,
interacting by exchanging or emitting photons.



Quantum Synchronization

There is interest in studying synchronization in the quantum case, for example

Quantum-classical transition of correlations of two coupled cavities
Tony E. Lee and M. C. Cross, Phys. Rev. A, 2013.

Quantum synchronization of quantum van der Pol oscillators with trapped ions, Tony E. Lee and H. R. Sadeghpour, Phys. Rev. Lett., 2013.

Quantum manifestation of a synchronization transition in optomechanical systems, Lei Ying, Ying-Cheng Lai, and Celso Grebogi, 2014.

Quantum signatures of chimera states, VM Bastidas, I Omelchenko, A Zakharova, E Schöll, T Brandes, PRE, 2015.

This talk

We propose and analyze an **experimentally realizable** model for **spontaneous** synchronization in a **macroscopic** ensemble of quantum systems.

New Journal of Physics **17**, 083063 (2015)
Zhu et al.

Single dipole

A single unit is a quantum system with two levels, ground level $|0\rangle$, and excited level $|1\rangle$

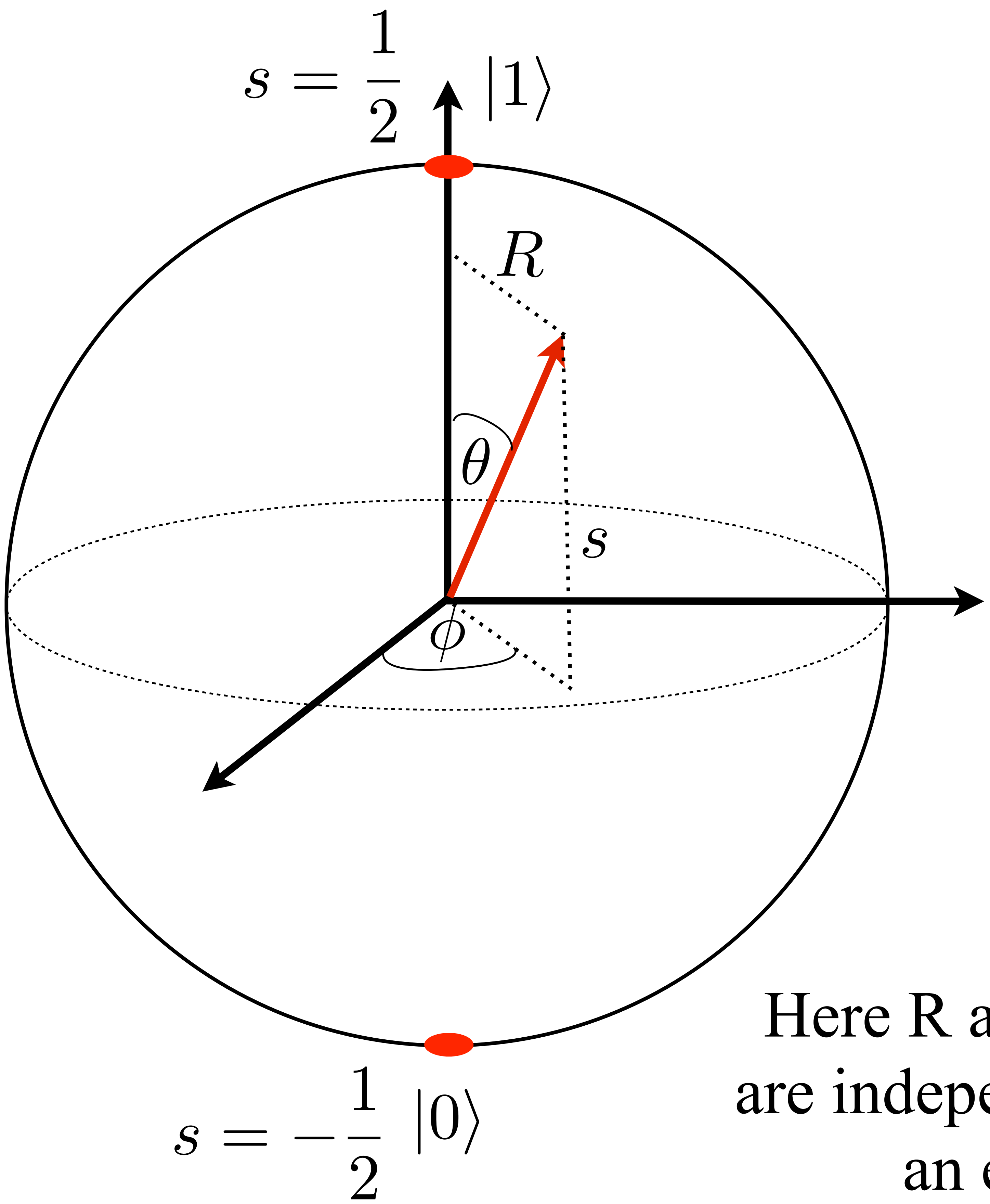
The state of the system is, in general, a superposition

$$|\psi\rangle = a|0\rangle + b|1\rangle$$

The normalization $|a|^2 + |b|^2 = 1$ allows us to parameterize the coefficients as

$$|\psi\rangle = \sin\left(\frac{\theta}{2}\right) e^{i\phi}|0\rangle + \cos\left(\frac{\theta}{2}\right)|1\rangle$$

Single dipole: Bloch sphere



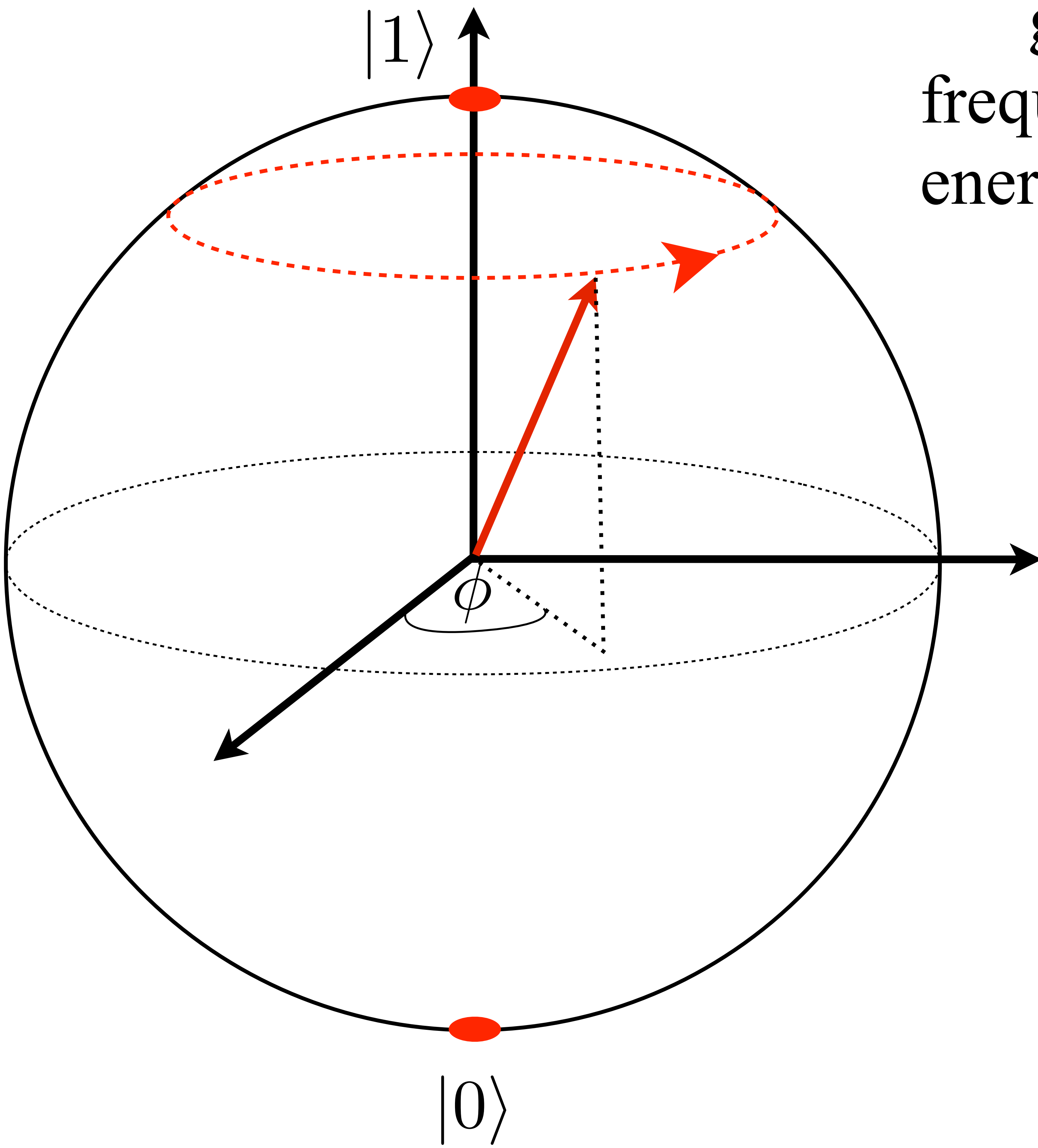
Alternative variables

$$s = \frac{1}{2}\cos(\theta)$$

$$R = \frac{1}{2}\sin(\theta)$$

Here R and s are dependent, but are independent when considering an ensemble average.

Single dipole oscillator

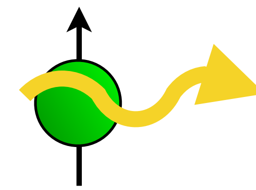


The phase of an isolated dipole grows with an angular frequency that depends on the energy difference between the two states:

$$\frac{d\phi}{dt} = \omega$$

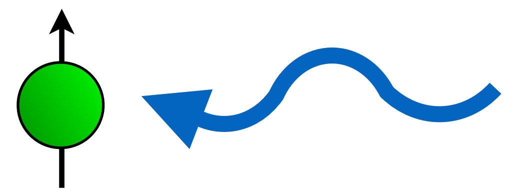
Dynamics of coupled dipoles

Emission of a photon



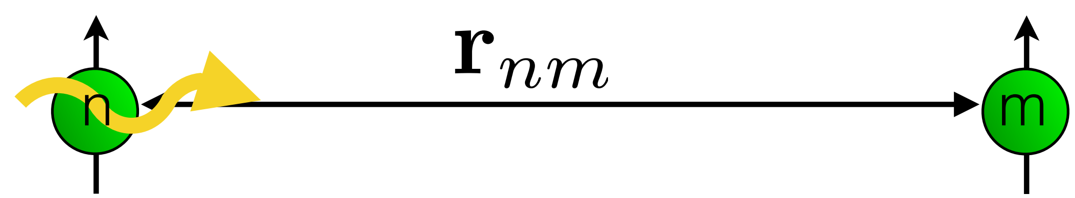
Brings state of dipole to $|0\rangle$, occurs at rate Γ .

Incoherent pumping



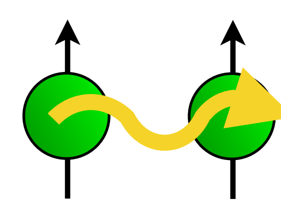
Brings state of dipole to $|1\rangle$, occurs at rate W.

Exchange of a photon



Promotes anti-synchronization, occurs at rate $g(\mathbf{r}_{nm})$

Cooperative emission of a photon



Promotes synchronization, occurs at rate $f(\mathbf{r}_{nm})$

N dipoles

Solving the system numerically or analytically for large N is currently impossible since the dimension of the Hilbert space **scales as 4^N**.

What to do?

Study very small systems ($N < 19$) exactly.

Study small systems with special symmetries and/or approximations (e.g., global coupling).

Study the "mean-field approximation".

