Interferometric wave source inversion for Photoacoustic Imaging

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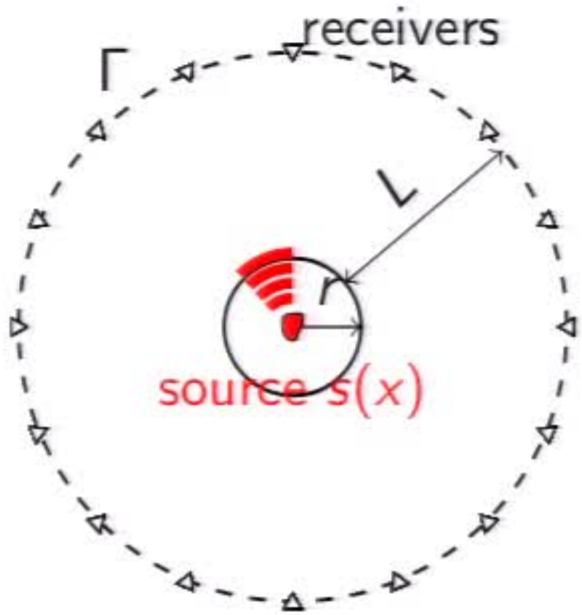
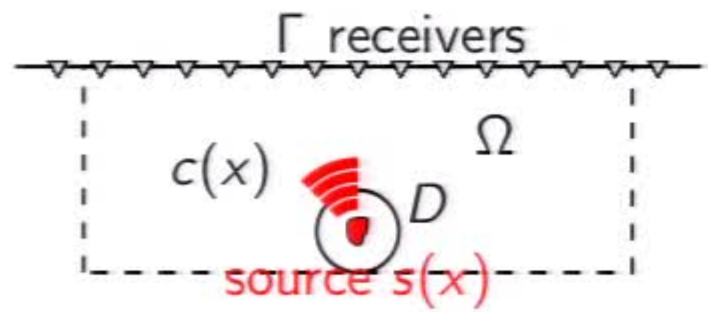
The direct source problem





In frequency domain the wave field $u(x, \omega)$ is solution of

$$\begin{cases} \left(\Delta + \dfrac{\omega^2}{c^2(x)}\right) u(x, \omega) = s(x) \text{ on } \mathbb{R}^d \\ \text{Sommerfeld condition at } \infty \end{cases}$$

Applications

Earth science :

- Passive geo-seismic source imaging
- Active geo-seismic imaging

Medical imaging :

- Ultrasound medical imaging
- Microwave imaging
- thermo-acoustic/opto-acoustic imaging

Military purpose :

- Active/passive sonar
- EM sources identification

Forward operator

If we have N_r receivers and N_ω frequencies, we have $N := N_r N_\omega$ complex numbers.

Data

$$d_i := u(x_i, \omega_i), \quad i = 1, \ldots, N$$

Forward operator

$$\begin{aligned} \tilde{F} : L^2(\Omega) &\longrightarrow \mathbb{C}^N \\ s &\longmapsto d \end{aligned}$$

Limitation

These methods work if and only if

$$\left\| F - \tilde{F} \right\|_{\mathcal{L}\left(L^2(\Omega),\mathbb{C}^N\right)} << \left\| \tilde{F} \right\|_{\mathcal{L}\left(L^2(\Omega),\mathbb{C}^N\right)}$$

and this requires an excellent approximation of the speed map $c(x)$. In this case the error is controlled by

$$\frac{\|s_{LS} - s\|}{\|s\|} \leq \kappa(F)^2 \frac{||\tilde{F} - F||}{||F||}$$

Sensitivity to a change of speed

If c is constant, then the forward map ca be written in 3D as

$$(\tilde{F}s)_i := \int_\Omega \frac{e^{\frac{i\omega_i}{c}|y-x_i|}}{4\pi|y-x_i|} s(y) \mathrm{d}y$$

We can see that un error on $1/c$ of order $1/(\omega_{\max} dist(s,\Gamma))$ completely destroy the data. On other terms, one can show that if c_0 does not respect

$$\frac{c-c_0}{c_0} << \frac{c_0}{\omega_{\max} dist(s,\Gamma)} \quad (\approx 5\% \text{ in geoseismic setting.})$$

then the operator error

$$\frac{|||\tilde{F}-F|||}{|||F|||} \approx 1$$

at least.

Sensitivity to a change of receivers position (defocusing)

