Noise Induced Switching and Extinction in Systems with Delay

Ira B. Schwartz
US Naval Research Laboratory
Washington, DC 20375

Co-authors:

Lora Billings
Montclair State University

Tom Carr
Southern Methodist University

Mark Dykman
Michigan State University

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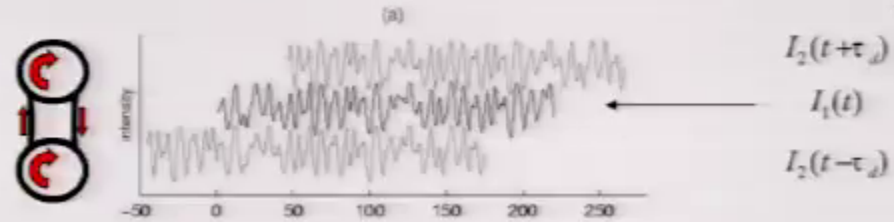
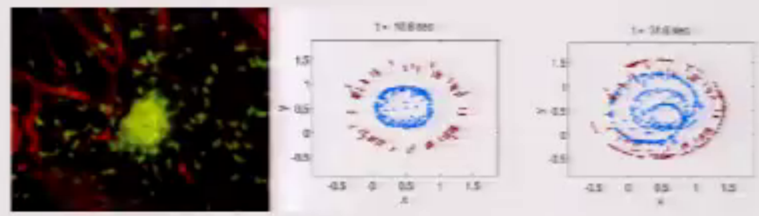
Motivation

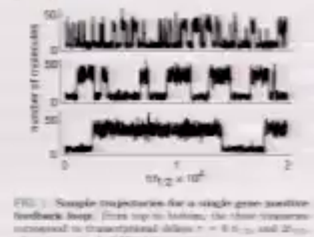
Many physical systems are characterized by delay and noise. Their evolution depends not only on the values of their dynamical variables at current times time, but also on previous history.

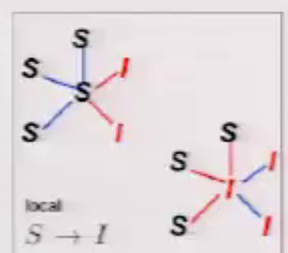
Examples:

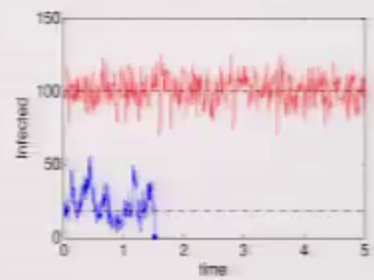
- Coupled optical systems such as fiber ring lasers-switching [Ray 2006, Franz 2007, Uchida 2008, Soriano 2013, Marandi 2014].
- Swarms of large numbers [Lammermann 2013], [Swzakowska, MS71 Tuesday,5:30pm].
- Neural and gene networks [Marcus 1989, Gupta 2014].
- Extinction: delay related to temporary immunity or the time between conception and birth [Taylor 2009, Blyuss 2010, Lindley 2014, [Shaw MS33 Monday 1:45pm].











Problem and approach

- Two types of delay:
  - Retarded dissipation = delayed backaction of a thermal reservoir
    Dykman and Schwartz, PRE (2012)
    Franosch et al 2011
  - Dynamical delay
    Delay due to a finite time of propagation-communication, coupled lasers, ...
    Maturation or latency in populations

$< f(t)f(t') >= 4\Gamma k_B T \delta(t-t')$

$< f(t)f(t') > \propto \Gamma k_B T \tau_D^{1/2} (t-t')^{-3/2}$

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- Problem: Study rates of rare events in systems with delay.
  The trajectories most likely to be followed in rare events.

- Goal : Predict scaling behavior of the rates of interstate switching and population extinction

Fluctuation induced escape - Classical and No Delay

In noise driven systems:
One noise realization leads to a one system trajectory - switching to a new state or extinction.
[Onsager and Malchup (1953), Feynman & Hibbs (1965)]

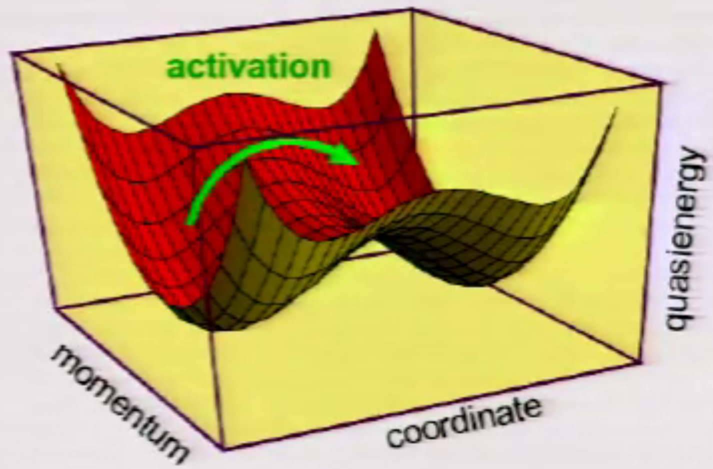
To find the rate of occurrence of a rare event, look for the most probable realization of the noise trajectories that bring the system to that state.