This talk

Mean field approximation

By neglecting quantum correlations, one obtains a system of 3N coupled nonlinear ordinary differential equations for the ensemble averages of

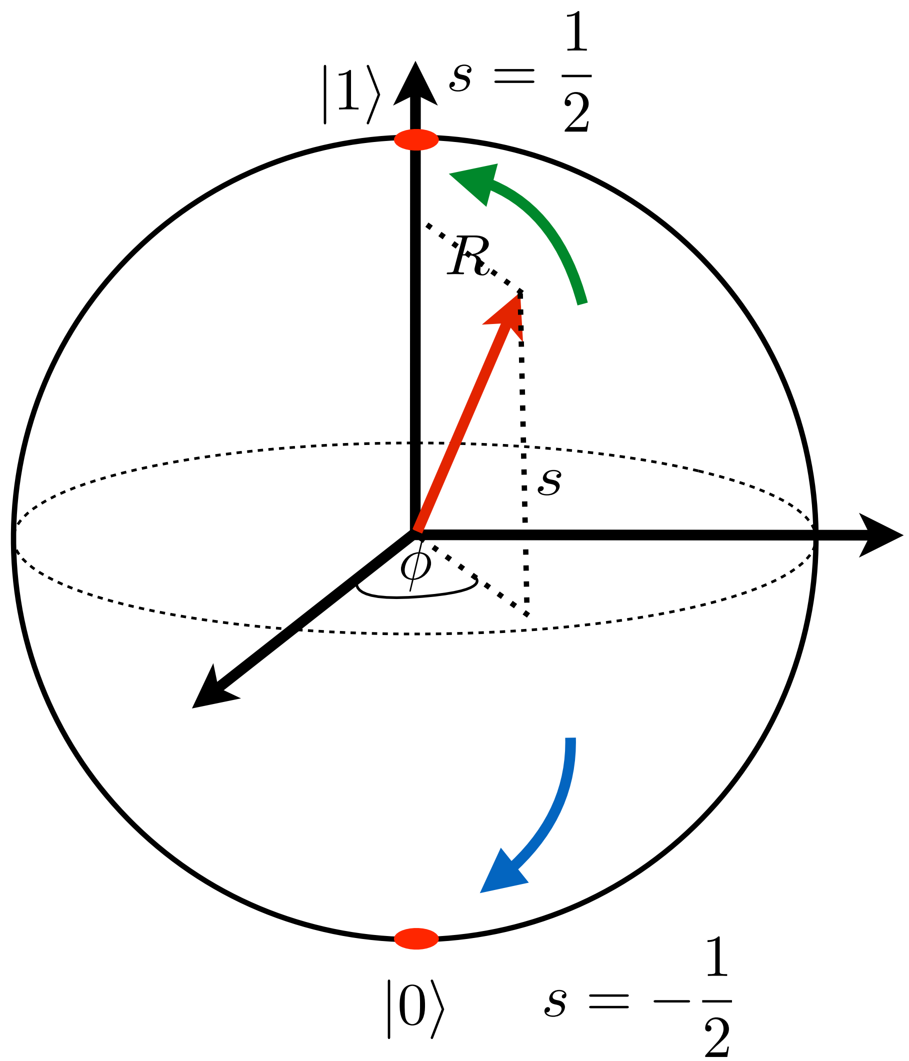
$$R_n , s_n , \phi_n$$

Mean-field description

$$\frac{ds_n}{dt} = -\Gamma R_n \sum_{m\neq n} R_m[f_{nm}\cos(\phi_m-\phi_n) - g_{nm}\sin(\phi_m-\phi_n)] \boxed{-\Gamma\left(\frac{1}{2}+s_n\right)} + \boxed{W\left(\frac{1}{2}-s_n\right)},$$

$$\frac{dR_n}{dt} = -\frac{1}{2}(\boxed{\Gamma}+\boxed{W})R_n + \boxed{\Gamma s_n \sum_{m\neq n} R_m[f_{nm}\cos(\phi_m-\phi_n) - g_{nm}\sin(\phi_m-\phi_n)],}$$

$$\boxed{\frac{d\phi_n}{dt} = -\omega_n + \frac{\Gamma s_n}{R_n}\sum_{m\neq n} R_m[g_{nm}\cos(\phi_m-\phi_n) + f_{nm}\sin(\phi_m-\phi_n)].}$$



~ Sakaguchi-Kuramoto model

Decay: $s_n \to -1/2, R \to 0$

Pumping: $s_n \to +1/2, R \to 0$

Decay of R is prevented if oscillators are in sync

Compare with exact quantum solution

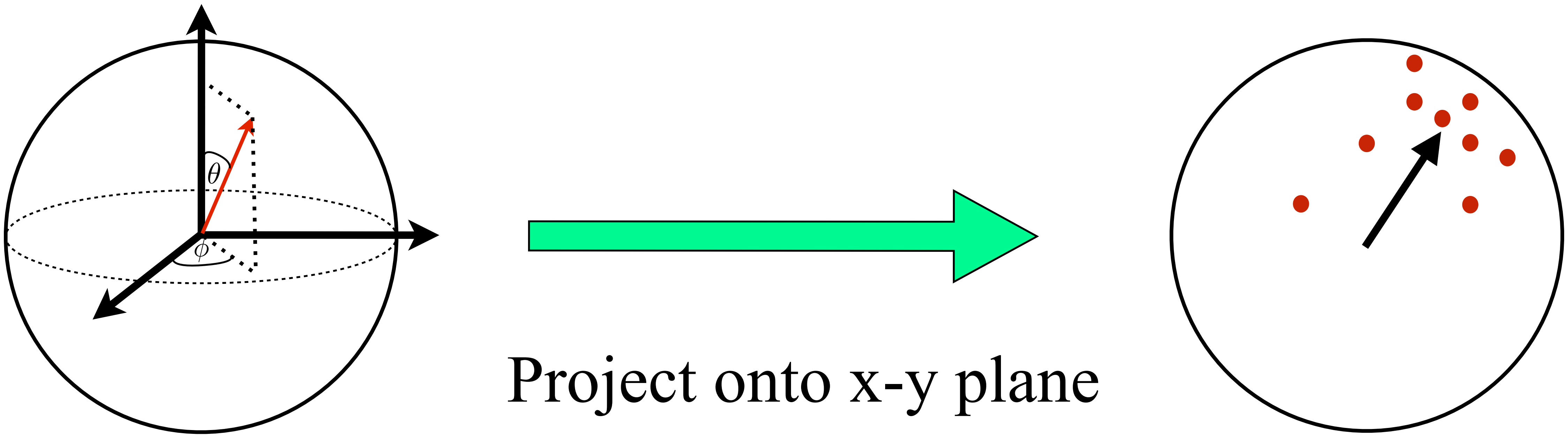
In order to be able to compare with the exact quantum solution, we focus first on the easiest case:

Global coupling, $f(\mathbf{r}_{nm}) = f$

No heterogeneity, $\omega_n = 0$

Steady-state solution.

Order parameter



$$Ze^{i\psi} = \frac{1}{N}\sum_{n=1}^{N} R_n e^{i\theta_n}$$

Steady-state solution

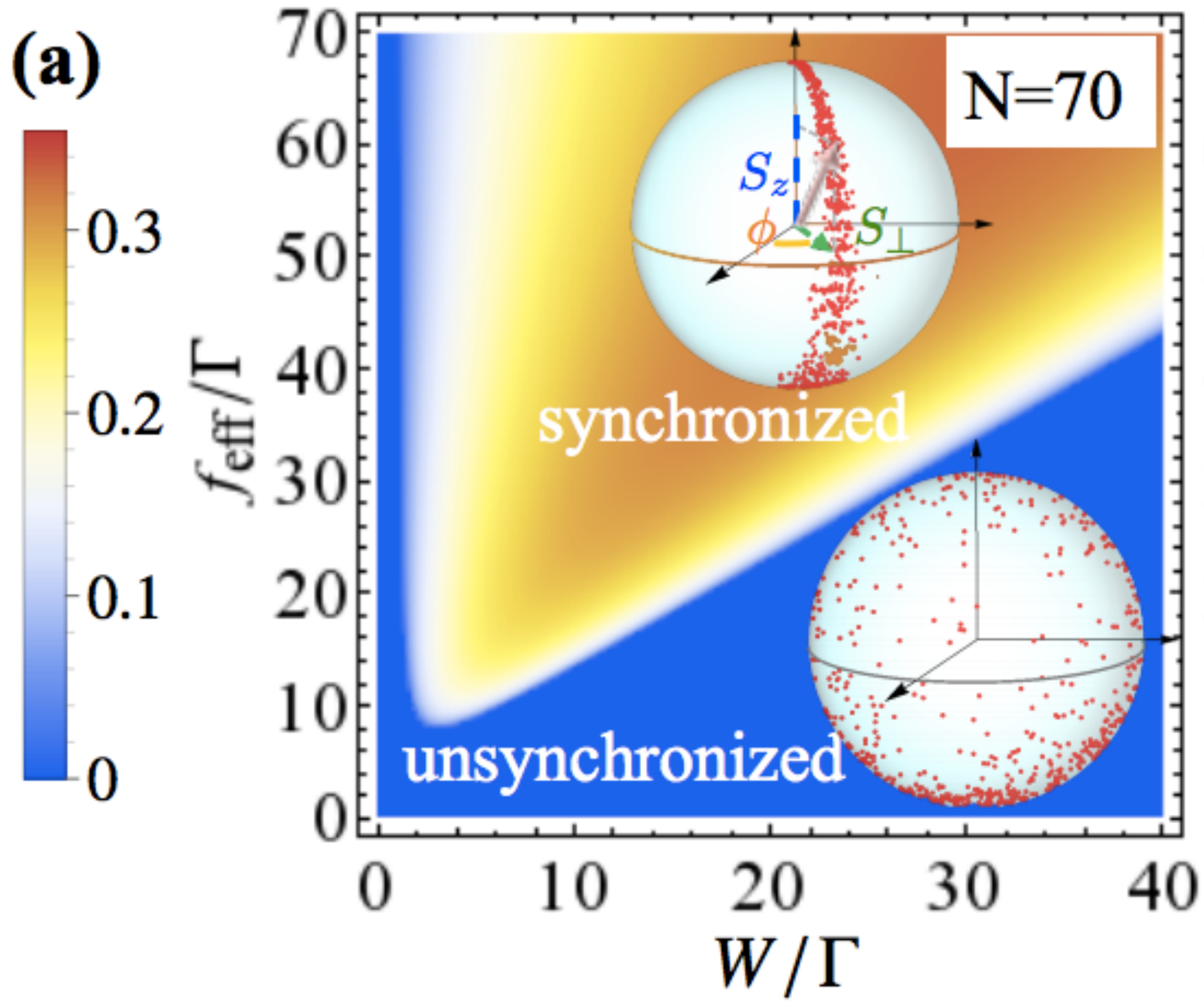
We look for a solution of the form

$$s_n = s, \qquad \dot{s} = 0, \qquad R_n = R, \qquad \dot{R} = 0, \qquad \phi_n = \Omega t$$

$$\boxed{\begin{aligned} Z = R &= \frac{\sqrt{\Gamma f(W-1) - (W+1)^2}}{\sqrt{2}f}, \\ s &= \frac{W+1}{2f}, \\ \Omega &= \frac{g(\Gamma+1)}{2f}. \end{aligned}}$$