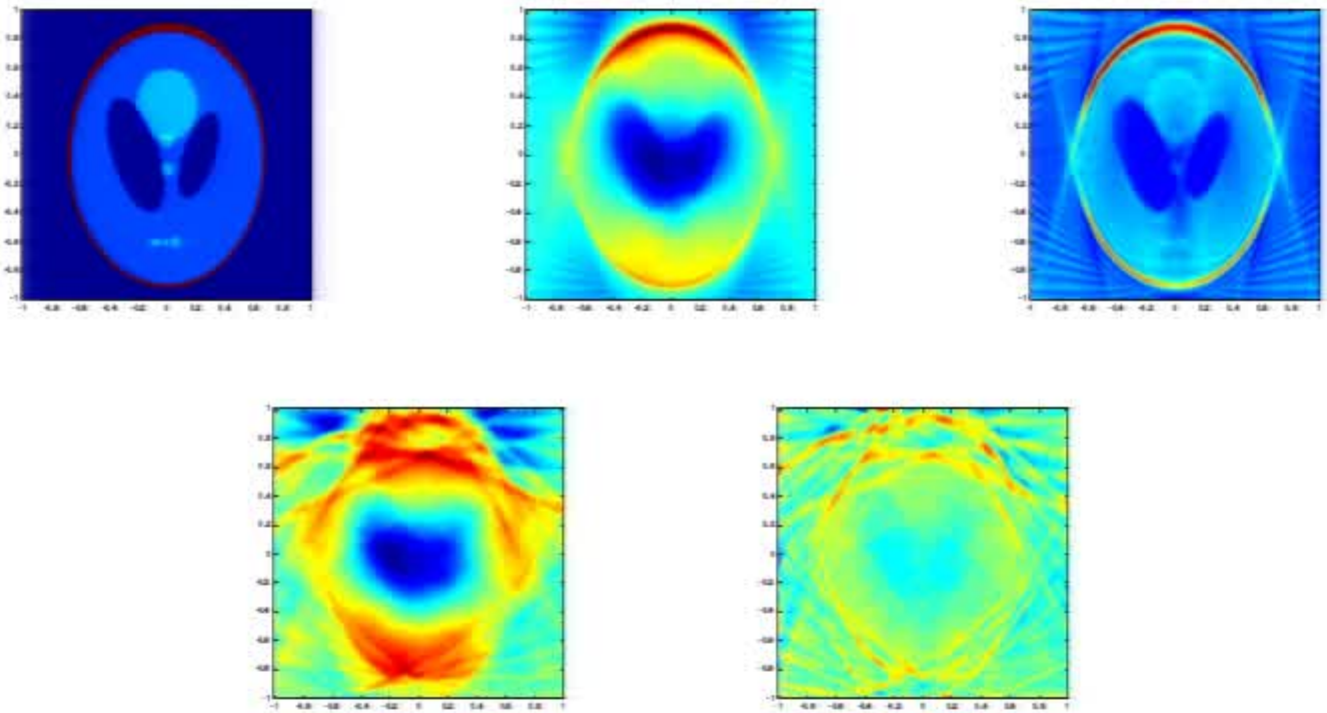


Figure: Source inversion with $c = 1$, aperture $= 80$, depth $= 10$, $\omega_{\max} = 128$. (1) The source. (2) Back projection solution. (3) Least squares solution. (4) Back projection solution with 2% of error on the receivers location. (5) Least squares solution with 2% of error on the receivers location.

Interferometric data

Instead of using the data $u(x_i, \omega_i)$ which is very sensitive to uncertainties, we may use a correlation between two data :

$$D_{i,j} := u(x_i, \omega_i)\overline{u(x_j, \omega_j)}$$

Which may be more robust. The intuition is that if $u(x_i, \omega_j) \approx A(x_i)e^{i\omega_j T(x_i)}$ where $T(x_i)$ is the travel time between the source and the receiver, an error on $T(x_j)$ is dramatic but as

$$u(x_i, \omega_i)\overline{u(x_j, \omega_j)} \approx A(x_i)A(x_j)e^{i(\omega_i T(x_i) - \omega_j(T(x_j))}$$

then the error on T might be almost the same on x_i and x_k if these receivers are close to each other.

Interferometric inversion

Introduce a sparse selector $E \in \mathcal{S}_N(\{0,1\})$ the associated semi-norm defined on hermitian matrices

$$|A|_{E,2}^2 = \sum_{i,j} E_{i,j} A_{i,j}^2,$$

We introduce the functional

$$J_E(s) := \frac{1}{2} |(Fs)(Fs)^* - dd^*|_{E,2}^2$$

which is an order 4 polynomial non convex cost function that we want to minimize.

Lifting method

Instead of minimizing

$$J_E(s) := \frac{1}{2}|Fss^*F^* - dd^*|^2_{E,2}$$

One can minimize the quadratic least square cost function

$$\tilde{J}_E(M) := \frac{1}{2}|FMF^* - dd^*|^2_{E,2}$$

which is convex and then extract s as the first eigenvector of M.

Interpretation of s_{CINT} formula (Interferometric back propagation)

$$s_{CINT} = diag\nabla\tilde{J}_E(0)$$

This is the first step of a gradient descent for $\tilde{J}_E$.

Interferometric inversion

Introduce a sparse selector $E \in \mathcal{S}_N(\{0,1\})$ the associated semi-norm defined on hermitian matrices

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Papanicolaou's imaging formula

Papanicolaou et Al. proposed the following formula in 2005 :

$$s_{CINT}(x) = \sum_{|x_i - x_j| < \alpha} \sum_{|\omega_i - \omega_j| < \beta} \overline{G_{\omega_i}(x_i, x)} G_{\omega_j}(x_j, x) u(x_i, \omega_i) \overline{u(x_j, \omega_j)}$$

where G is the green function for the forward problem. He sees it as a square of a back-propagation. α and β are some parameter to calibrate.

It is a way to back propagate the interferometric data.

