Coherence-Pattern Guided Compressive Sensing with Unresolved Grids

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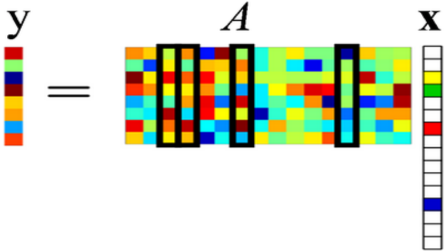
Compressive sensing

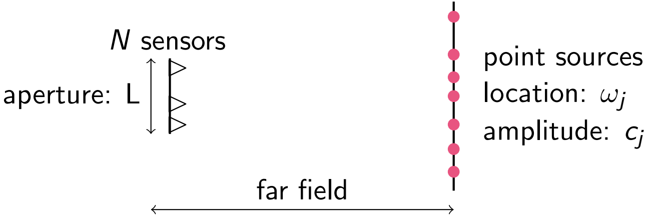
Find sparse solution to an underdetermined linear system:



- Pioneering work: Candès, Romberg and Tao 2004, Donoho 2004, ...
- A: random rows of DFT matrix, i.i.d. gaussian, ...

Benefit to imaging: save number of measurements/sensors

Source localization with sensor array



Source locations and amplitudes: $\{(\omega_j, c_j), j = 1, \ldots, s\}$

Sensor locations: $t_k \in (0, L), k = 1, \ldots, N$

Signal model: at the sensor located at t_k

$$y_k = \underbrace{\sum_{j=1}^{s} c_j e^{-2\pi i t_k \omega_j}}_{\text{signal received by the } k\text{th sensor}} + \underbrace{e_k}_{\text{measurement noise}}$$

[1] Fannjiang, Strohmer and Yan 2010

Resolution limit

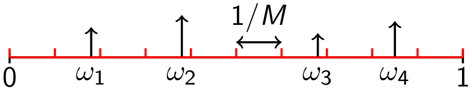
$$\text{Rayleigh Length (RL)} = \frac{1}{\text{Aperture}} = \frac{1}{L}$$

Without additional information, we can only hope to recover sources separated by one RL.

Grid model

Source located on the continuum of a bounded domain: i.e. $\omega_j \in [0,1]$

$$y_k = \sum_{j=1}^{s} c_j e^{-2\pi i t_k \omega_j} + e_k, \quad k = 1, \ldots, N$$



Discretization: approximate ω_j by the closest point on a regular grid $\mathcal{G} = \{(m-1)/M, m = 1, \ldots, M\}$.

Amplitudes: Write $x = \{x_m\}_{m=1}^{M} \in \mathbb{C}^M$ where $x_m = c_j$ whenever $(m-1)/M$ is the closest grid point of ω_j and zero otherwise.

Linear inverse problem

$$y = Ax + e$$

- Sensing matrix $A \in \mathbb{C}^{N \times M}$ with

$$A_{k,m} = e^{-2\pi i t_k (m-1)/M}$$

$$k = 1, \ldots, N, \;\; m = 1, \ldots, M.$$

- e = measurement noise + gridding error

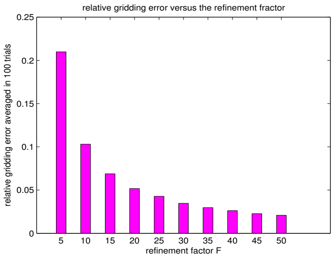
Gridding error

Refinement factor

$$F = \frac{\text{RL}}{\text{grid spacing}} = M/L\text{: \# grid points within one RL}$$

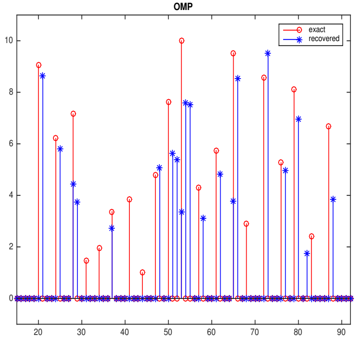
Griding error

- arises from approximating sources by nearest grid points
- almost inversely proportional to refinement factor F