Agreement with the quantum solution

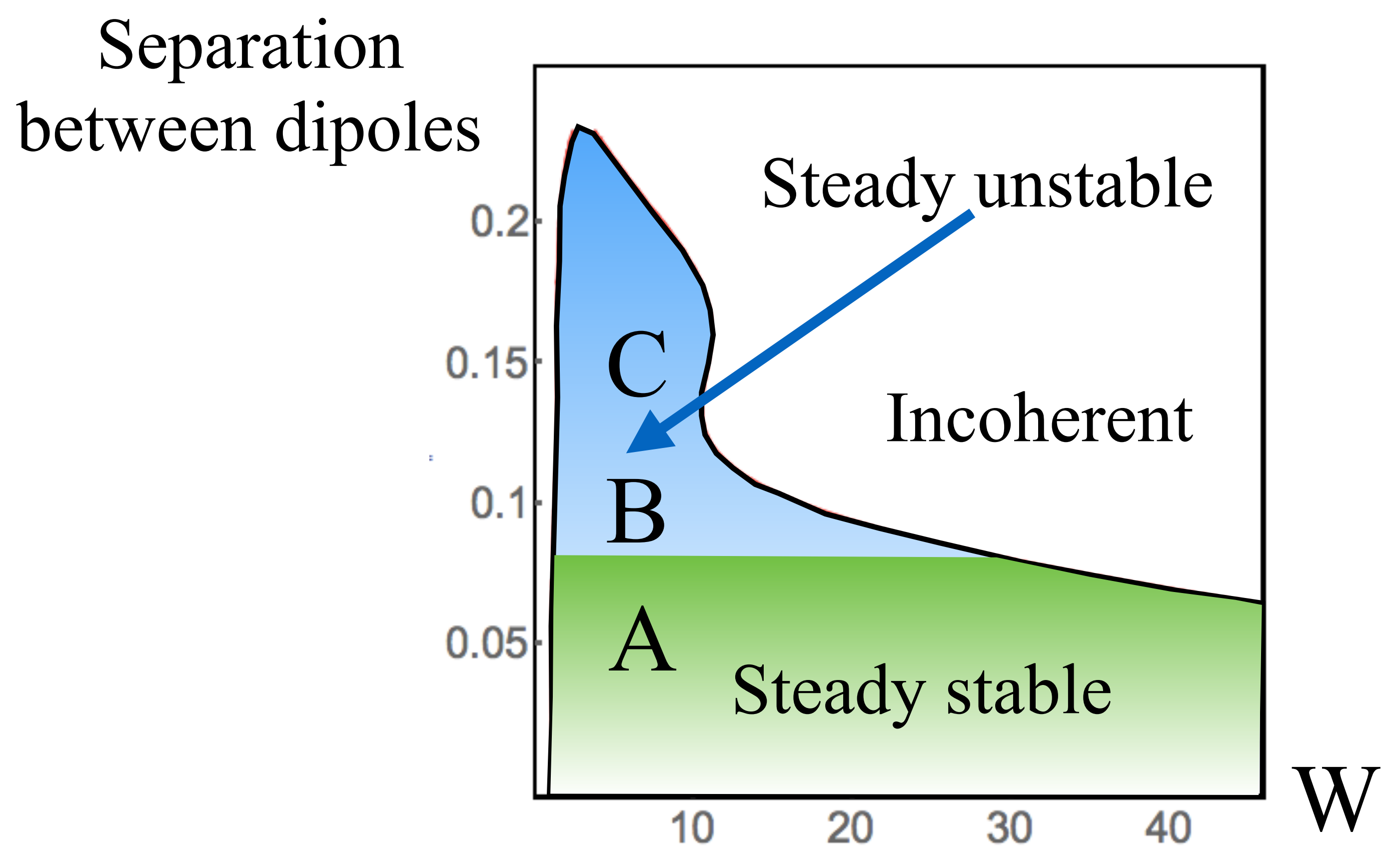
Mean-field

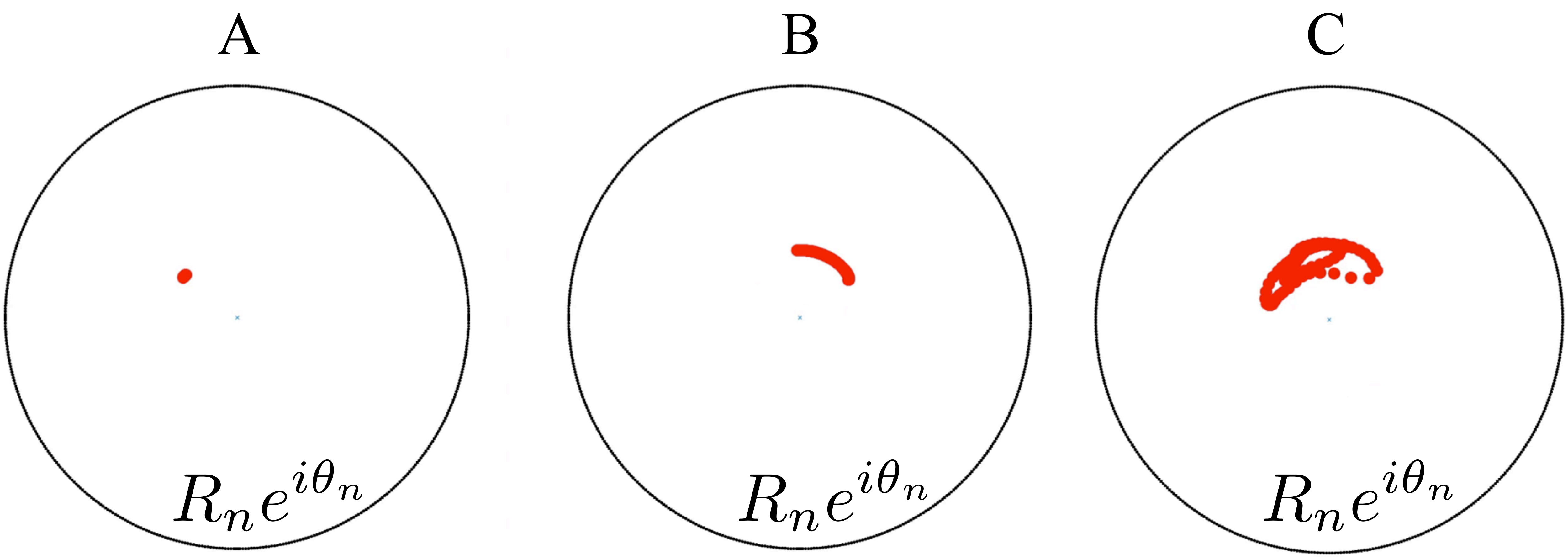


Potentially rich dynamics

The mean field solution
has rich dynamics

Nonstationary synchronized solutions

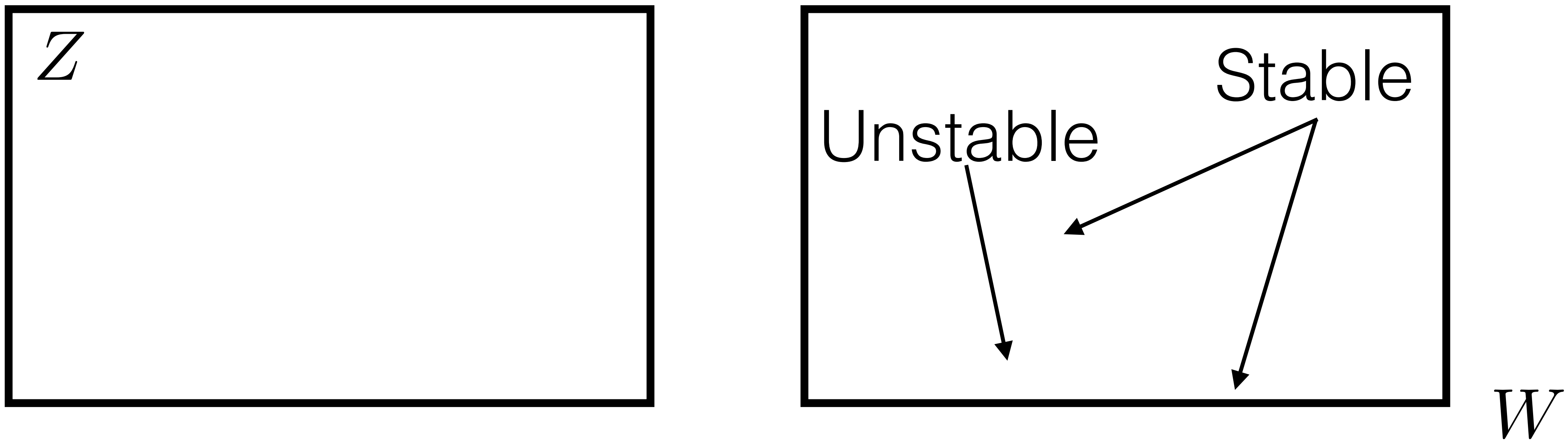




Bistability

For the special case of frequencies with a Lorentzian distribution (with width Δ) , we can use Kuramoto's self-consistent analysis method.

Bistability

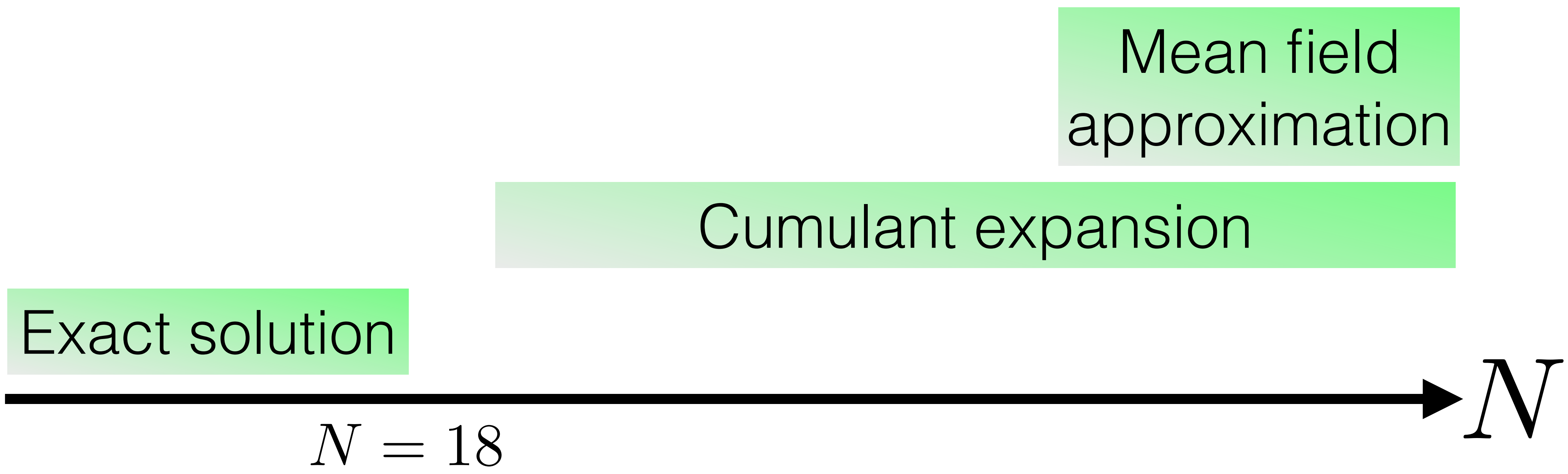


Condition for bistability:

$$g^2\Delta > \frac{(1+W+\Delta)(1+W)^2(1+2\Delta+W)^2}{(W-1)^2},$$

Frequency heterogeneity $(\Delta > 0)$ and photon exchange ($g > 0$) are both necessary!

Challenge: is this all real?