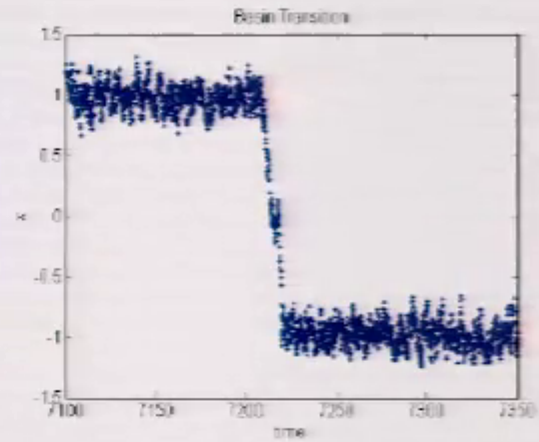
Escape/Extinction rate:

$$W \propto \exp(-R/D)$$

R is the minimum action

D is the noise intensity.





There are many such path realizations!

Goal: Extend theory that includes the delay.

Outline

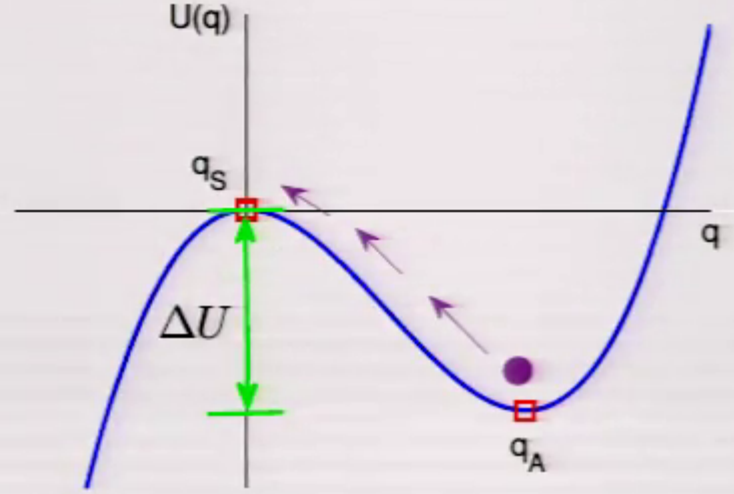
- Variational formulation to stochastic escape for systems with delay.
- Derive rates of switching and extinction.
- Examples to compare our theory with numerical simulations.
- Odds and ends and extensions
- Conclusions

The general problem setup

The stochastic dynamics are described by the Langevin equation:

$$\dot{\mathbf{q}}(t) = \mathbf{K}(\mathbf{q}(t), \mathbf{q}(t-\tau)) + \hat{G}(\mathbf{q}(t))\mathbf{f}(t).$$

state $\mathbf{q} = (q_1, q_2, \ldots, q_N)$, delay τ



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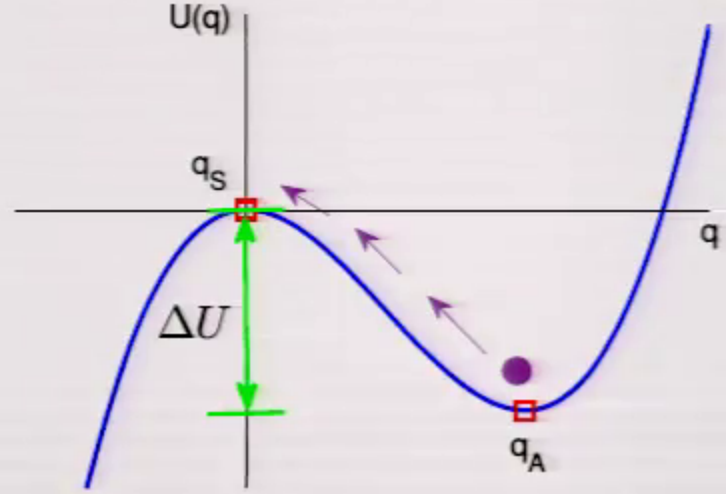


$\mathbf{K}(\mathbf{q}_A) = \mathbf{K}(\mathbf{q}_S) = 0$ (attractor $\mathbf{q}_A$ and saddle $\mathbf{q}_{\mathcal{S}}$),
$\mathbf{f}(t)$ is the noise vector, and $\hat{G}(\mathbf{q}(t))$ scales the noise.