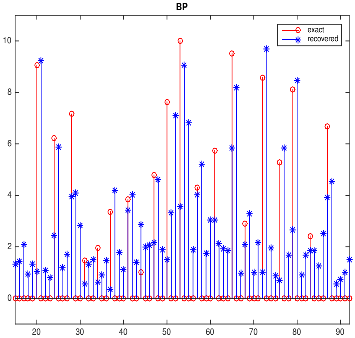


Reconstruction on coarse grid: spacing = RL

OMP



L_1 minimization



minimum separation $\geq$ 3 RL, noise-free

Compressive imaging

Goal: stably recover s sources from $\mathcal{O}(s)$ or $\mathcal{O}(s^2)$ sensors

Condition: Sensing matrix A satisfies either condition:

- Restricted Isometry Property (RIP)
- Incoherence: Coherence of $A := \mu(A) = \max_{j \neq \ell} \mu(j, \ell) \sim 1/\sqrt{N}$

$$\mu(j, \ell) = \frac{| < A(:, j), A(:, \ell) > |}{\|A(:, j)\|_2 \cdot \|A(:, \ell)\|_2}$$

[Foucart and Rauhut 2013] Suppose

1. grid spacing = RL, e.g., $1/M = 1/L$,
2. $\{t_k\}$ are independently and uniformly chosen from $[0, L]$,

then A satisfies RIP with high probability if $N \geq \mathcal{O}(s \ln^4 M)$.

Dilemma

Grid spacing = RL

Sensing matrix A satisfies RIP and incoherence but gridding error is large

Grid spacing $\ll$ RL

Gridding error is small but A is highly coherent.

Compressive imaging on fine grid

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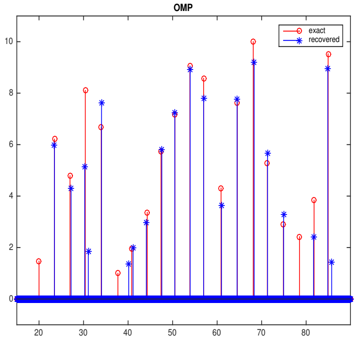
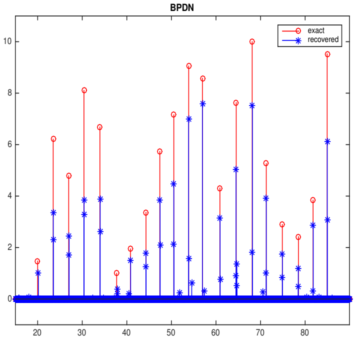


L_1 minimization

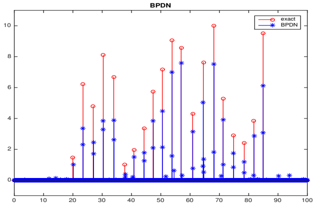


minimum separation $\geq$ 3 RL, F$=$ 50, SNR $=$ 20

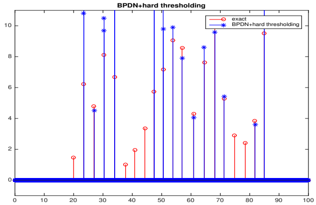
Post-processing of L_1 minimization

- Hard thresholding

L_1 solution

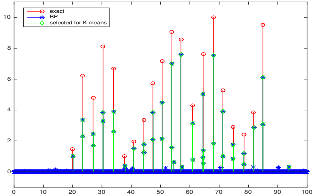


Select the s largest spikes

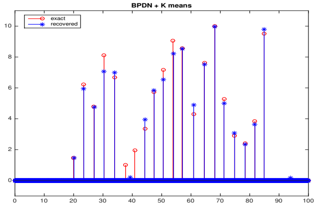


- K means clustering

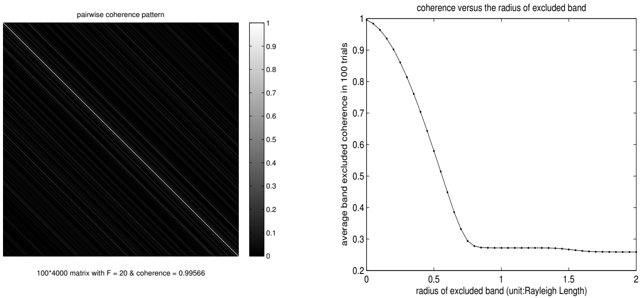
Select the $2s$ largest spikes



K means



Coherence pattern of A on fine grid



Left: A^*A; right: average $\mu(j,\ell)$ versus separation of the jth and the ℓth column.

$$\mu(A) = \max_{j \neq \ell} \mu(j,\ell) = 0.996 \approx 1 \text{ when } F = 20.$$

- large pairwise coherence only occurs at adjacent columns.
- pairwise coherence is small if two columns are separated by 1 RL.