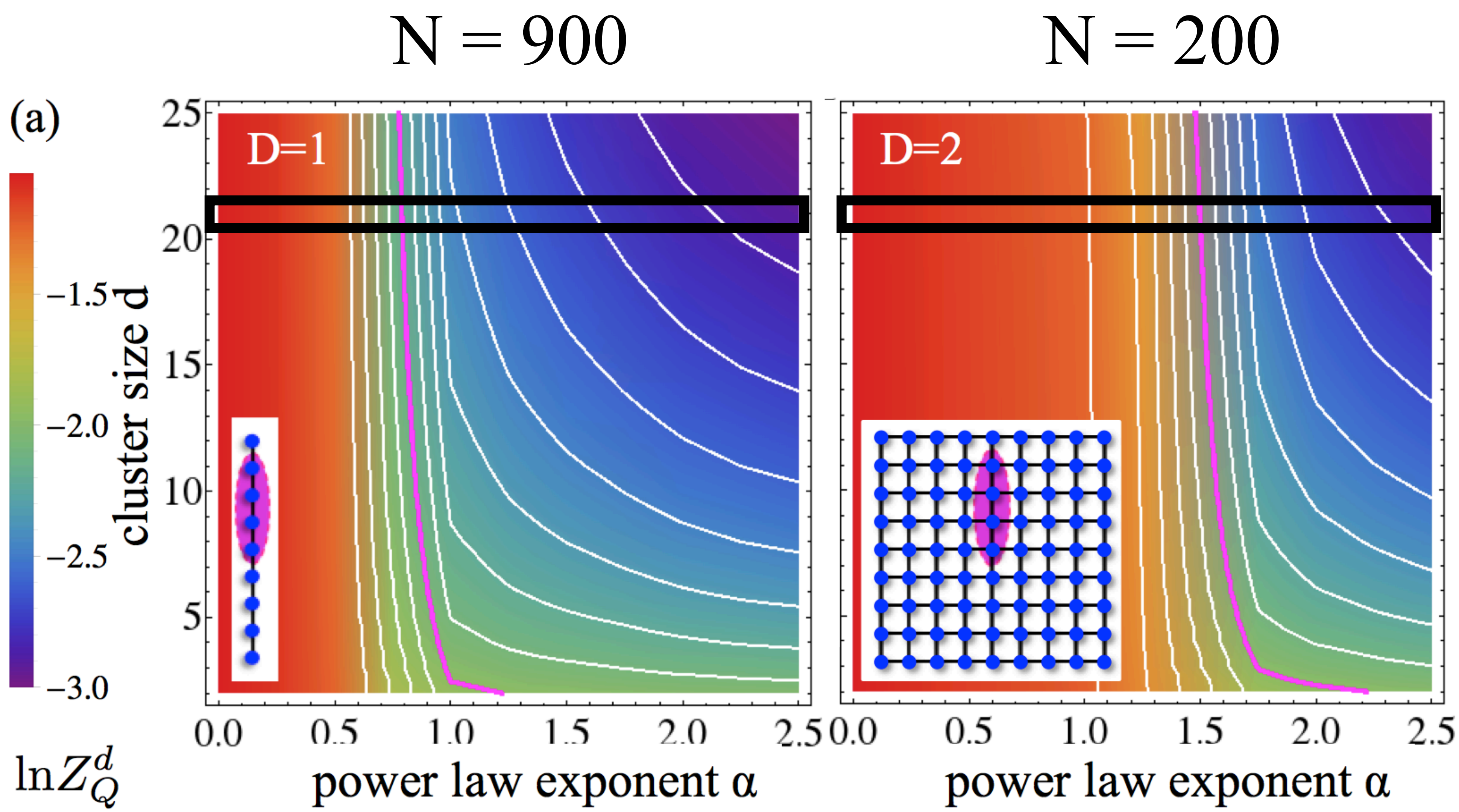


Currently working with the cumulant expansion.

Synchronization with long-range coupling

Power-law interactions, different frequencies

$$f(\mathbf{r}_{nm}) \propto \frac{1}{|\mathbf{r}_{nm}|^{\alpha}} \qquad g(\mathbf{r}_{nm}) = 0$$



Synchronization is possible when interactions are long-range

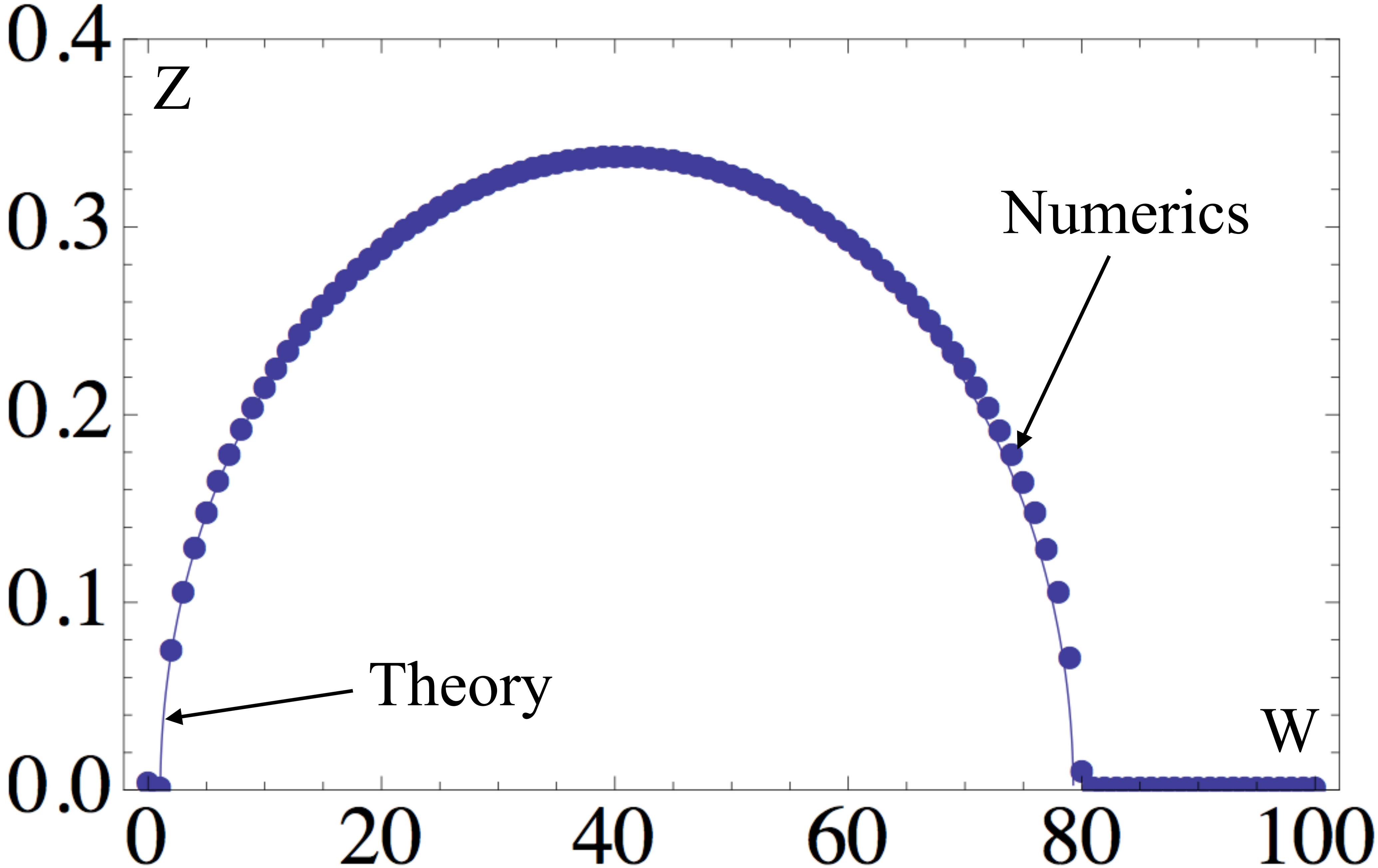
Summary

A system of quantum dipoles can be studied at the mean-field level with the techniques used to study classical synchronization.

The steady syncronization is robust to oscillator heterogeneities and long-range coupling.

The mean-field dynamics suggests there could be rich synchronization dynamics in the quantum system.

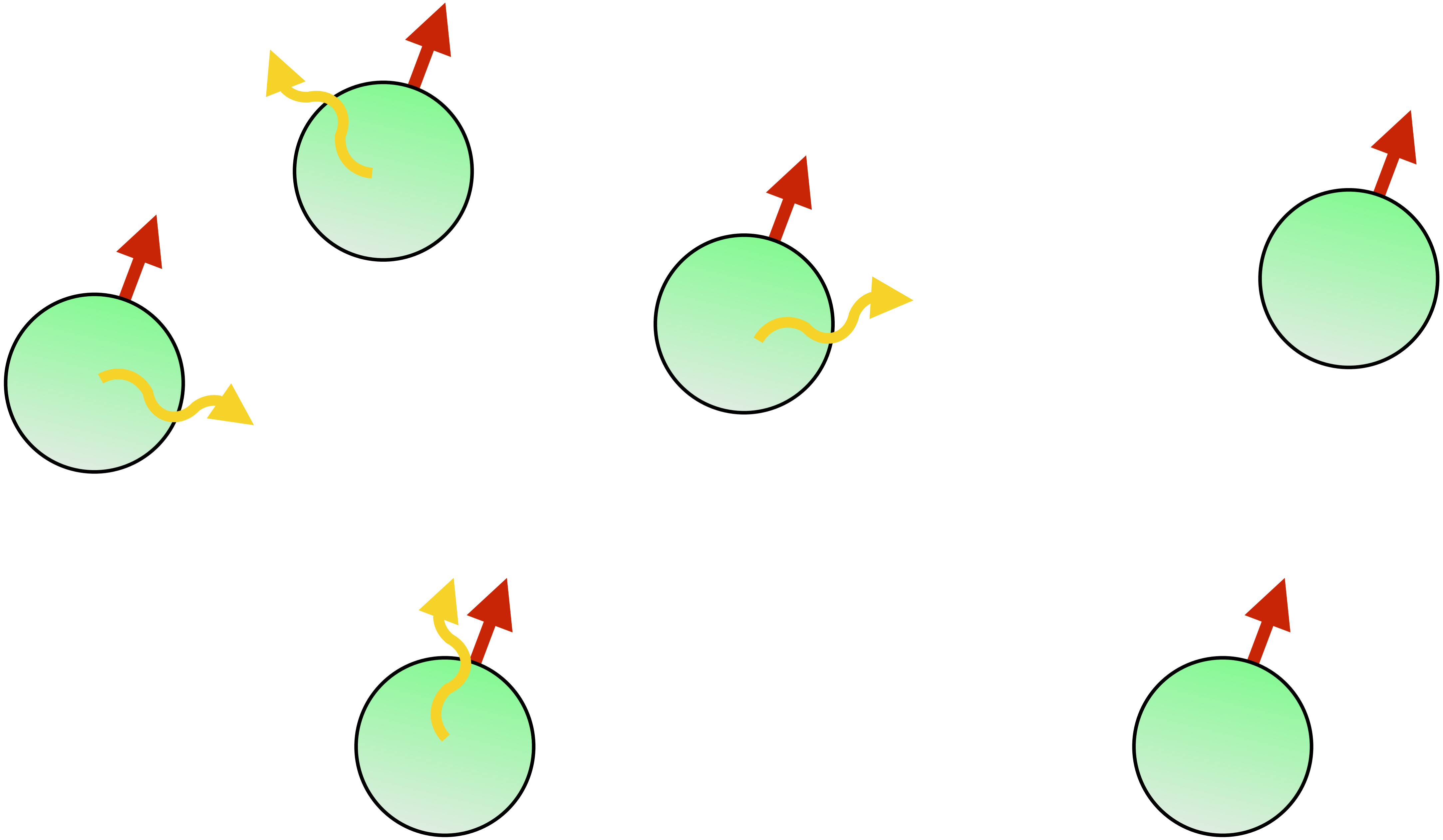
Steady-state solution



(N = 100)