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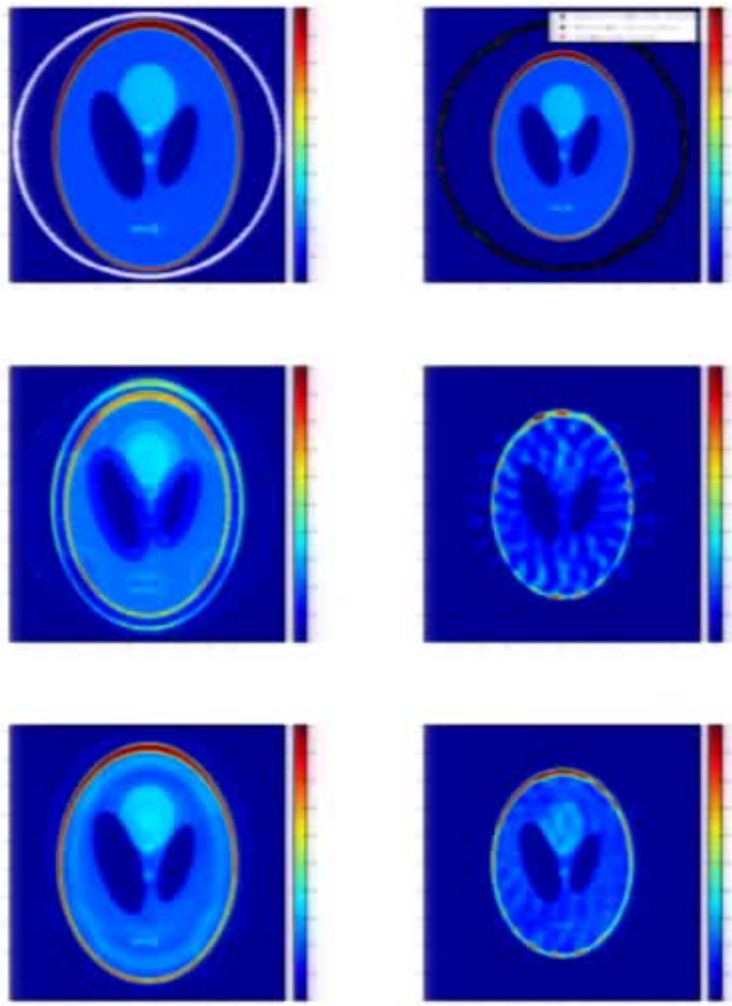
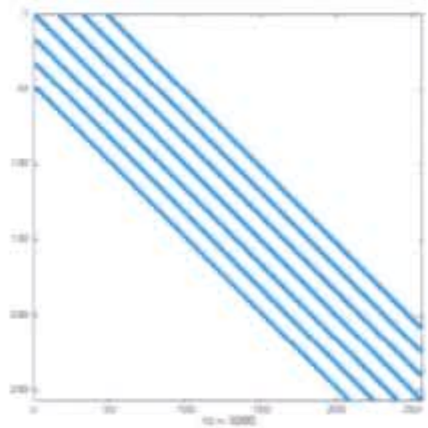
Autofocus effect



with $E_{i,j} = \chi_{\{|x_i - x_j| \leq \alpha, |\omega_i - \omega_j| \leq \beta\}}$.

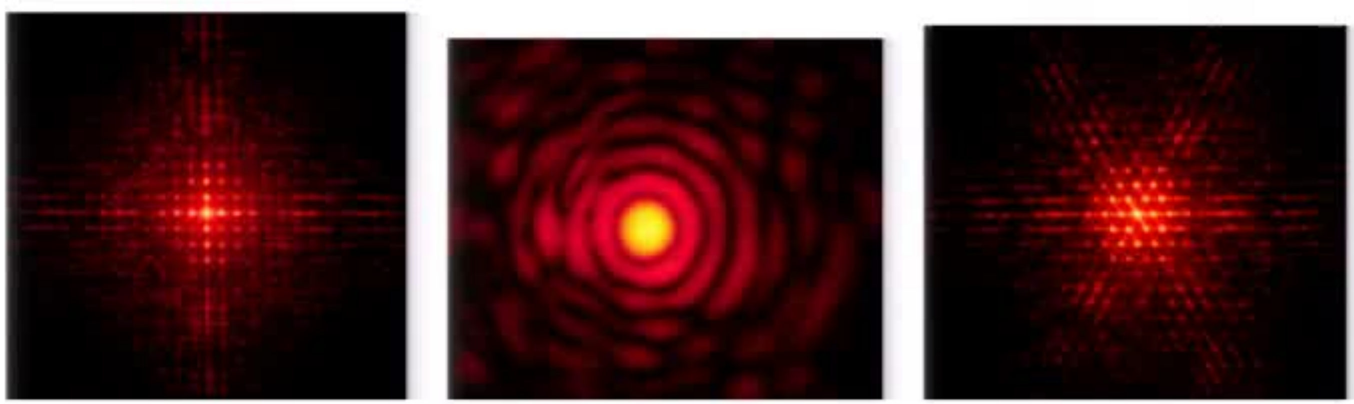
About the sparse filter



Intuitive sparse filter E :

Is it the best choice ? It there a rule too choose pairs of data to compute correlations ?

If $E = Id$ it is the diffraction inversion case. The data looks like :



Far field approximation in constant speed

In 3D, the Helmholtz Green function is

$$G_c(x,\omega) = \frac{e^{i\frac{\omega}{c}|x|}}{4\pi|x|}$$

and satisfies

$$G_c(x-y,\omega) \approx G_c(x,\omega)\exp\left(-i\frac{\omega}{c}\frac{x}{|x|}\cdot y\right)$$

in far field approximation. Then the forward operator satisfies

$$(\tilde{F}s)_i \approx G_c(x_i,\omega_i)\widehat{s}\left(\frac{\omega_i}{c}\frac{x_i}{|x_i|}\right)$$

Phase difference cancellation

Theorem : Filter for constant speed error

Suppose that the hypothetic linear forward operator is given by

$$F_\varepsilon s(x,\omega) = G_{\frac{c}{1+\varepsilon}}(x,\omega)\widehat{s}\left(\frac{(1+\varepsilon)\omega}{c}\frac{x}{|x|}\right),$$

where ε is a real error on the slowness $1/c$. Fix a positive real number η and suppose that the sparse selector E satisfies

$$E_{i,j} = 1 \Rightarrow |\omega_i|x_i| - \omega_j|x_j|| \leq \eta c.$$

Then the dilated source given by $\widehat{s_\varepsilon}(\xi) = \widehat{s}((1+\varepsilon)\xi)$ satisfies

$$|F_\varepsilon[s_\varepsilon](F_\varepsilon[s_\varepsilon])^* - dd^*|_{E,2}^2 \leq C\varepsilon^2\eta^2.$$