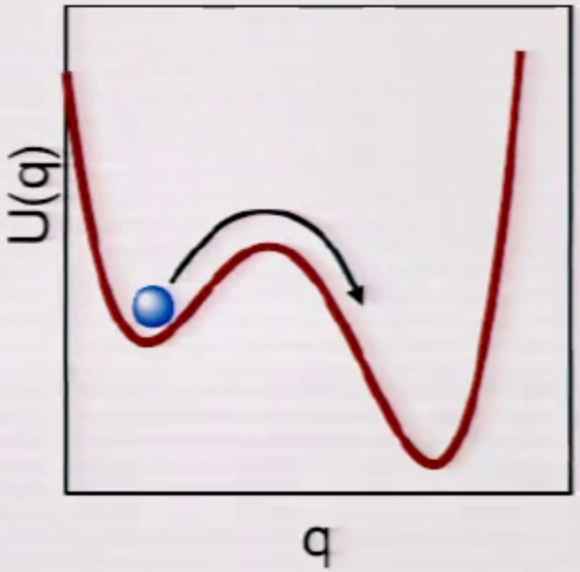
The probability density is given by

$$\mathcal{P}_{\mathbf{f}}[\mathbf{f}(t)] \propto \exp\left(-\mathcal{R}_{\mathbf{f}}/D\right), \quad \mathcal{R}_{\mathbf{f}}[\mathbf{f}(t)] = \frac{1}{4}\int_{-\infty}^{\infty} dt\, dt'\, \mathbf{f}(t)\hat{\mathcal{F}}(t-t')\mathbf{f}(t')$$

Noise $\mathbf{f}(t)$
Gaussian, stationary, and weak on average

$\hat{\mathcal{F}}(t) \equiv$ inverse of the pair correlator of $\mathbf{f}(t)$,
$\int dt'\, \hat{\mathcal{F}}(t-t')\hat{\phi}(t'-t'') = 2D\hat{\mathcal{I}}\delta(t-t'')$
D is the noise intensity.

Fluctuation characterization near the attractor



Examine linearized system near $\mathbf{q}_A$:

$$\dot{\mathbf{X}}(t) = \hat{\mathcal{K}}^{(1)}\mathbf{X}(t) + \hat{\mathcal{K}}^{(2)}\mathbf{X}(t-\tau),$$

$$\mathcal{K}^{(1)} = \partial \mathbf{K}(\mathbf{q}_A, \mathbf{q}_A)/\partial \mathbf{q}, \qquad \mathcal{K}^{(2)} = \partial \mathbf{K}(\mathbf{q}_A)/\partial \mathbf{q}_\tau.$$

Gaussian noise $\mathbf{f}(t)$ characterization:

$$\phi_{nm}(t) = \langle f_n(t) f_m(0) \rangle,$$

$$\Phi_{nm}(\omega) = \frac{1}{2}\int_{-\infty}^{\infty} dt \exp(i\omega t)\phi_{nm}(t).$$

Fluctuations spend most of the time around $\mathbf{q}_A$

Fluctuation amplitude and correlations about $\mathbf{q}_A$ depend explicitly on delay

Assume stability does not change for small delay.

Fluctuations near the attractor (in Fourier space)

$$\delta\mathbf{q}(\omega) = \hat{\mathcal{G}}(\omega)\mathbf{f}(\omega),$$

$$\hat{\mathcal{G}}(\omega) = -\left(i\omega\mathcal{I} + \hat{\mathcal{K}}^{(1)} + \hat{\mathcal{K}}^{(2)} e^{i\omega\tau}\right)^{-1} \hat{G}(\mathbf{q}_A),$$

With correlations:

$$\langle \delta q_n(\omega)\delta q_m(\omega') \rangle = 4\pi [\hat{\mathcal{G}}(\omega)\hat{\Phi}(\omega)\hat{\mathcal{G}}^\dagger(\omega)]_{nm}\delta(\omega + \omega').$$

Variational Problem for Large Rare Fluctuations

Switching or escape rate $W \propto \exp\left(-\mathcal{R}/D\right)$

$$\mathcal{R} = \min_{\mathbf{q},\mathbf{f},\chi}\left(\mathcal{R}_{\mathbf{f}}[\mathbf{f}(t)] + \int_{-\infty}^{\infty} dt\chi(t)[\dot{\mathbf{q}}(t) - \mathbf{K}\left(\mathbf{q}(t), \mathbf{q}(t-\tau)\right) - \hat{G}(\mathbf{q})\mathbf{f}(t)]\right).$$

$\chi(t)$ is the Lagrange multiplier.

Minimize $\mathcal{R}$ with the constraint that the noise drives the system to the target state

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Minimize $\mathcal{R}$ with the constraint that the noise drives the system to the target state

- Optimal path - the most probable path to follow in a large rare fluctuation
- Determined by the most probable force realization of the noise.
- Used to compute the mean escape times

Boundary conditions:

$$\lim_{t\to-\infty} \mathbf{q}(t) = \mathbf{q}_A \qquad \lim_{t\to\pm\infty} \mathbf{f}(t) = 0$$
$$\lim_{t\to+\infty} \mathbf{q}(t) = \mathbf{q}_S \qquad \lim_{t\to\pm\infty} \chi(t) = 0$$