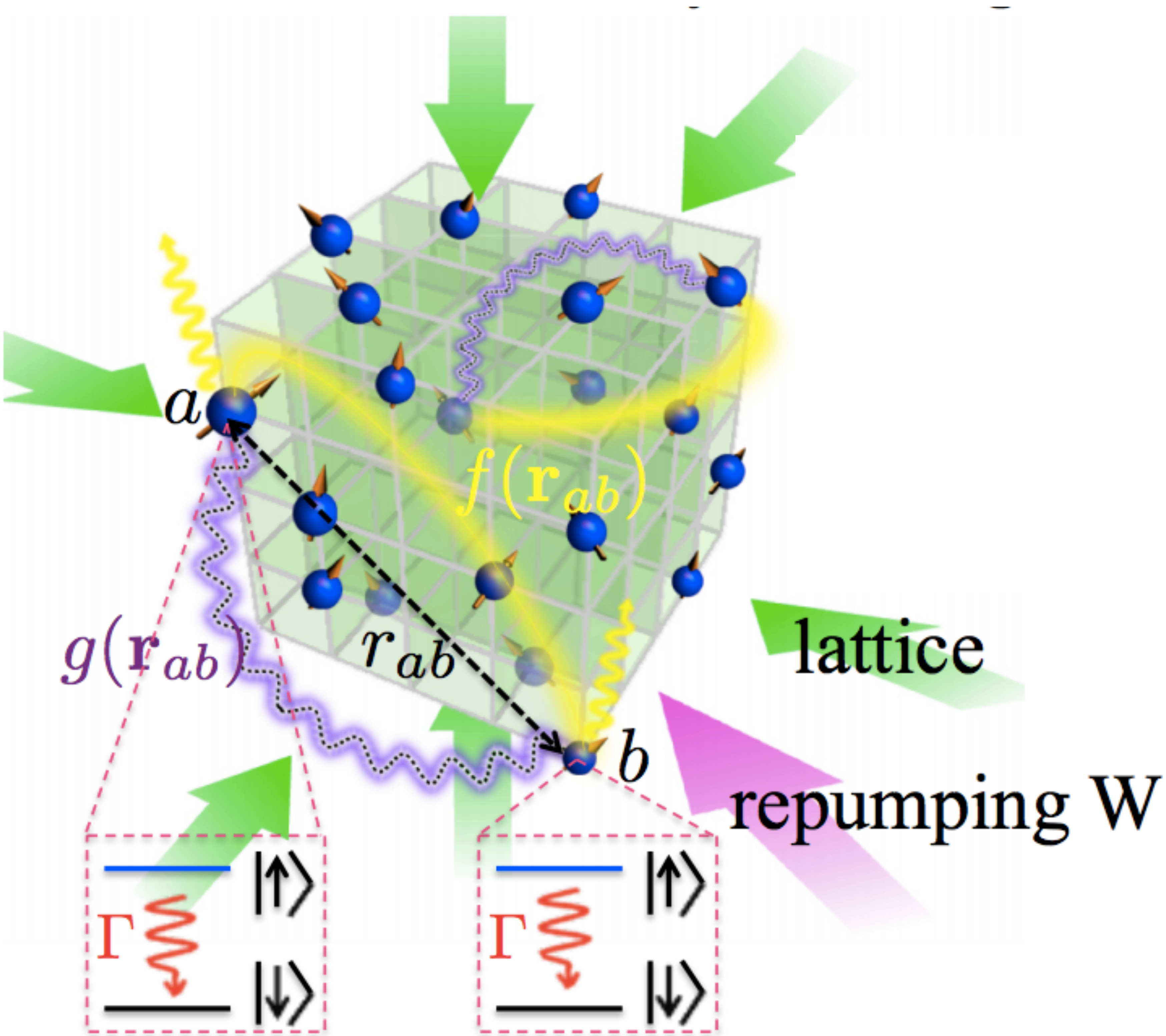
System: interacting dipoles

Atoms or molecules interacting by exchanging or emitting photons.



System: interacting dipoles

Dipoles = atoms or molecules held in an optical lattice



Classical Synchronization

State of system described by state vector $\mathbf{x}(t)$

Evolution of system and coupling described by ODEs

$$\frac{d\mathbf{x}_n}{dt} = F(\mathbf{x}_n; \mathbf{x}_1, \ldots, \mathbf{x}_N)$$

Phase equations, Kuramoto model, etc

Quantum Synchronization

State of system described by vector in Hilbert space $|\psi\rangle$

Evolution of system and coupling described by Schrödinger's equation

$$i\hbar \frac{d|\psi\rangle}{dt} = H_{\text{eff}}|\psi\rangle$$

?

Questions

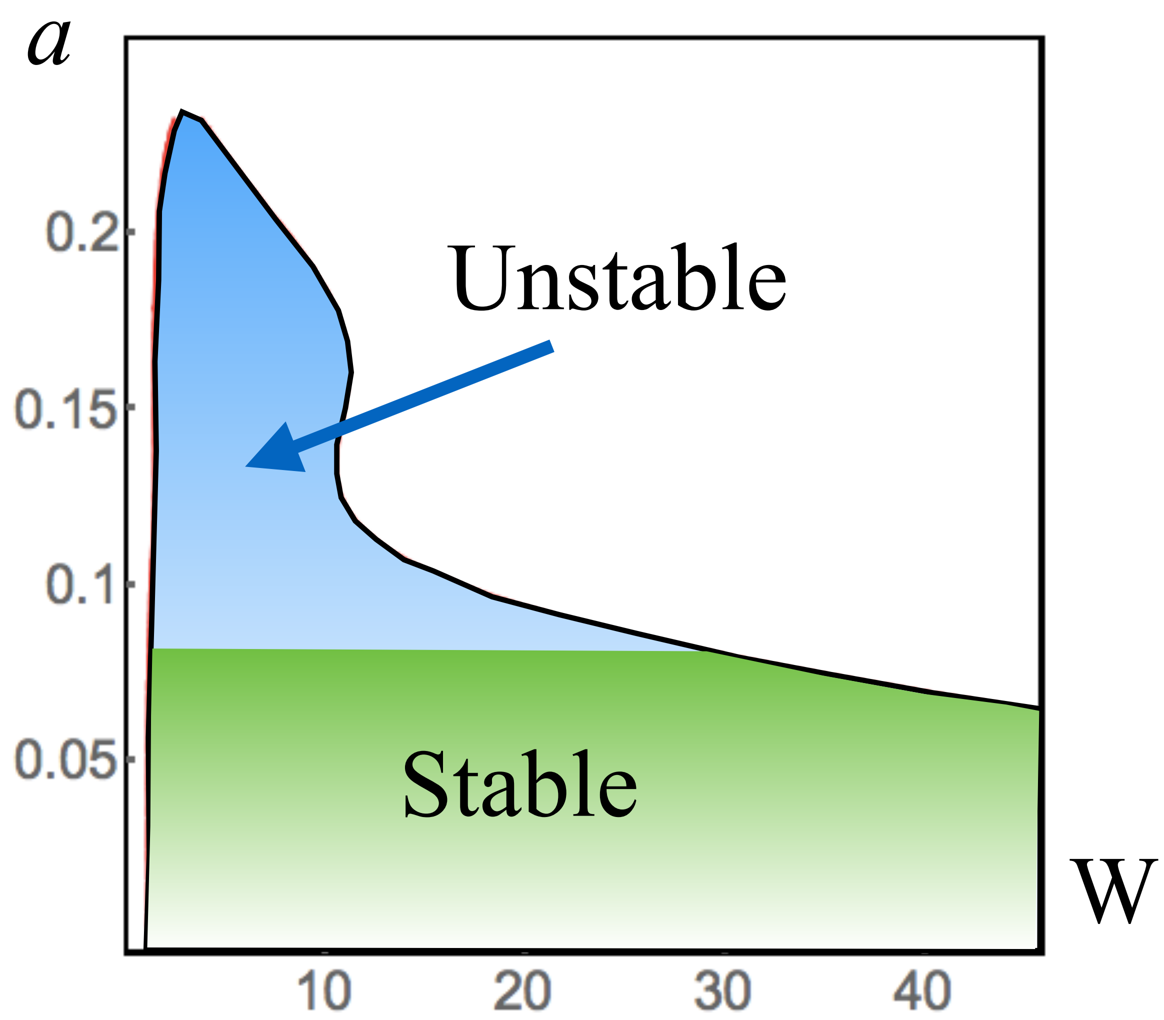
How well does the mean-field solution agree with the quantum solution?

Within the mean-field description, are there other solutions? What about stability?

Can the final synchronized state be considered a quantum phenomenon (for example, having entangled states)?

Stability of steady sync

A linear stability analysis of the steady synchronized solution gives



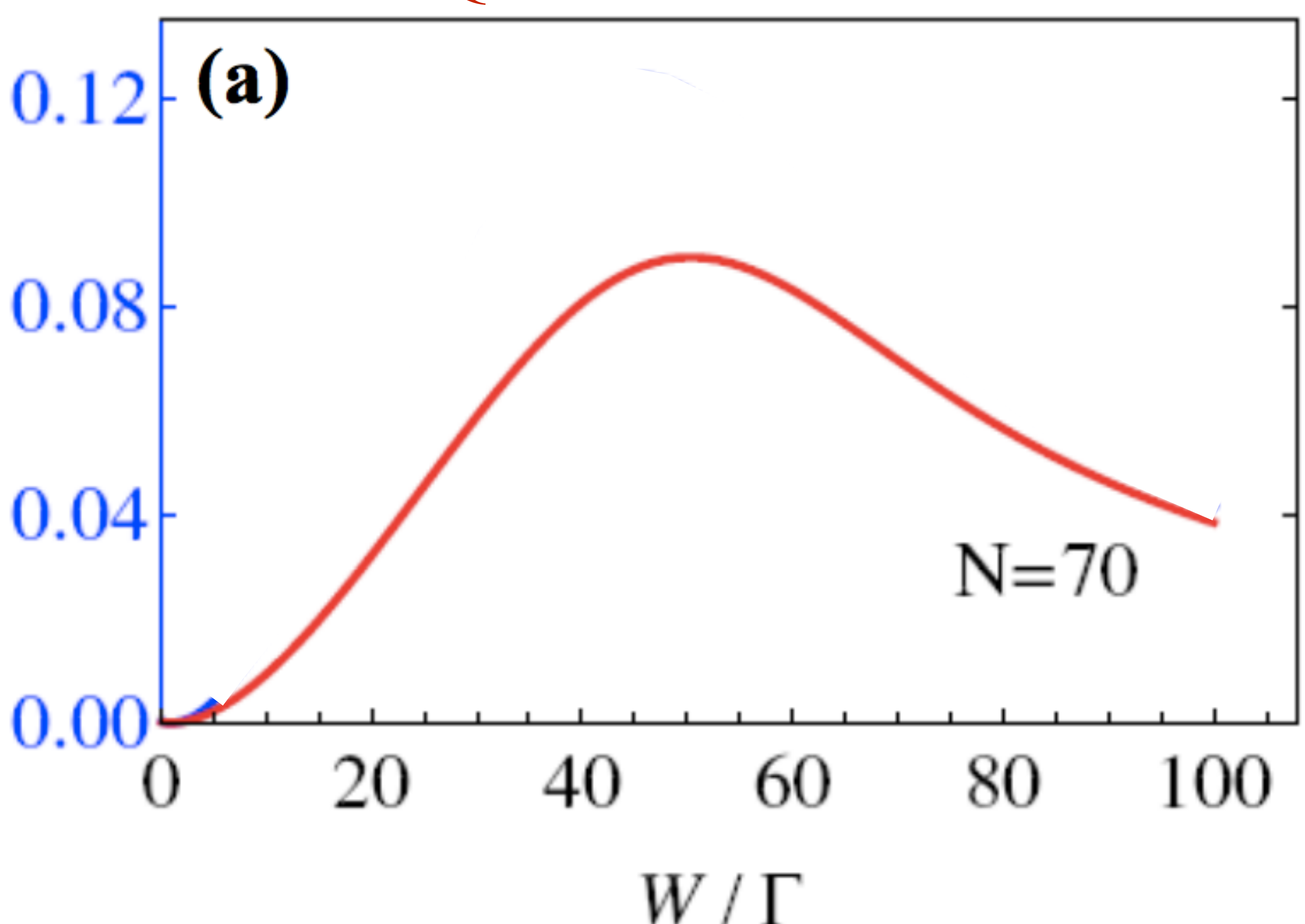
(N = 103)

$a =$ 2π distance between dipoles/photon wavelength
smaller a means larger f

Is this synchronization "quantum"?

Physicists have measures of the "quantumness" of a system. A state with non-zero "Quantum discord" behaves non-classically: a local measurement can disturb the whole system.