Summary of our work

- Define coherence band
- Propose techniques of band exclusion and local optimization
- Embed these techniques into standard compressive sensing algorithms
- Prove approximate support recovery

Coherence band

Coherence band: Let $\eta \in (0,1)$. Define the η-coherence band of Column k to be the set

$$B_\eta(k) = \{i \mid \mu(i,k) > \eta\},$$

and the η-coherence band of the column set S to be the set

$$B_\eta(S) = \cup_{k \in S} B_\eta(k).$$

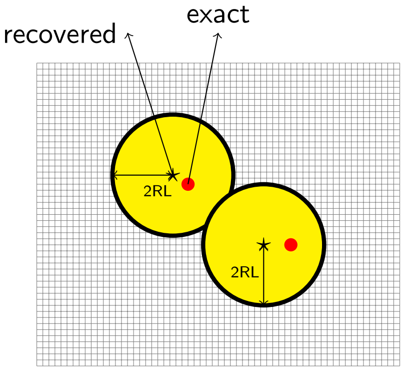
Double coherence band:

$$B_\eta^{(2)}(k) := B_\eta(B_\eta(k)) = \cup_{j \in B_\eta(k)} B_\eta(j)$$
$$B_\eta^{(2)}(S) := B_\eta(B_\eta(S)) = \cup_{k \in S} B_\eta^{(2)}(k)$$

Technique I: Band exclusion(BE)

Idea: exclude the double coherence band of recovered objects

Example:



Band Excluding Orthogonal Matching Pursuit (BOMP)

Algorithm 1. BOMP

Input: $A, y, s, \eta > 0$

Initialization: $x^0 = 0, r^0 = y$ and $S^0 = \emptyset$

Iteration: For $n = 1, ..., s$

1) $i_{\max} = \arg\max_i |\langle r^{n-1}, A(:, i)\rangle|, i \notin B_\eta^{(2)}(S^{n-1})$
2) $S^n = S^{n-1} \cup \{i_{\max}\}$
3) $x^n = \arg\min_z \|Az - y\|_2$ s.t. $\text{supp}(z) \in S^n$
4) $r^n = y - Ax^n$

Output: x^s.

Theorem (Fannjiang and L.)

Let x be s-sparse and $\eta > 0$ be fixed. Suppose that

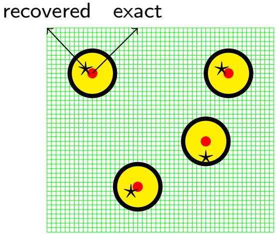
$$B_\eta(i) \cap B_\eta^{(2)}(j) = \emptyset, \quad \forall i, j \in supp(x),$$

$$\eta(5s-4)\frac{x_{\max}}{x_{\min}} + \frac{5\|e\|_2}{2x_{\min}} < 1$$

where $x_{\max} = \max_k |x_k|, \quad x_{\min} = \min_k |x_k|$. *Let $\hat{x}$ be the BOMP reconstruction. Then every nonzero component of $\hat{x}$ is in the η-coherence band of a unique nonzero component of x.*

- separation of sources $\sim$ 3 RL
- approximate support recovery $\sim$ 1 RL
- compression: for moderate SNR

$$\eta = \frac{1}{\sqrt{N}} \qquad N \text{ (\# sensor)} \sim s^2 x_{\max}^2 / x_{\min}^2$$



Spectral compressive sensing

Duarte and Baraniuk 2011

Model Based Compressive Sensing

IHT: $x^{n+1} = T^s(x^n + A^*(y - Ax^n))$

SIHT: $x^{n+1} = T^s_{\text{model based}}(x^n + A^*(y - Ax^n))$

Coherence-inhibiting structured sparse approximation is implemented by the heuristics of selecting the s largest, well separated entries.