Delayed Hamiltonian Equations of Motion

The variational equations for the coupled most probable fluctuational paths

- Equations of Motion: $\delta\mathcal{R}/\delta\chi(t) = 0$ (causal)

$$\dot{\mathbf{q}}(t) = \mathbf{K}\left(\mathbf{q}(t), \mathbf{q}(t-\tau)\right) + \hat{G}(\mathbf{q}(t))\mathbf{f}(t).$$

- Optimal Noise: $\delta\mathcal{R}/\delta f(t) = 0$ (causal)

$$\frac{1}{2}\int_{-\infty}^{\infty} dt'(\hat{\mathcal{F}}(t-t')\mathbf{f}(t') - \hat{G}^{\dagger}(\mathbf{q}(t))\chi(t)) = 0.$$

- Optimal Lagrange multiplier: $\delta\mathcal{R}/\delta q(t) = 0$ (acausal)

$$\dot{\chi}(t) = -\partial_{\mathbf{q}(t)}\Big[\chi(t)\mathbf{K}(\mathbf{q}(t), \mathbf{q}(t-\tau)) + \chi(t+\tau)\mathbf{K}(\mathbf{q}(t+\tau), \mathbf{q}(t))\Big] - \partial_{\mathbf{q}(t)}\Big(\chi(t)\hat{G}(\mathbf{q}(t))\mathbf{f}(t)\Big)$$

Perturbation Theory

Assume $\boldsymbol{K}$ is a smooth and $\mathbf{q}(t-\tau) \equiv \mathbf{q}_\tau$ is close to $\mathbf{q}(t)$

$$||\boldsymbol{K}(\mathbf{q},\mathbf{q}) - \boldsymbol{K}(\mathbf{q},\mathbf{q}_\tau)|| \lessapprox \varkappa ||\mathbf{q}-\mathbf{q}_\tau||$$

$$\mathcal{R}[\mathbf{q},\mathbf{f},\chi] = \mathcal{R}^{(0)}[\mathbf{q},\mathbf{f},\chi] + \mathcal{R}^{(1)}[\mathbf{q},\mathbf{f},\chi],$$

Minimize $\mathcal{R}^{(0)}$ to get

$$\mathbf{q}^{(0)}(t), \chi^{(0)}(t), \mathbf{f}^{(0)}(t)$$

Evaluate $\mathcal{R}^{(1)}$ using zero order solution

First order result:

$$R_j^{(1)} \approx \tau \int dt \chi^{(0)}(t)[(\dot{\mathbf{q}}\,\partial_{\mathbf{q}'})\mathbf{K}(\mathbf{q},\mathbf{q}')]_{\mathbf{q}=\mathbf{q}'=\mathbf{q}^{(0)}(t)}.$$

One-dimensional examples

We consider two systems driven by white Gaussian noise.

$$\dot{q}(t) = K\left(q(t), q(t-\tau)\right) + G(q)f(t)$$

1) $K_{\text{attr}}(q(t), q(t-\tau)) = -q^2(t) + q(t) - \gamma q(t-\tau),$

2) $K_{\text{rpl}}(q(t), q(t-\tau)) = -q^2(t) - \gamma q(t) + q(t-\tau),$

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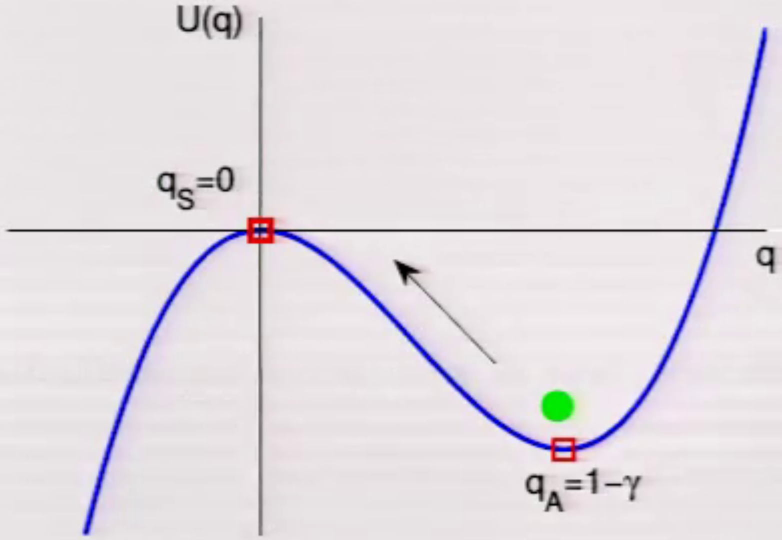
Let $0 < \gamma < 1$

Same for no noise and delay:
$K = (1-\gamma)q(t) - q^2(t)$

$q_A = 1-\gamma$ and $q_S = 0$



When noise is added, the particle escapes by passing over the barrier.

$\eta = 1-\gamma$ is the distance to the bifurcation point.

One-dimensional example – additive noise results

Switching rate: $R_{\text{attr}} \approx \frac{1}{6}(1-\gamma\tau)(1-\gamma)^3$ and $R_{\text{rpl}} \approx \frac{1}{6}(1+\tau)(1-\gamma)^3$.