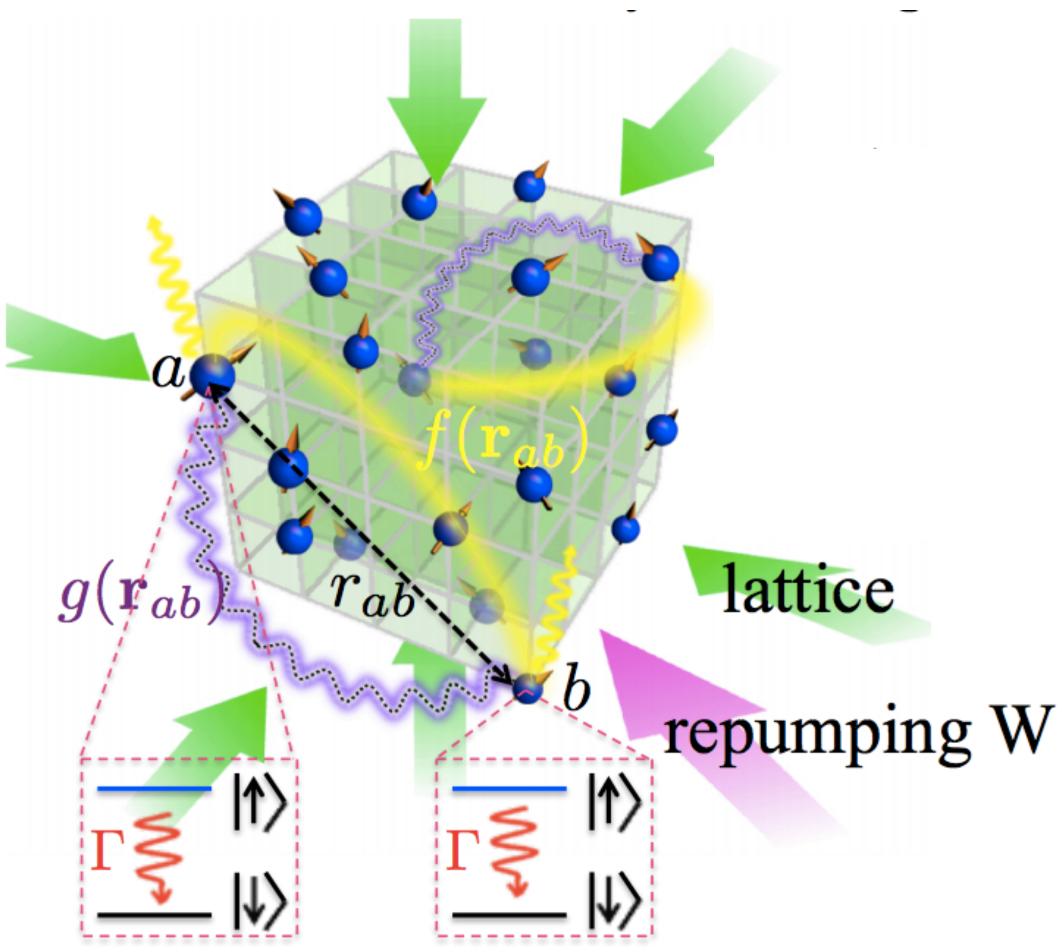
Quantum discord



Correlates with synchronization

This system could be realized in experiments at JILA

Dipoles = atoms or molecules held in an optical lattice



An experiment that verifies the details of the model [e.g., expressions for $f(\mathbf{r}_{nm})$, $g(\mathbf{r}_{nm})$] has been conducted at JILA.

Summary

A system of quantum dipoles can be studied at the mean-field level with the techniques devised by Kuramoto to study classical synchronization.

The synchronization still has quantum features.

Additional comparisons of the mean-field dynamics with quantum calculations and/or experiments might be possible soon.

Questions

How well does the mean-field solution agree with the quantum solution?

Within the mean-field description, are there other solutions? What about stability?

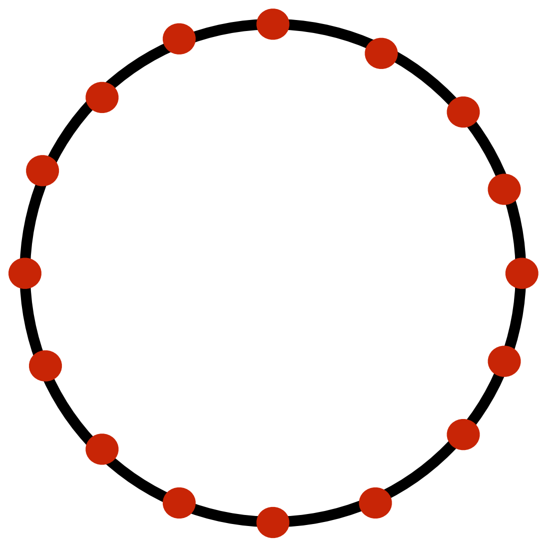
Can the final synchronized state be considered a quantum phenomenon (for example, having entangled states)?

Quantum Synchronization

New Journal of Physics **17**, 083063 (2015)

B. Zhu, J. Schachenmayer, M. Xu, F. Herrera, J. G. Restrepo, M. Holland, A. M. Rey

Other solutions of the mean-field equations?



We looked for traveling wave solutions

$$s_n = s, \qquad \dot{s} = 0, \qquad R_n = R, \qquad \dot{R} = 0,$$

$$\phi_n = \Omega t + 2\pi n k / N$$

($k = 0$ is the solution we found above)

For realistic values of the couplings $g(\mathbf{r}_{nm})$ and $f(\mathbf{r}_{nm})$ these solutions are suppressed.

However, we found that the steady-state solution can become unstable for small enough f, which in practice occurs for a large separation between dipoles

Phase oscillator models have been adapted to study different features of real-world systems, including time delays, network coupling, more general coupling functions, etc...

Amplitude can be added (e.g., Stuart-Landau oscillators).

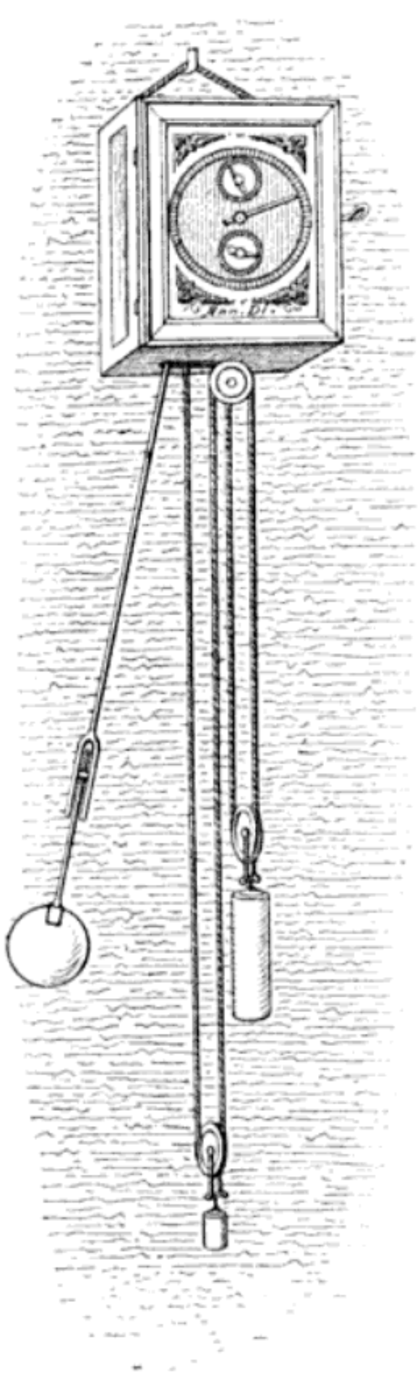
However, usually it is not possible to derive the phase oscillator equations from first principles because

- the existence of governing ODEs is only assumed (e.g., pedestrians)
- the change of variables needed to reduce ODEs to phase description is not explicitly calculated

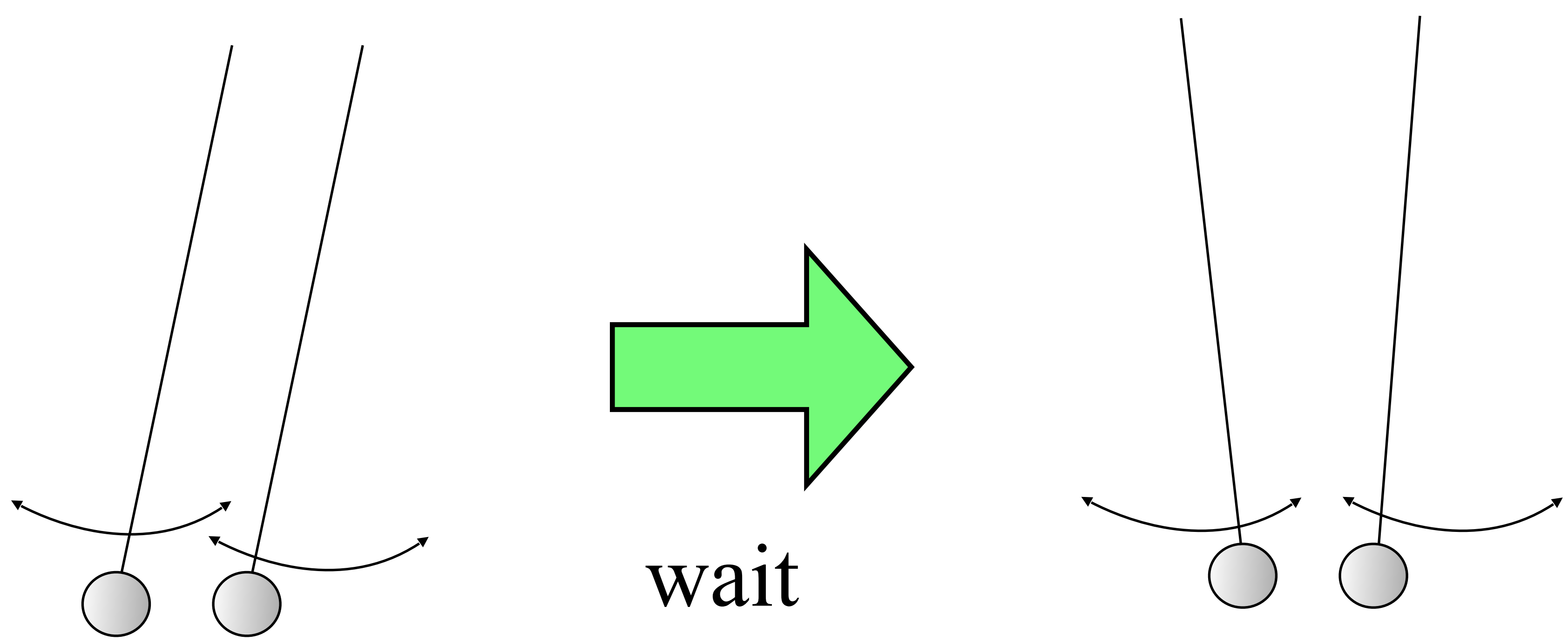
We will present a system where a phase oscillator model can be explicitly derived