Phases differences cancellation

Proof : For any i,j such that $E_{i,j} = 1$,

$$\left| G_{\frac{c}{1+\varepsilon}}(x_i, \omega_i) \overline{G_{\frac{c}{1+\varepsilon}}(x_j, \omega_j)} - G_c(x_i, \omega_i) \overline{G_c(x_j, \omega_j)} \right| =$$

$$\frac{\left| \exp\left(i \frac{(1+\varepsilon)}{c} (\omega_i |x_i| - \omega_j |x_j|) \right) - \exp\left(i \frac{1}{c} (\omega_i |x_i| - \omega_j |x_j|) \right) \right|}{16\pi^2 |x||x_j|}$$

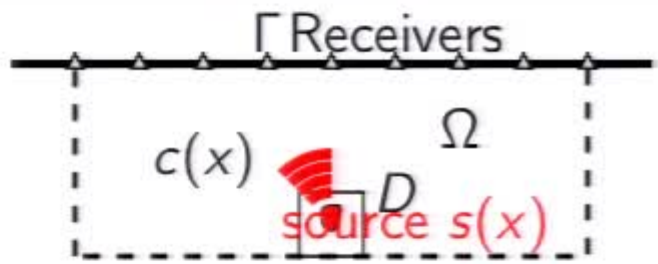
$$= \frac{\left| \sin\left(\frac{\varepsilon}{2c} (\omega_i |x_i| - \omega_j |x_j|) \right) \right|}{8\pi^2 |x||x_j|},$$

$$\leq \frac{\varepsilon \left| \omega_i |x_i| - \omega_j |x_j| \right|}{16\pi^2 c |x||x_j|},$$

$$\leq \frac{\varepsilon}{c} \left| \omega_i |x_i| - \omega_j |x_j| \right| |G_c(x_i, \omega_i)||G_c(x_j, \omega_j)|.$$

$$\leq \varepsilon \eta |G_c(x_i, \omega_i)||G_c(x_j, \omega_j)|.$$

Sparse filter for horizontal setting



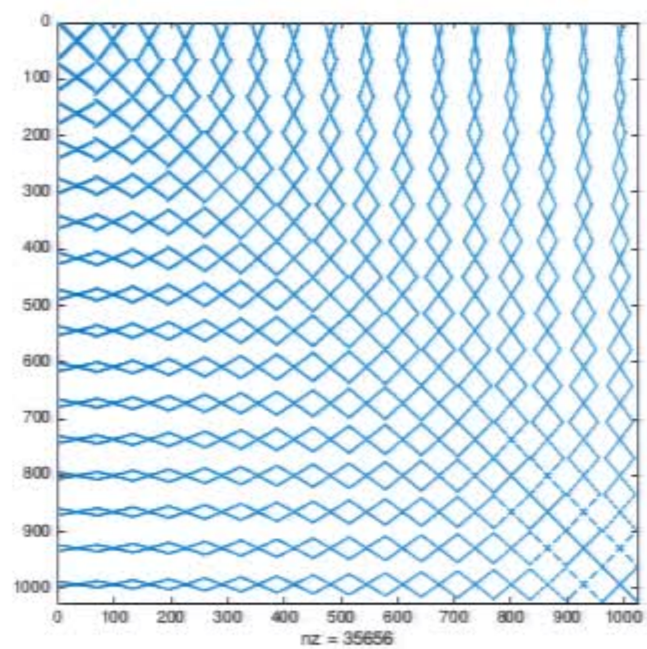


Figure: Sparse filter E for an horizontal array for 64 receivers and 16 frequencies for a speed error configuration

Error on receivers location

Filter for receivers location error

Suppose that the hypothetic linear forward operator F_h is given by

$$F_h s(x,\omega) = G_c(x+h(x),\omega)\widehat{s}\left(\frac{\omega}{c}\frac{x+h(x)}{|x+h(x)|}\right), \qquad (1)$$

where the error on the receivers location satisfies $\|h\|_{L^\infty(\Gamma)} \leq \varepsilon L$ and $\|h\|_{Lip(\Gamma)} \leq \varepsilon$. Fix a positive real number η and suppose that the sparse selector E satisfies

$$E_{i,j} = 1 \Rightarrow \max(\omega_i,\omega_j)|x_i - x_j| \leq \eta c \;\text{ and }\; L|\omega_i - \omega_j| \leq \eta c.$$

Then the true source s satisfies

$$|F_h[s](F_h[s])^* - dd^*|^2_{E,2} \leq C\varepsilon^2\eta^2$$

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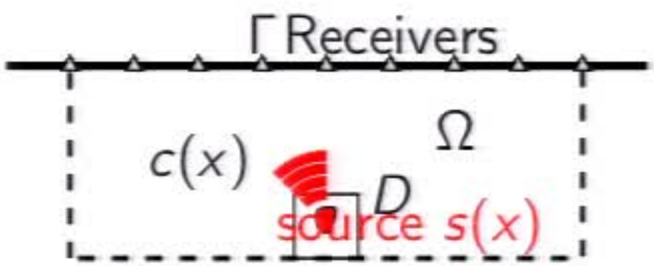