Mean time to switch: $T_{\text{sw}} = \frac{2\pi}{1-\gamma}\exp(R/D)$

The pre-factor uses theory in which there is no delay. [Kramers (1940)]

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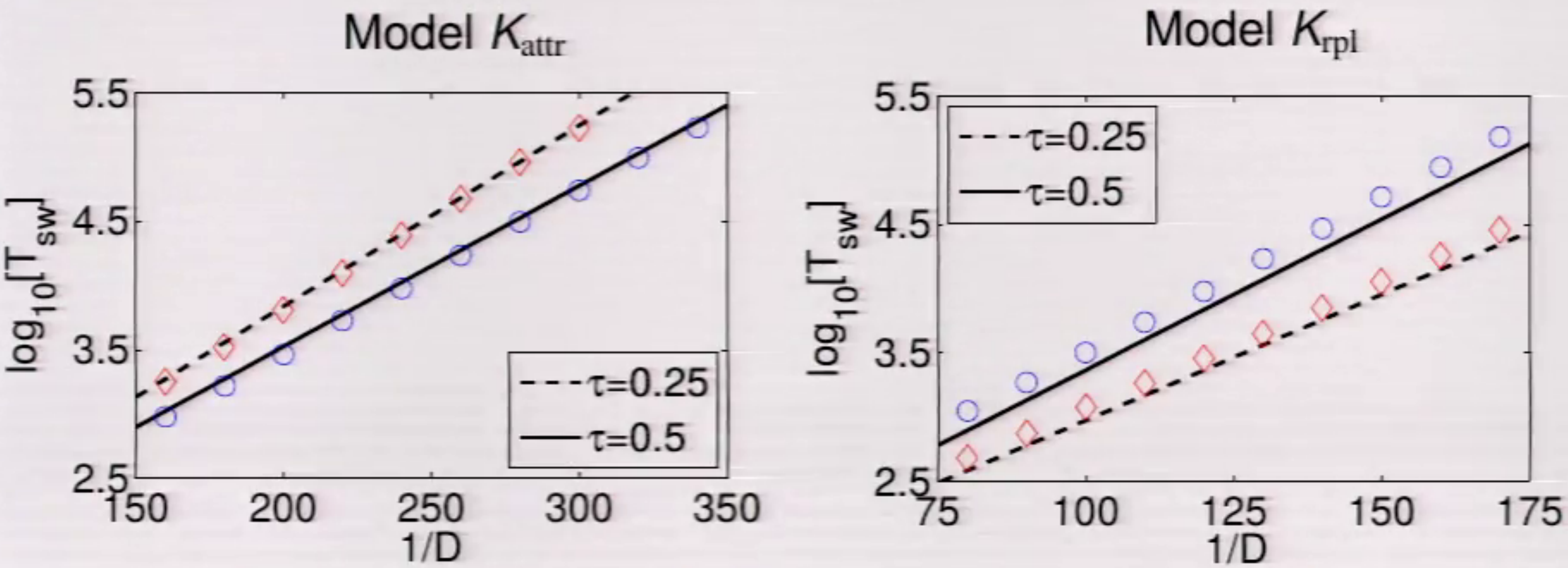


Theory – solid and dashed lines; MC Simulations – data points. $\gamma = 0.4$

One-dimensional example – multiplicative noise

We consider the same two systems driven by multiplicative white Gaussian noise, $G(q) = \sqrt{q(t)}$.

$$\dot{q}(t) = K\left(q(t), q(t-\tau)\right) + \sqrt{q(t)}\, f(t)$$

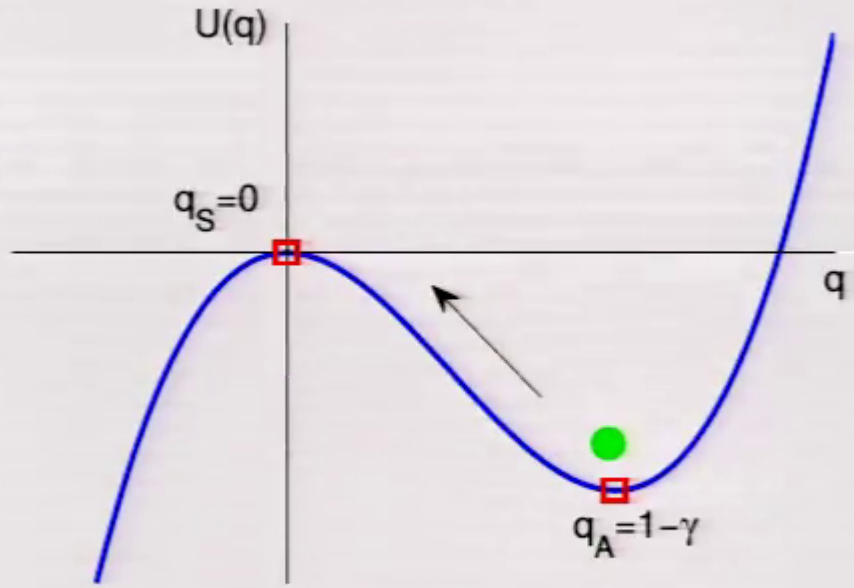
1) $K_{\mathrm{attr}}(q(t), q(t-\tau)) = -q^2(t) + q(t) - \gamma q(t-\tau),$

2) $K_{\mathrm{rpl}}(q(t), q(t-\tau)) = -q^2(t) - \gamma q(t) + q(t-\tau),$

Mean time of extinction:
after the system has reached the extinction state, we must make sure it will not leave it.