Synchronization & clocks

Pendulum clock invented by C. Huygens in 1656



Different clocks had frequencies differing by about 15 seconds per day

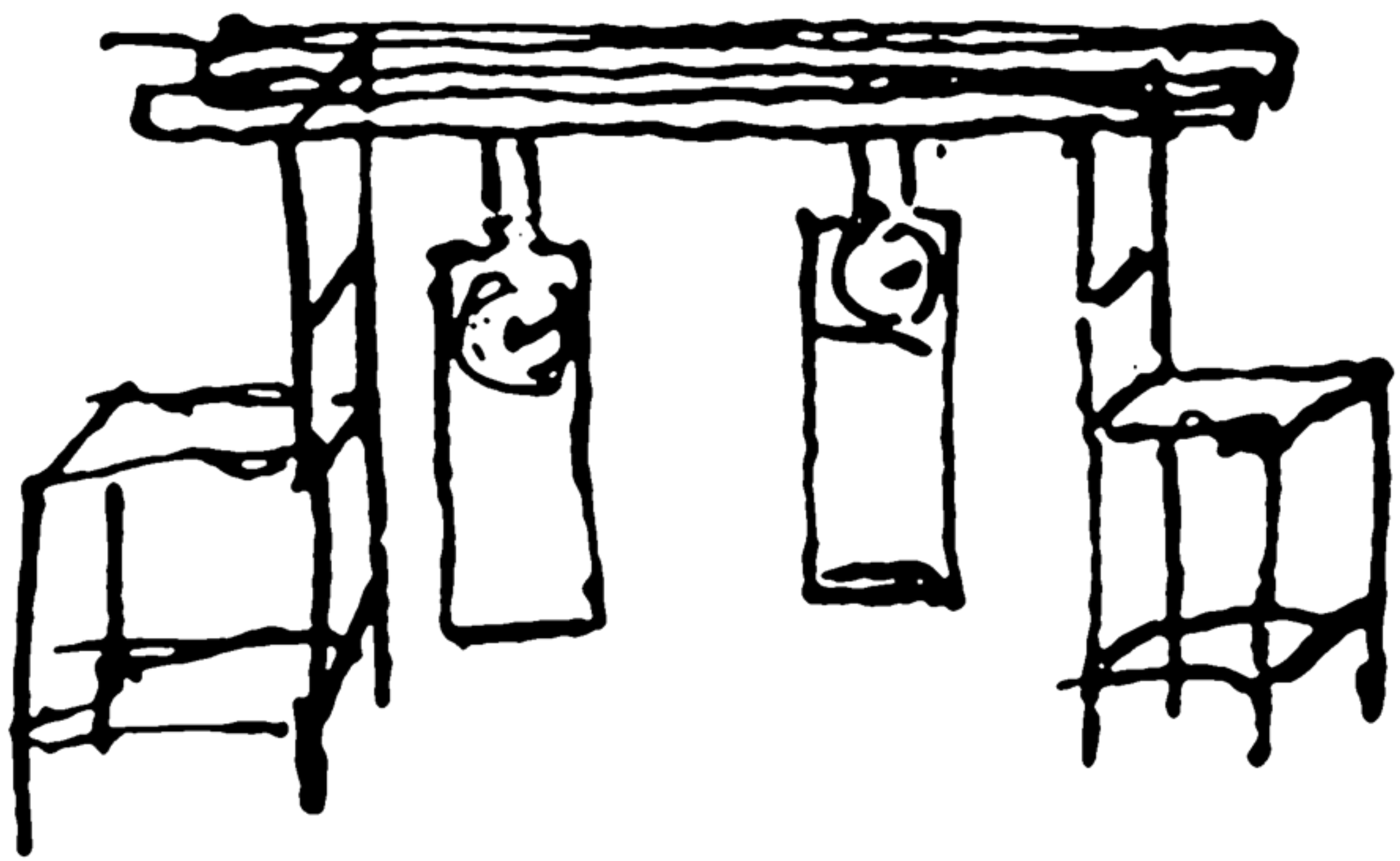


Initially equal phase

Different phases

Synchronization & clocks

In 1665, Huygens observed that two clocks suspended from a common beam would oscillate with the same frequency



Most precise clocks are here at JILA

The Ye group's most recent strontium lattice optical atomic clock is so sensitive that its timekeeping is affected by gravitational changes due to height differences of as little as 2 cm.

Classical Synchronization

Examples of synchronization

Mechanical clocks.

Cellular clocks in the brain.

Pedestrians on a bridge.

Electric circuits.

Pacemaker cells in the heart.

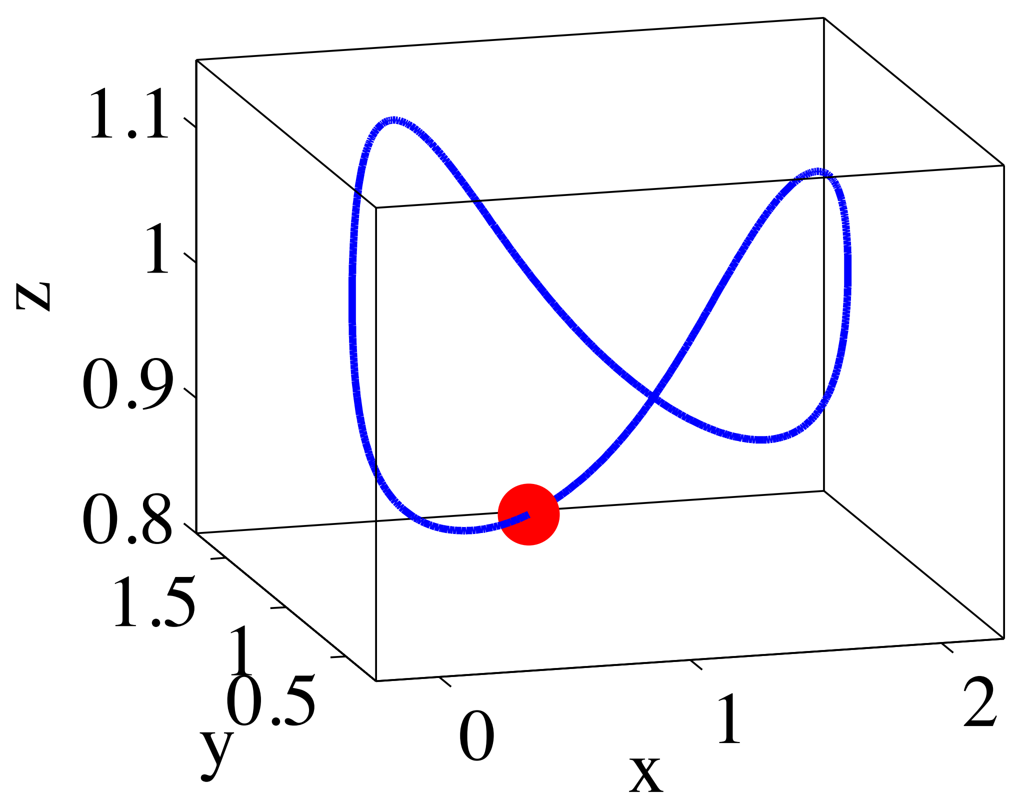
Coupled phase oscillators

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x})$$ System of ODEs that

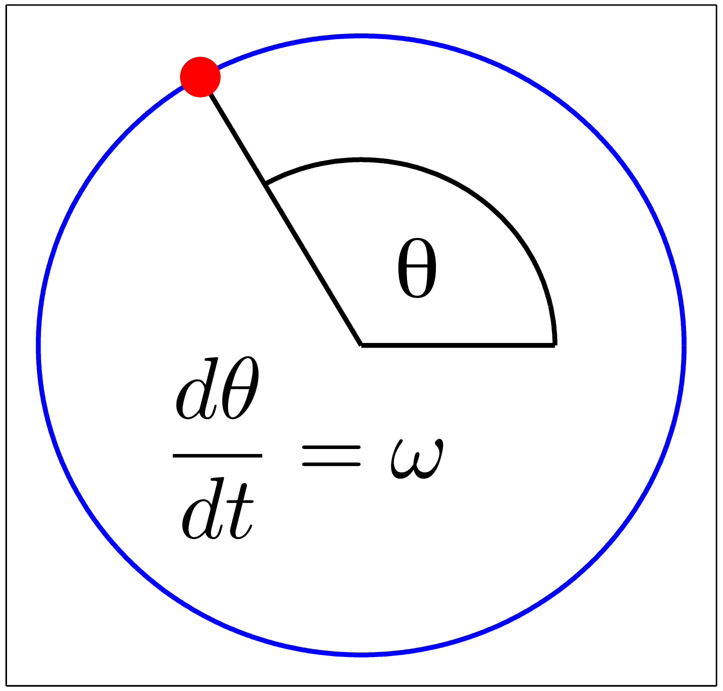
1) Have a strongly attracting limit cycle

2) Are *weakly coupled*: coupling doesn't deform the limit cycle

Such oscillators can be described by just a phase angle (no amplitude)



Change of variables



Coupled phase oscillators

Kuramoto derived the equations

$$\frac{d\theta_n}{dt} = \omega_n + \sum_{m=1}^{N} H_{nm}(\theta_m - \theta_n)$$

where ω_n, H_{nm} depend on the original ODEs

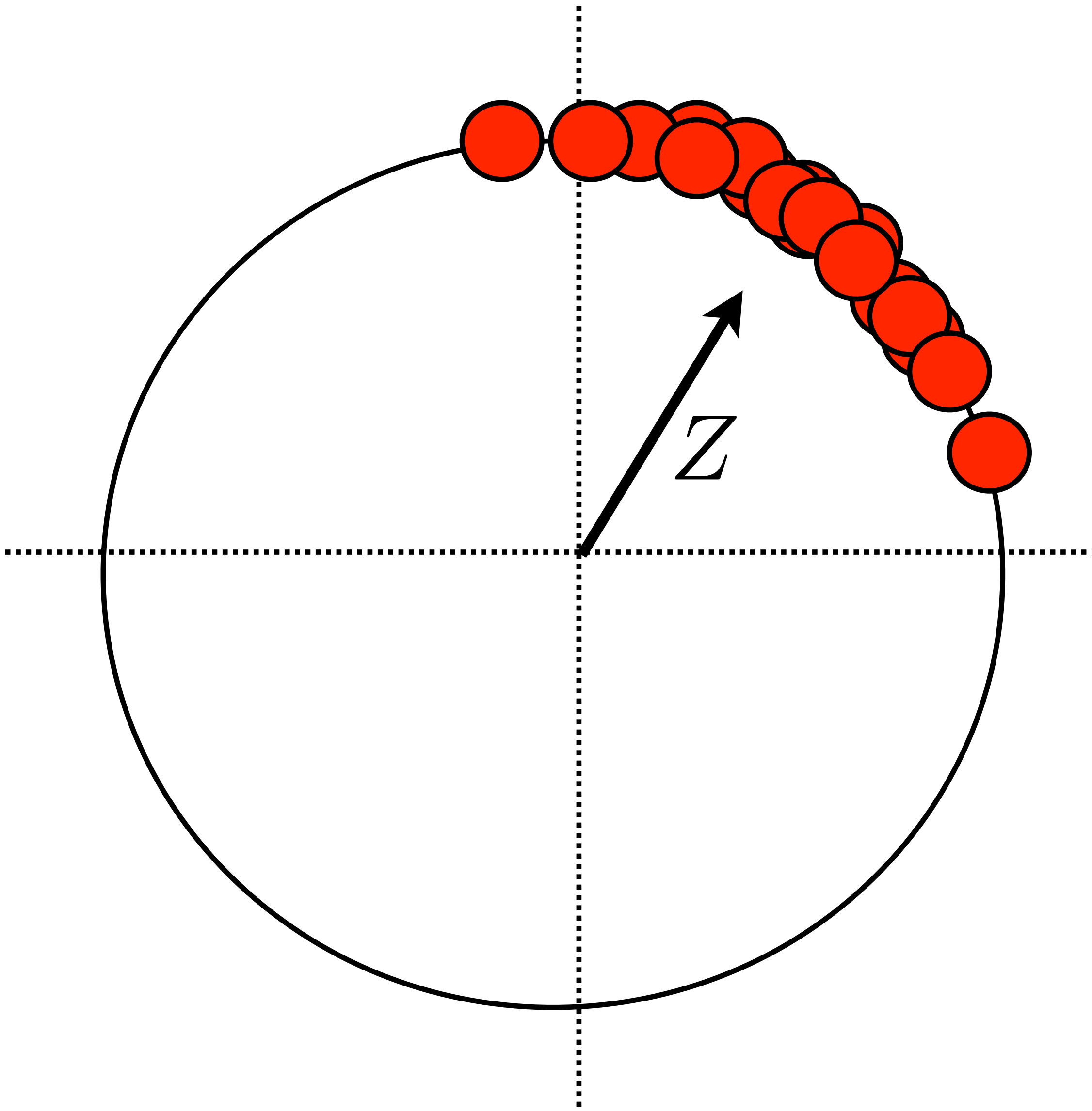
Kuramoto model: simplest choice

$$\boxed{\frac{d\theta_n}{dt} = \omega_n + \frac{K}{N}\sum_{m=1}^{N} \sin(\theta_m - \theta_n)}$$