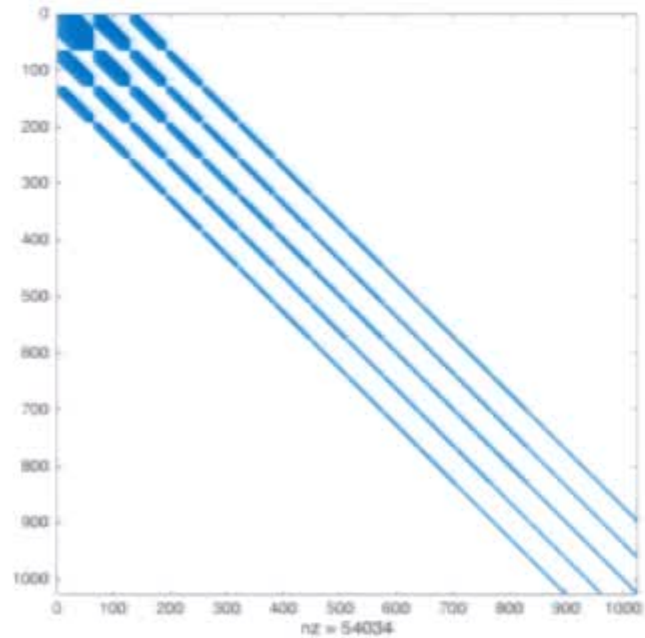

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Numerical results

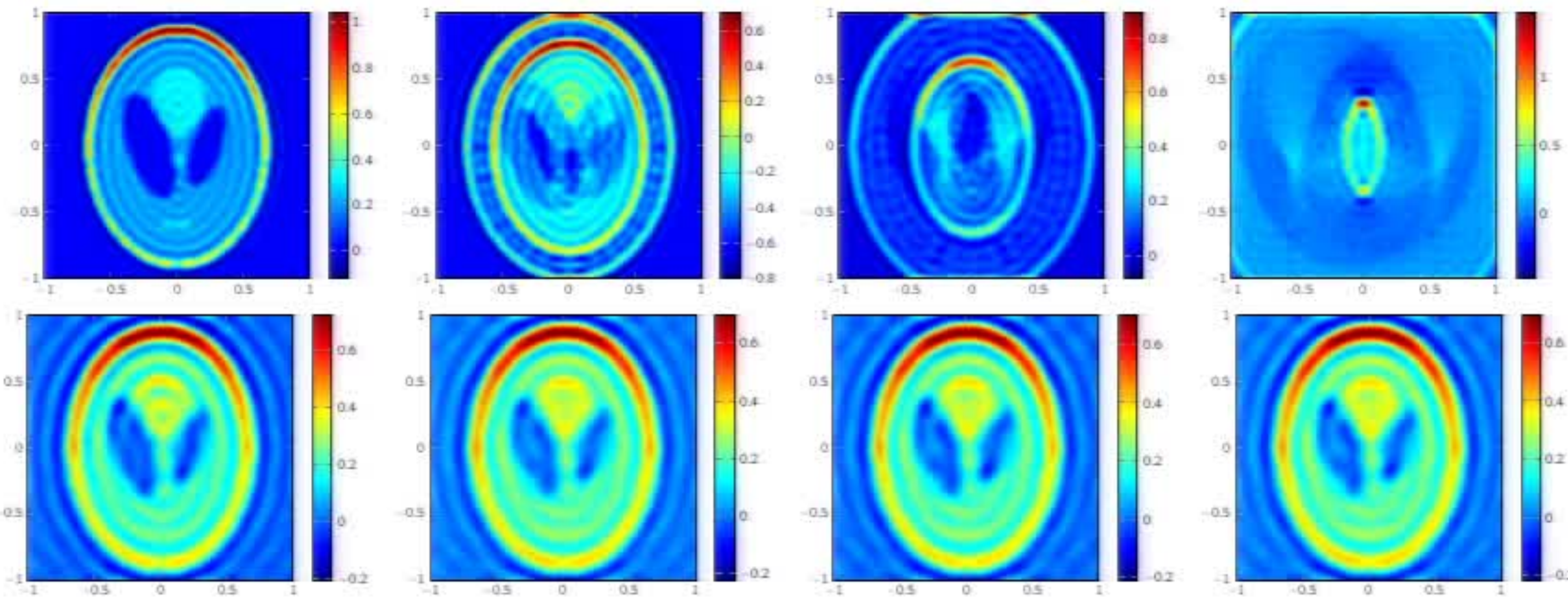

Figure: Least squares solutions and interferometric solution with constant error on the wave speed. From top to bottom, $\varepsilon = 0,\ 0.1\%,\ 0.2\%,\ 0.5\%$ and 1%. Tested with 128 receivers on $C(0, 100)$, 128 frequencies from 2 to 64 and with $\eta = 1$.

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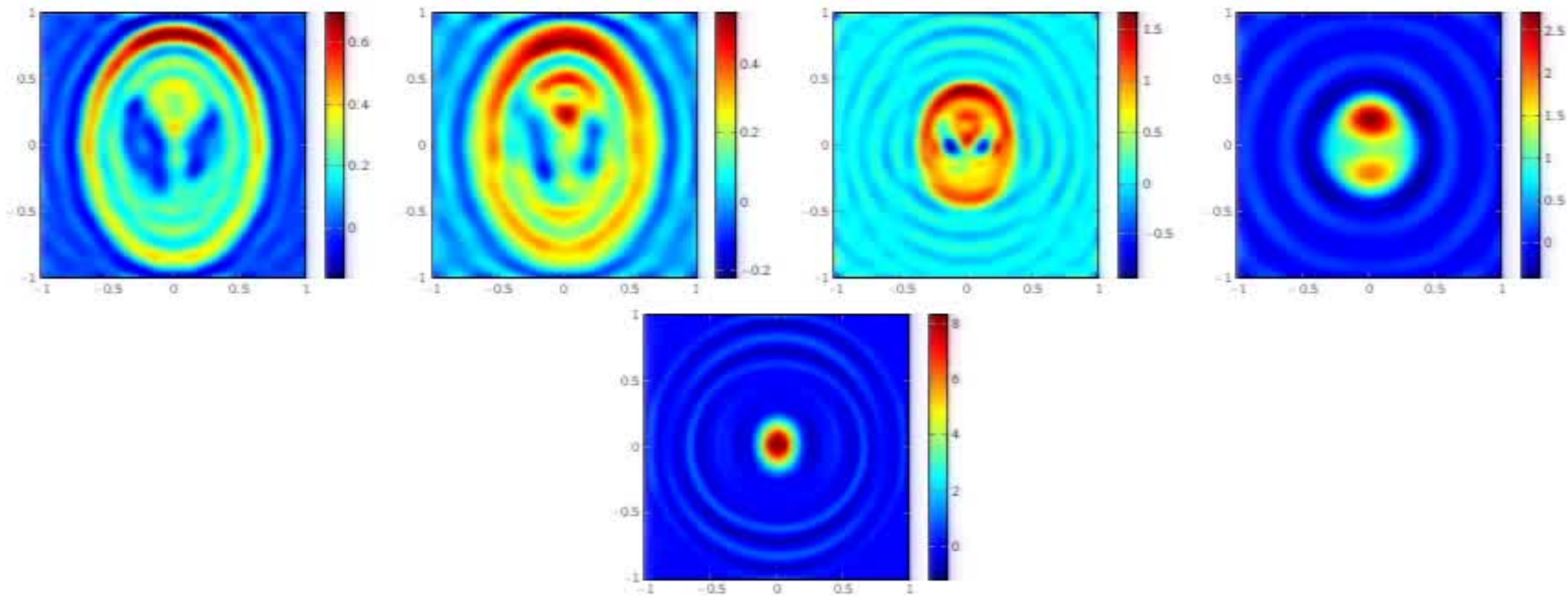