Careful with K_{rpl}

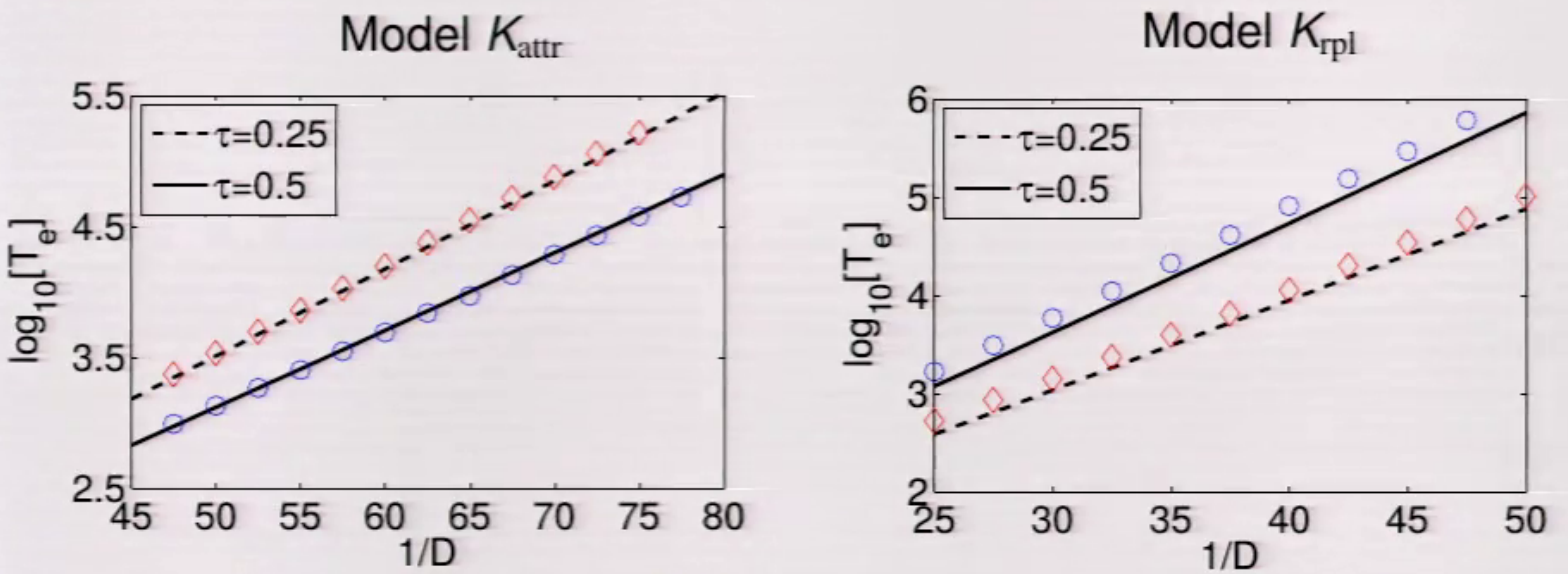
Delayed population growth problem
Think of chickens and eggs...



One-dimensional example – multiplicative noise results

Extinction rate: $R_{\text{attr}} \approx \frac{1}{2}(1-\gamma\tau)(1-\gamma)^2$ and $R_{\text{rpl}} \approx \frac{1}{2}(1+\tau)(1-\gamma)^2$

Mean time to extinction: $T_{\text{e}} = \frac{\sqrt{2\pi D}}{(1-\gamma)^2}\exp(R/D)$