Sakaguchi-Kuramoto model

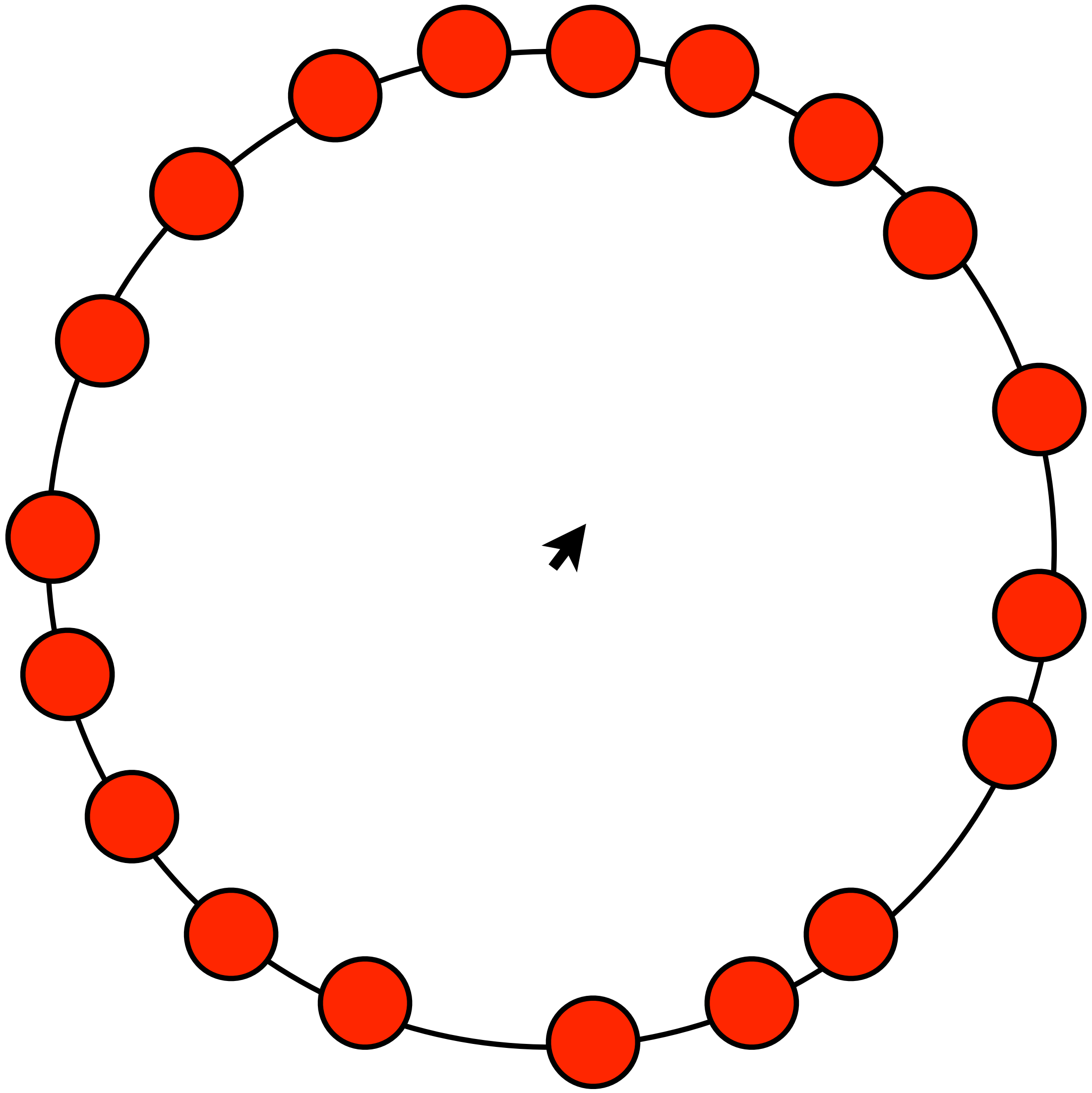
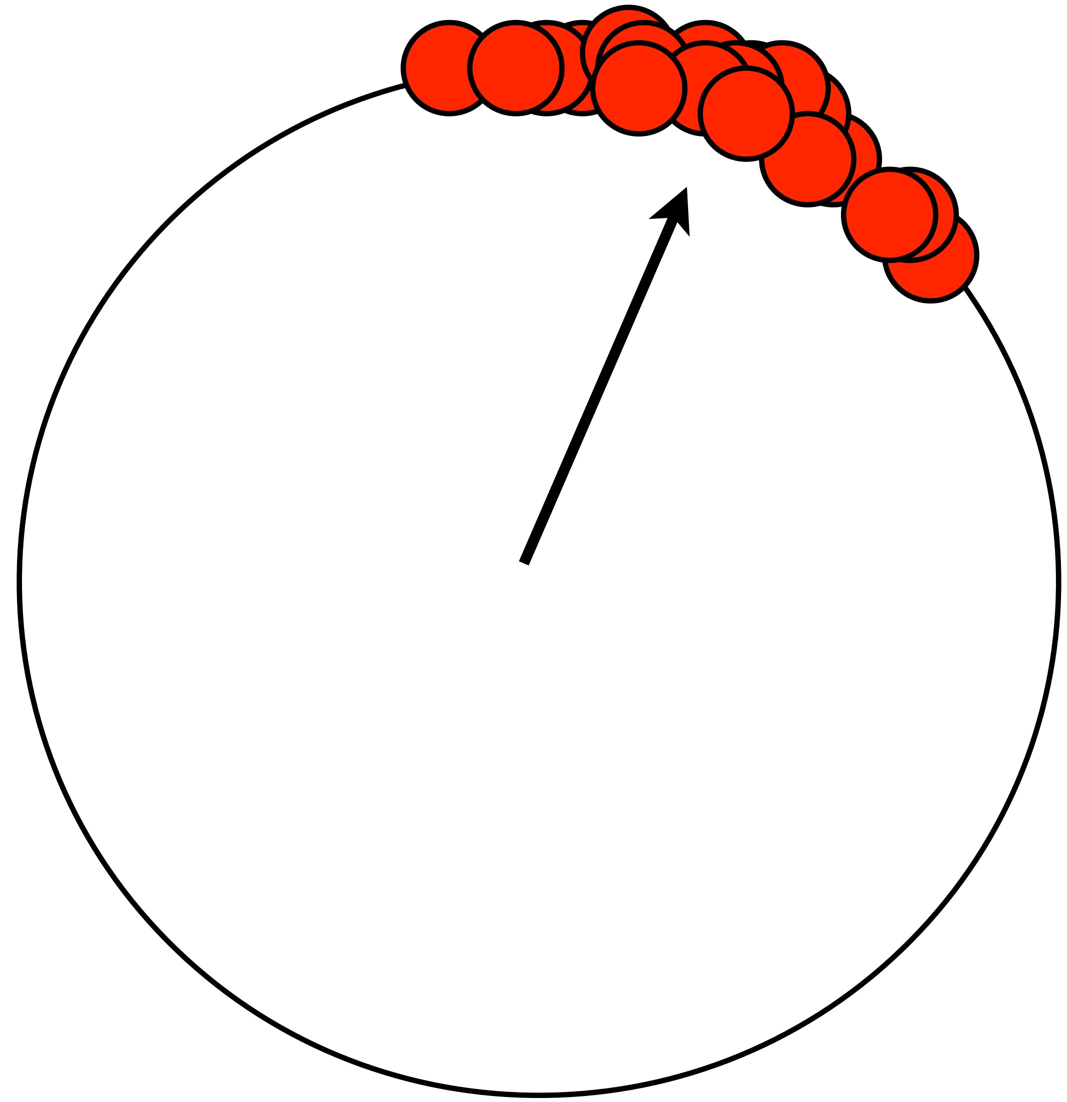
$$\boxed{\frac{d\theta_n}{dt} = \omega_n + \frac{K}{N}\sum_{m=1}^{N} [f \sin(\theta_m - \theta_n) + g \cos(\theta_m - \theta_n)]}$$

Order parameter to measure synchronization



$$Ze^{i\psi} = \frac{1}{N}\sum_{n=1} e^{i\theta_n}$$

$Ze^{i\psi}$ = average position of oscillators in complex plane



$Z \approx 0$

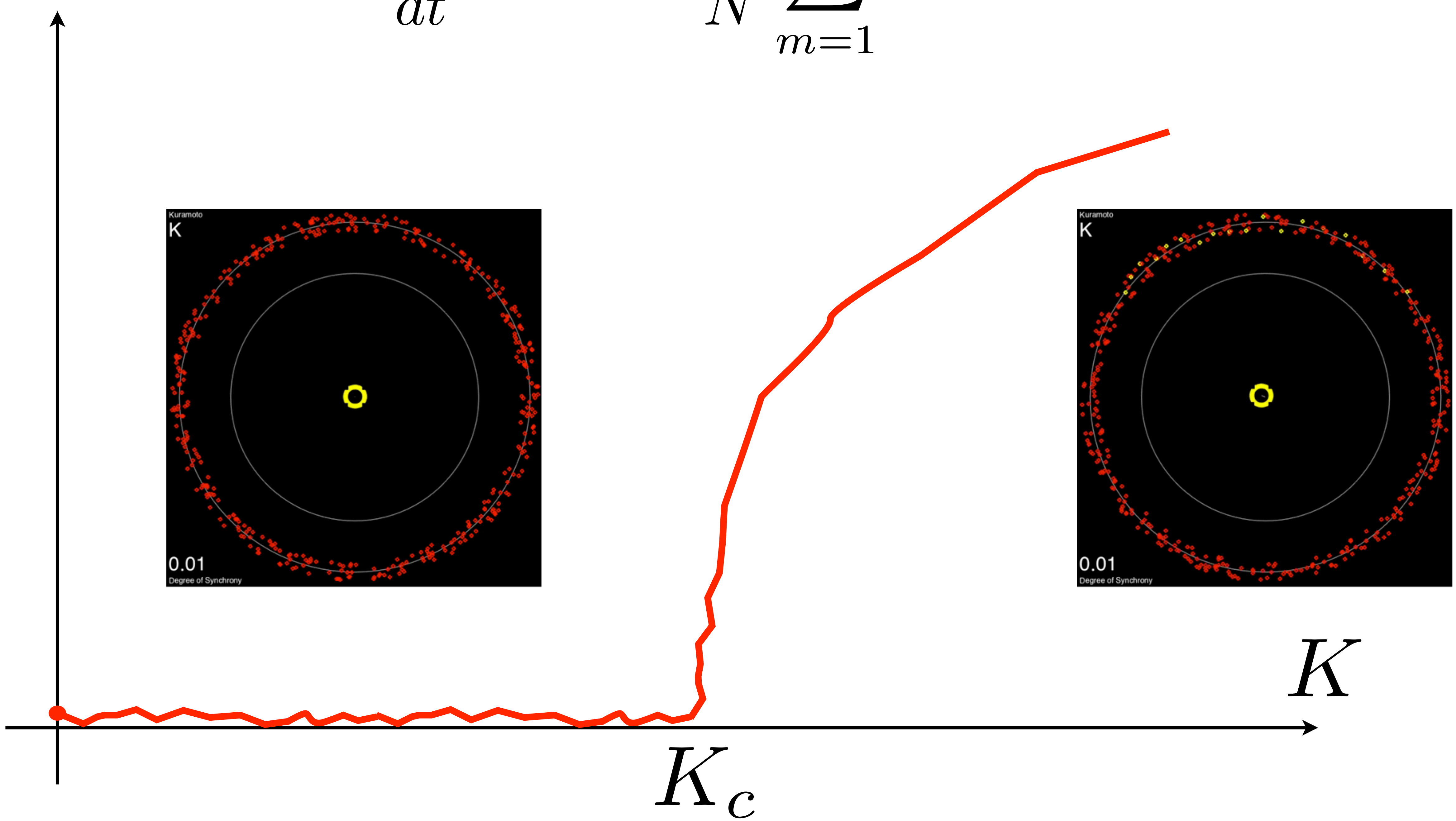
Incoherent

$Z > 0$

Synchronized

Synchronization transition

$$\frac{d\theta_n}{dt} = \omega_n + \frac{K}{N}\sum_{m=1}^{N}\sin(\theta_m - \theta_n)$$



Demonstration

(Lancaster University, Physics Dept.)