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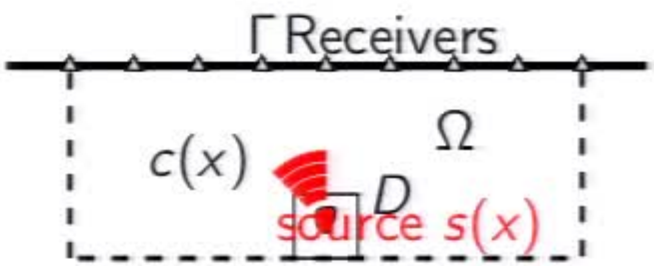

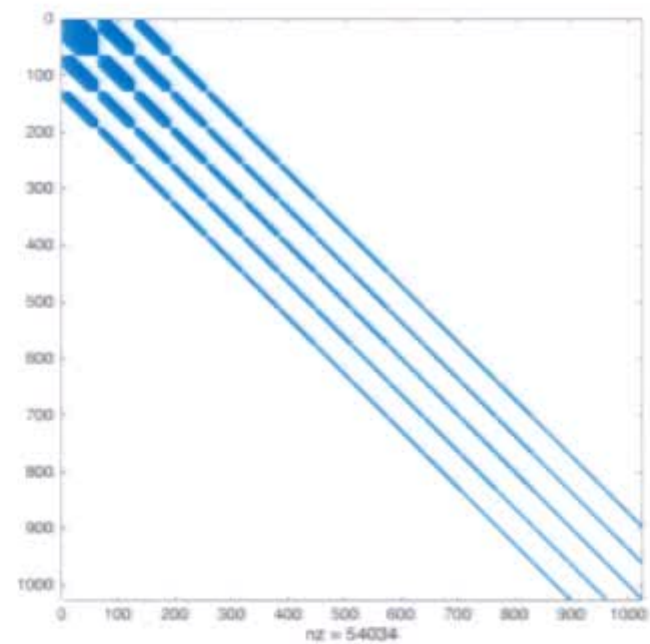

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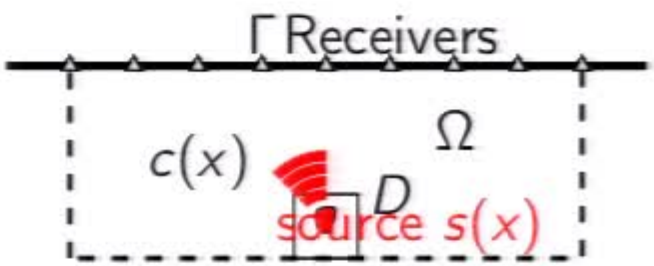