Technique II: Local optimization(LO)

Algorithm 2. Local Optimization (LO)

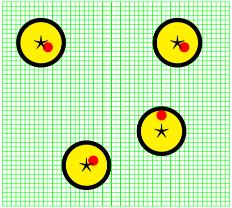
Input:$A, y, \eta > 0, S^0 = \{i_1, \ldots, i_k\}$

Iteration: For $n = 1, 2, ..., k$

1) $x^n = \arg\ \min_z \|Az - y\|_2$
$\text{supp}(z) = (S^{n-1} \backslash \{i_n\}) \cup \{j_n\},\ j_n \in B_\eta(\{i_n\})$

2) $S^n = \text{supp}(x^n)$

Output: S^k



- LO is a residual reduction technique:

$$r(S^k) \le r(S^{k-1}) \le \ldots \le r(S^1) \le r(S^0)$$

where $r(S) = \min_{\text{supp}(z) \subset S} \|Az - y\|$.

Band-excluding, Locally Optimized Orthogonal Matching Pursuit (BLOOMP)

Algorithm 3. BLOOMP

Input: $A, y, s, \eta > 0$

Initialization: $x^0 = 0, r^0 = y$ and $S^0 = \emptyset$

Iteration: For $n = 1, ..., s$

1) $i_{\max} = \arg\max_i |\langle r^{n-1}, a_i \rangle|, i \notin B_\eta^{(2)}(S^{n-1})$
2) $S^n = \mathrm{LO}(S^{n-1} \cup \{i_{\max}\})$
3) $x^n = \arg\min_z \|Az - y\|_2$ s.t. $\mathrm{supp}(z) \in S^n$
4) $r^n = y - Ax^n$

Output: x^s.

BLO-based CS algorithms

Greedy algorithms

BLO Subspace Pursuit

BLO CoSaMP

BLO Iterative Hard Thresholding

L_1 approach

BP-BLOT	constrained L_1 minimization
Lasso-BLOT	L_1 regularization

L_1 approach to recover sources on a continuum

Candes and Fernandez-Granda 2012

$$\|x_{\mathrm{rec}} - x\|_1 \leq \text{Constant} \cdot F^2 \cdot \text{ Noise}$$

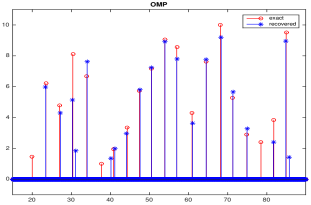
- Full Fourier measurements
- Minimum separation $\geq$ 4 RL

Tang, Bhaskar, Shah and Recht 2013

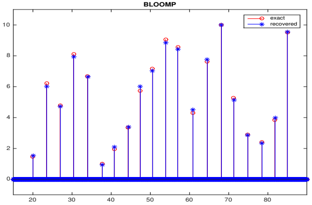
- Compressive Fourier measurements
- Exact recovery without noise
- Minimum separation $\geq$ 4 RL

minimum separation ≥ 3 RL, F$= 50$, SNR $= 20$

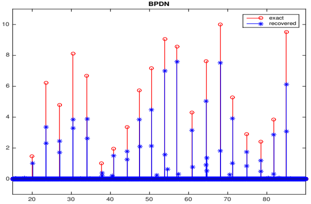
OMP



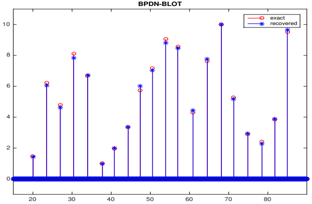
BLOOMP



BP



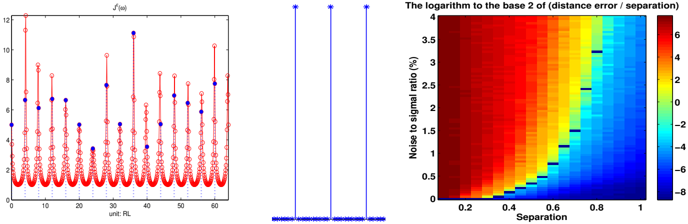
BP-BLOT



BLO-based algorithms can handle larger dynamic range $x_{\text{max}}/x_{\text{min}}$ and have better stability to noise.

