Linear Algebra as a Template for Applied Mathematics

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1 Why Linear Algebra Is My Favorite Subject to Teach

Its important applications

Opportunity to start at the beginning

Linear algebra helps develop the students'

- logical ability
- algorithmic ability
- verbal ability

abstraction and analogy

elementary geometry

big picture ability

It is more representative than other subjects of the way we do mathematics

2 Linear Algebra Is Different Things to Different People

Linear systems and related algorithms

Study of vectors and linear transformations

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Interplay between algebra and geometry

3 A Typical Calculus Question

What is the derivative of $\sin^2 x / (1 + \cos x)$?

$$
\begin{aligned}
\left(\frac{\sin^2 x}{1+\cos x}\right)' &= \frac{\left(\sin^2 x\right)' (1+\cos x) + \sin^2 x\,(1+\cos x)'}{(1+\cos x)^2} \\
&= \frac{2 \sin x \sin' x\,(1+\cos x) + \sin^2 x\,(-\sin x)}{(1+\cos x)^2} \\
&= \frac{2 \sin x \cos x\,(1+\cos x) - \sin^3 x}{(1+\cos x)^2}
\end{aligned}
$$

4 A Typical Linear Algebra Question

Are vectors

$$\begin{bmatrix} 21 \\ 22 \\ 23 \end{bmatrix}, \begin{bmatrix} 10 \\ 15 \\ 20 \end{bmatrix}, \begin{bmatrix} 97 \\ 100 \\ 103 \end{bmatrix}$$

linearly dependent?

5 Expressions Are More Alive

The subspace from the previous slide can be captured by the expression

$$\begin{bmatrix} \alpha \\ \frac{\alpha+\beta}{2} \\ \beta \end{bmatrix}$$

But also by the expression

$$\begin{bmatrix} \alpha \\ \beta \\ 2\beta - \alpha \end{bmatrix}.$$

So

$$\begin{bmatrix} \alpha \\ \frac{\alpha+\beta}{2} \\ \beta \end{bmatrix} = \begin{bmatrix} \alpha \\ \beta \\ 2\beta - \alpha \end{bmatrix}!$$

6 ...Which Is Good Training

for many situations

$$
\begin{aligned}
\frac{\partial \frac{1}{2} x^T A x}{dx} &= Ax \\
\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} &= 0 \\
\int_{\Omega} df &= \int_{\partial\Omega} f \\
|A| &= \frac{1}{3!}\delta^{ijk}_{rst} A^r_i A^s_j A^t_k
\end{aligned}
$$

7 Abstraction and Analogy

The differential equation

$$x^2u'' + xu' + \left(x^2 - n^2\right)u = f\left(x\right)$$

is

$$Ax = b.$$

Therefore

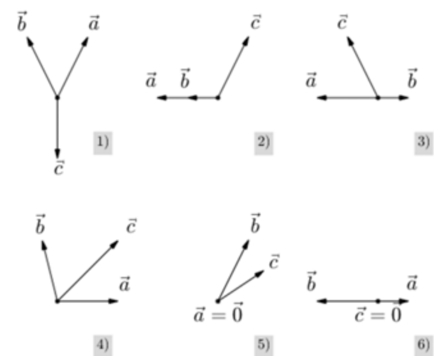
$$u\left(x\right) = u_p\left(x\right) + N\left(x\right)$$

More specifically, since $\dim N = 2$,

$$u\left(x\right) = u_p\left(x\right) + \alpha J_n\left(x\right) + \beta Y_n\left(x\right)$$

8 Overstepping analogy

Come up with three linearly independent vectors in the plane.



9 Overstepping the Analogy

Are the following polynomials linearly dependent?

$$\begin{cases} x^2 - 72x + 9 \\ 13x^2 + 54x - 113 \\ \pi x^2 - 17x + \sqrt{2} \\ ex^2 + \sqrt{19}x - 4 \end{cases}$$

10 Connection with Geometry

Linear Algebra can be seen as a meeting point between algebra and geometry.

11 My Favorite Problems

1. Construct a matrix with the following column space and null space:

$$R = \alpha \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \qquad N = \alpha \begin{bmatrix} 7 \\ 8 \\ 0 \\ 1 \end{bmatrix} + \beta \begin{bmatrix} 4 \\ 2 \\ 1 \\ 2 \end{bmatrix}$$

2. Evaluate

$$\begin{bmatrix} 110 & 55 & -164 \\ 42 & 21 & -62 \\ 88 & 44 & -131 \end{bmatrix}^{2017}$$