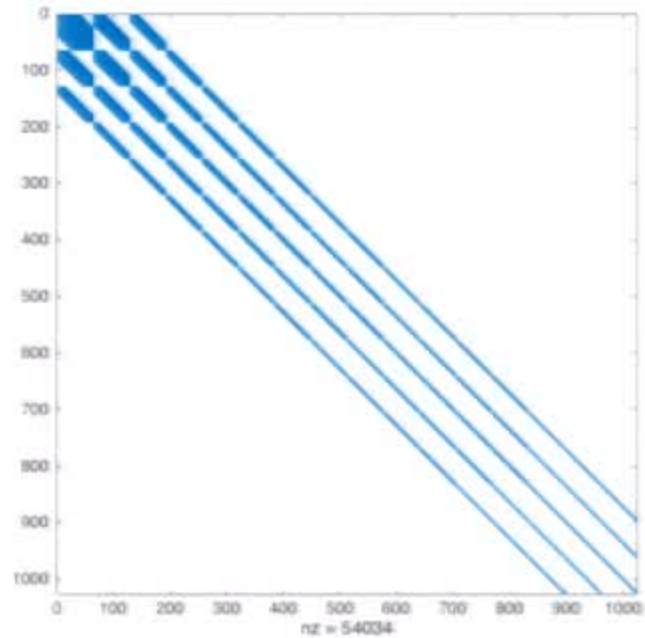

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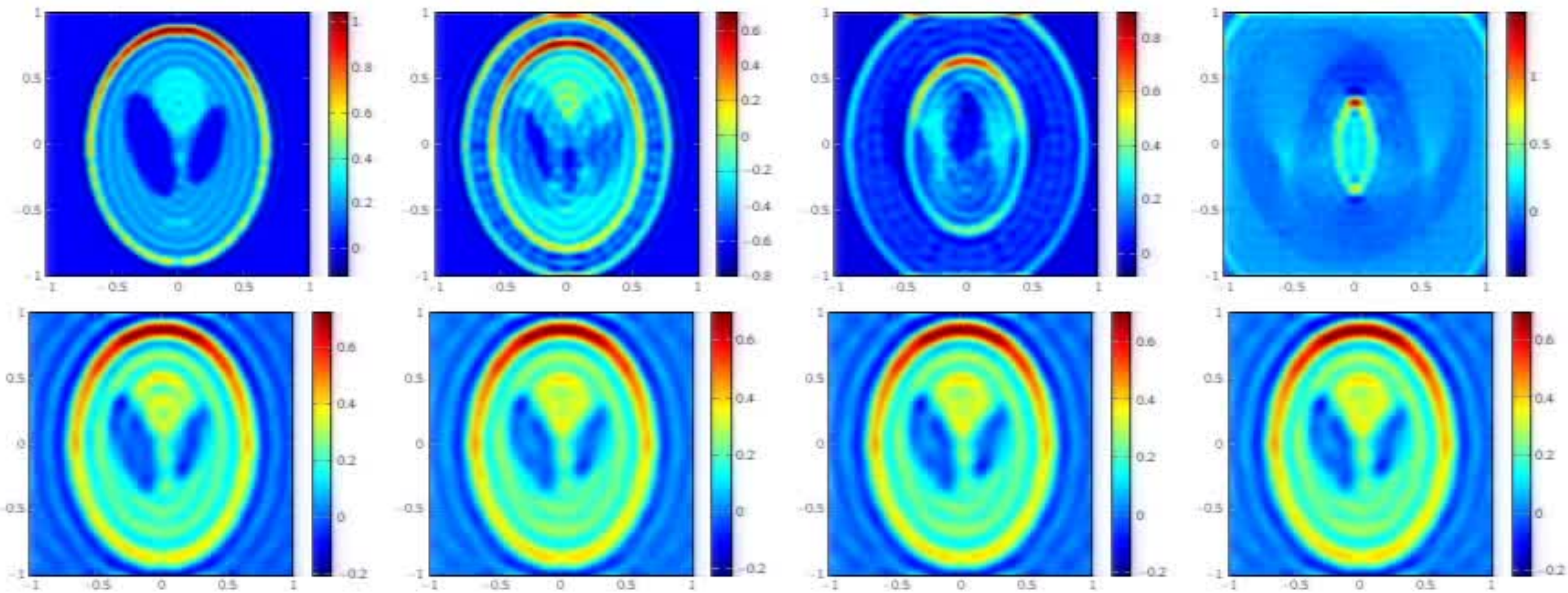

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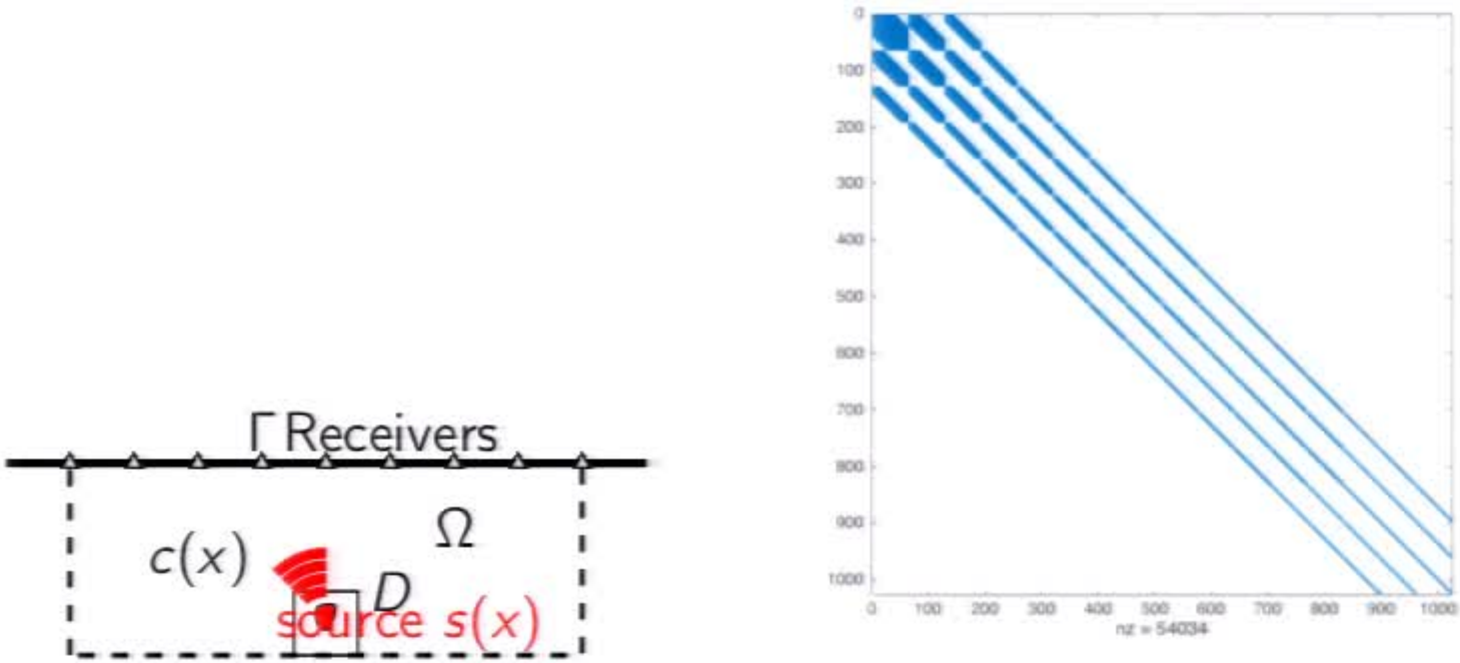