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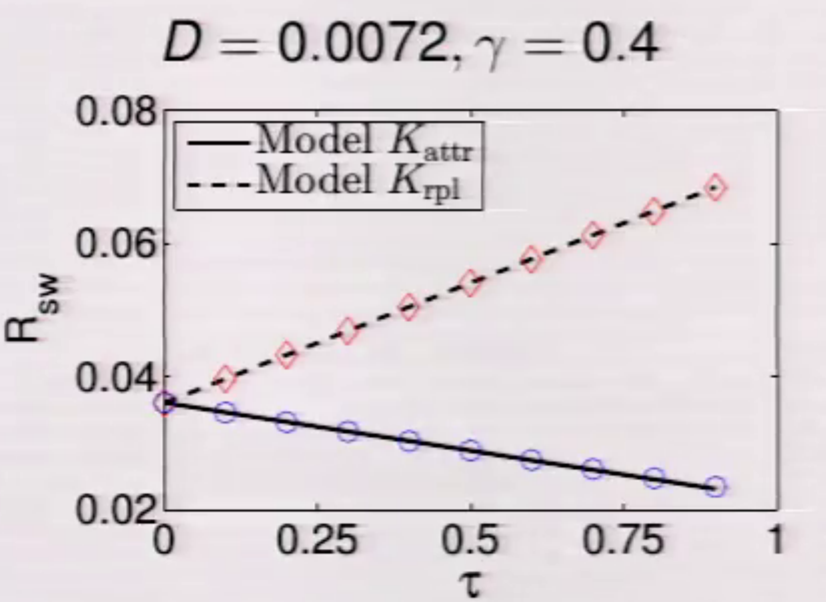
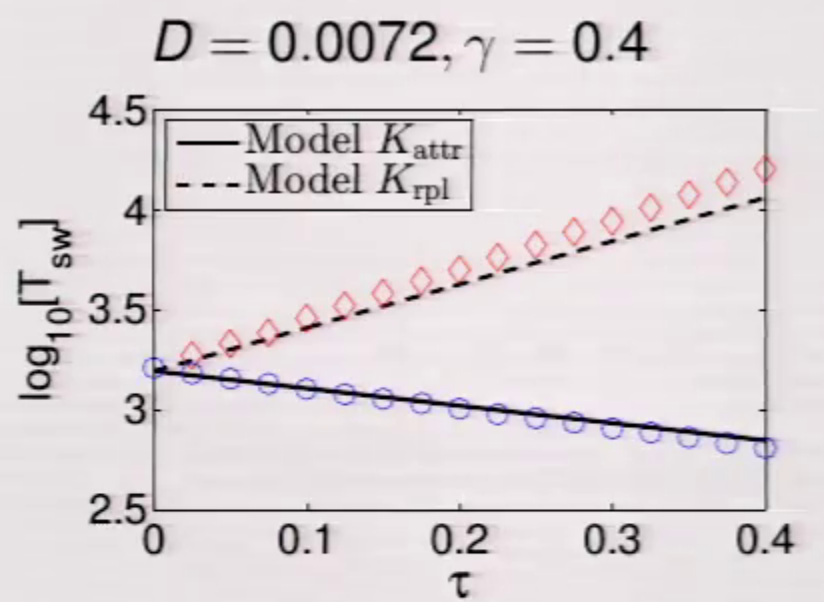
The simulations use the Ito formulation of the Milstein method.

Direct Variation and First Order Additive Theory
τ dependence

Using the K_{attr} and K_{rpl} delay models where $G(q) = 1$:



Lines/dashes are theory-Symbols are direct variation

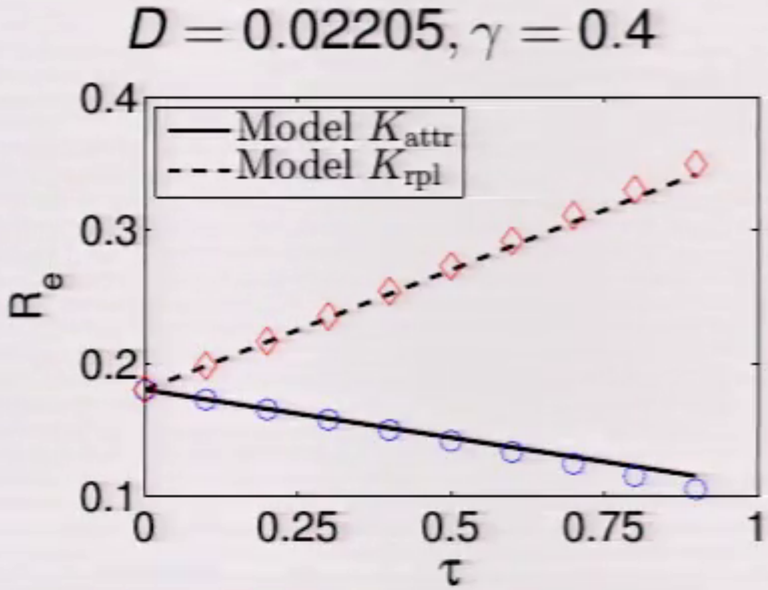


Lines/dashes are theory-Symbols are Monte Carlo

Direct Variation and First Order Multiplicative Theory τ dependence

Assume noise is white.
Singular multiplicative noise - models extinction

Using the K_{attr} and K_{rpl} delay models where $G(q) = \sqrt(q)$:



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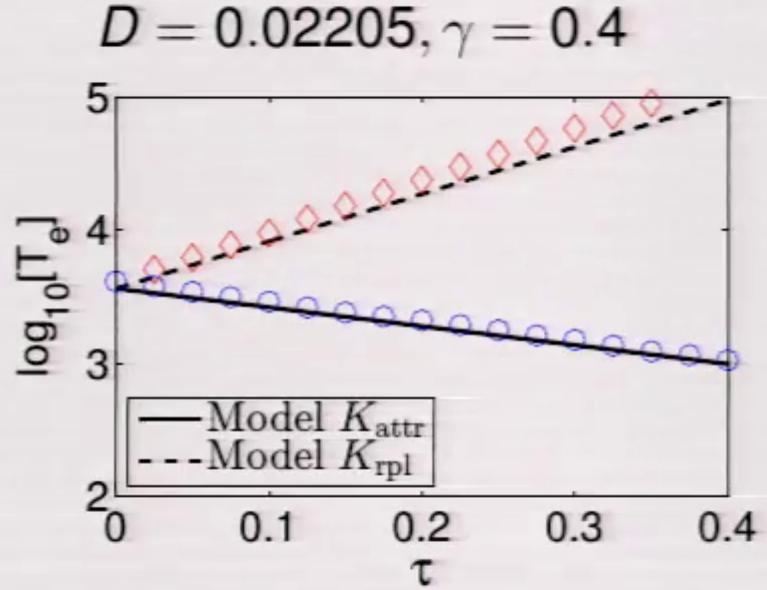

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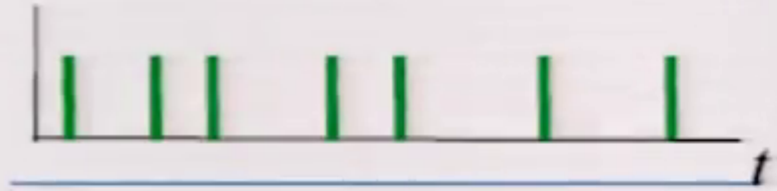
Generalizations to Non-Gaussion Noise

Variational theory may be extended to systems driven discrete noise

- Shot noise
- Poisson noise



Poisson Noise $\mathbf{f}(t) = g \sum \delta(t - t_n)$
Amplitude g mean frequncy ν

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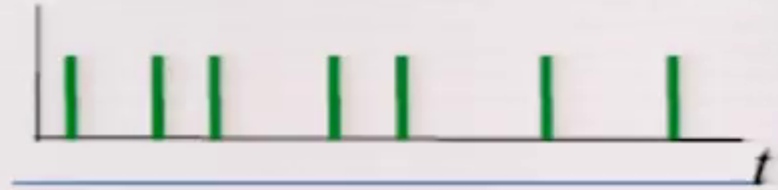

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Extend perturbation to Poisson only noise

Let $v = K(q,q)/gv >> 1$. Then

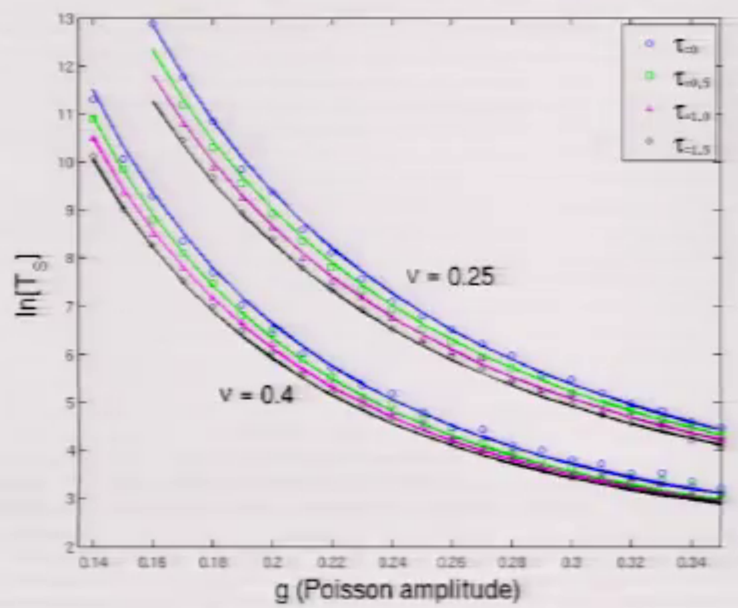
$$R_0 \approx \int_{\tilde{q}_a}^{\tilde{q}_S} \kappa(q) dq \kappa(q) \approx g^{-1}\{\ln v(q) + \ln[\ln v(q)]\}$$

In the limit $g\nu << 1$, action exhibits

non-power law scaling:

$$R \approx \frac{(1-\gamma\tau)(\tilde{q}_S - \tilde{q}_a)}{g} ln(\frac{\tilde{q}_S - \tilde{q}_a}{g\nu t_r})$$

Switching vs. g



Conclusions

- The problem of large rare fluctuations in dissipative and reaction systems with delay is reduced to a variational problem. The solution describes most probable paths followed in rare events and after a target state has been reached.
- The delayed Hamiltonian equations are acausal: the evolution along the trajectory depends both on the evolution before and after a given instant of time.
- Delay can stabilize or destabilize the escape process.
- Delay can enhance or diminish the effective barrier measured by the mean escape times along the optimal path.
- The exponent of the escape rate scales with the distance to the bifurcation point for both additive and multiplicative noise.