MUltiple SIgnal Classification (MUSIC) algorithm (Schmidt 1981)

- Full Fourier measurement
- Sources are recovered at the peaks of an imaging function



1. Sources separated $\geq$ 2 RL: stable recovery.
2. Super-resolution: The noise tolerance of MUSIC obeys a power law with respect to the minimum separation of sources.

[1]W. Liao and A. Fannjiang, "MUSIC for single-snapshot spectral estimation: stability and super-resolution," *ACHA* Vol. 40 No. 1, pp.33-67, 2016.

References

1. A. Fannjiang and W. Liao, "Coherence pattern-guided compressive sensing with unresolved grids", *SIAM Journal on Imaging Science*, Vol. 5, No. 1, 2012.
2. A. Fannjiang and W. Liao, "Mismatch and resolution in compressive sensing", *W*avelets and Sparsity XIV, Proceedings of SPIE, Vol. 8138, 2011.
3. A. Fannjiang and W. Liao, "Super-resolution by compressive sensing algorithms," *The Forty Sixth Asilomar Conference on Signals, Systems and Computers*, 2012.
4. W. Liao and A. Fannjiang, "MUSIC for single-snapshot spectral estimation: Stability and super-resolution," *Applied and Computational Harmonic Analysis* Vol. 40 No. 1, pp.33-67, 2016.

Thank you!

Compressive sensing with highly redundant dictionary

$$y = \Phi x + e = \Phi D\alpha + e$$

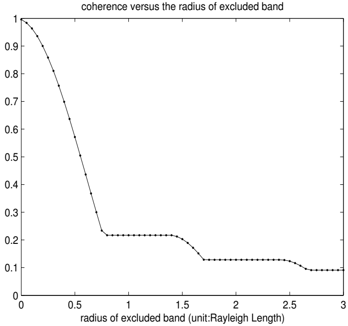
- Φ is i.i.d. Gaussian matrix
- D is an oversampled, redundant DFT frame

Goal: recover x
Performance metric:

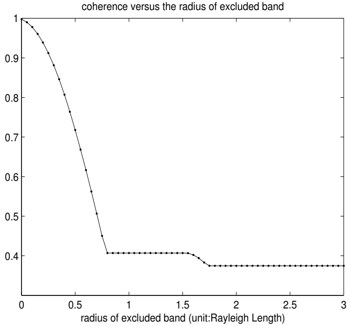
$$\frac{\|D(\alpha - \alpha_{\mathrm{rec}})\|}{\|D\alpha\|}$$

Coherence band

Coherence bands of the DFT frame D and $A = \Phi D$



DFT frame D



$A = \Phi D$

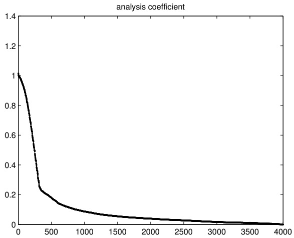
Analysis approach: frame-based L_1 minimization

Candès, Eldar, Needell and Randal 2010

$$\min_z \|D^* z\|_1 \quad \|\Phi z - y\|_2 \leq \varepsilon$$

Assumptions:

- Frame adapted restricted isometry property $\surd$
- Sparsity or compressibility of analysis coefficients $\times$



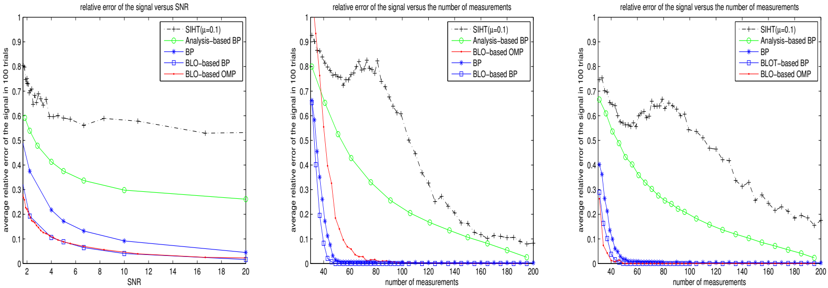
Unless with a tight frame, analysis coefficients have long tail.

Comparison

Stability and Compressibility

Error versus SNR

Error versus # measurement



$x_{\max}/x_{\min}$: 1 1 10