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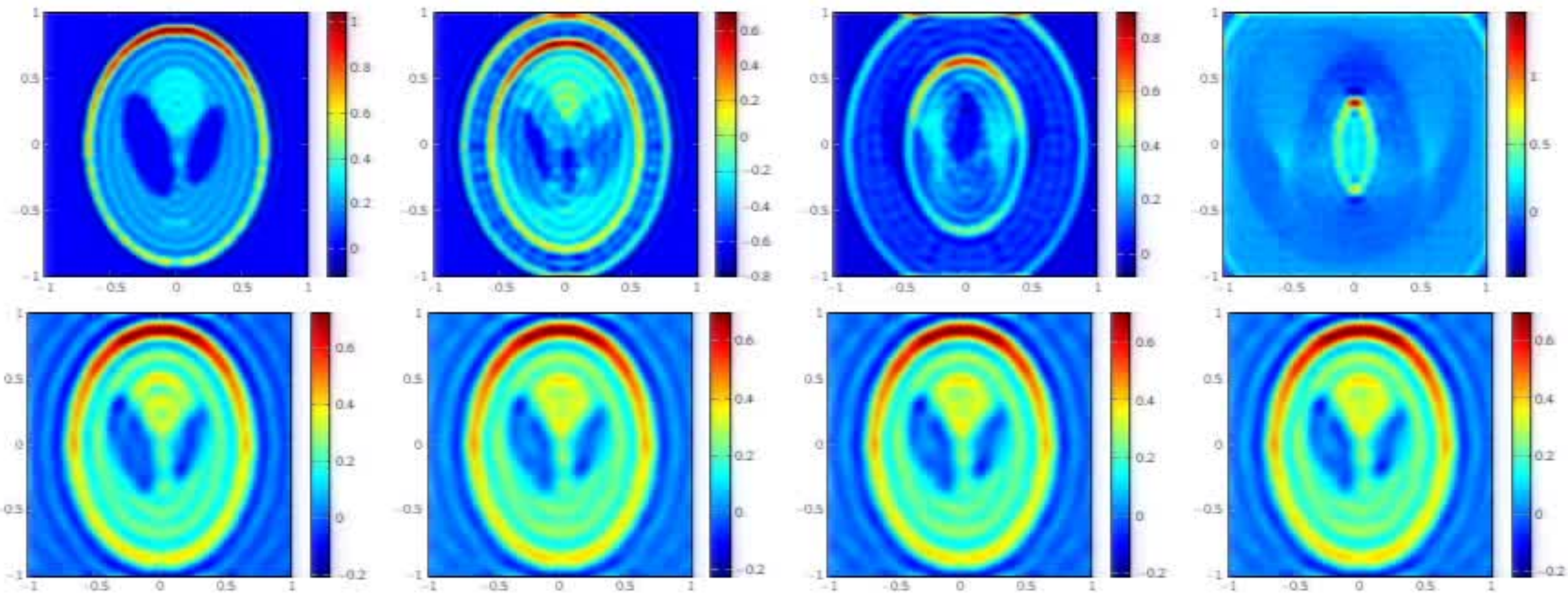

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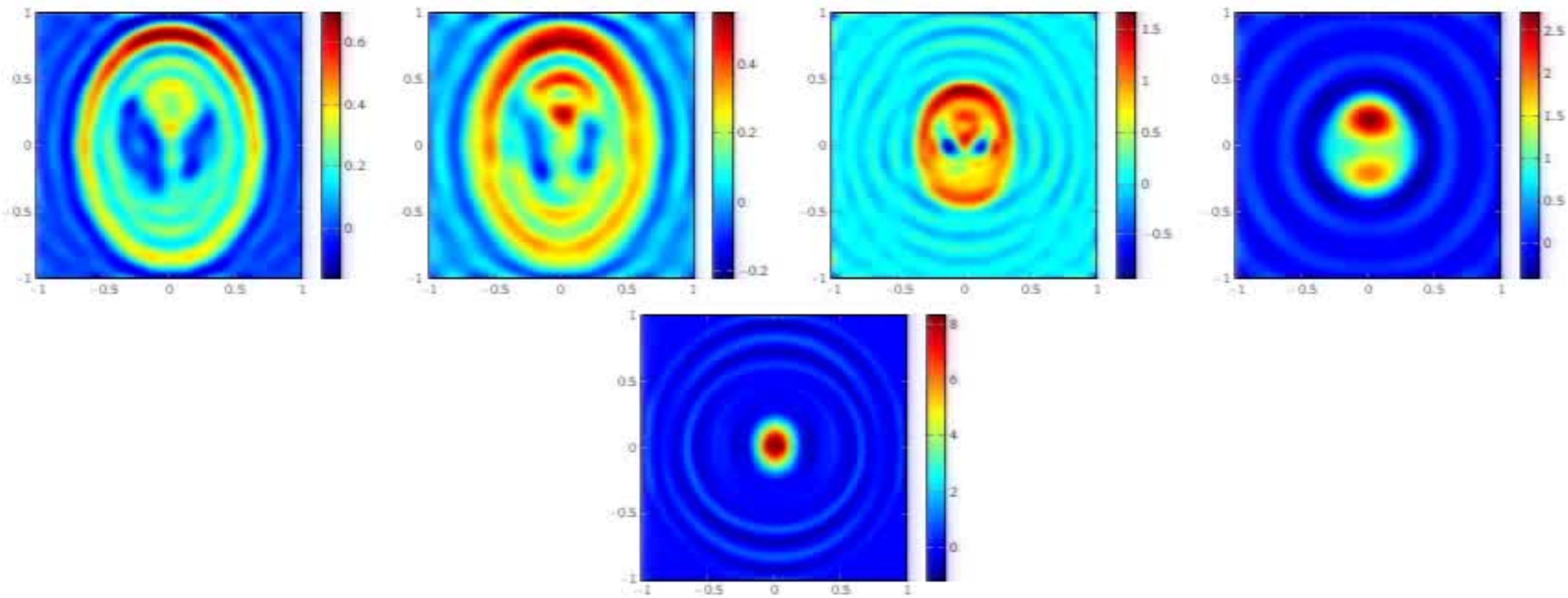

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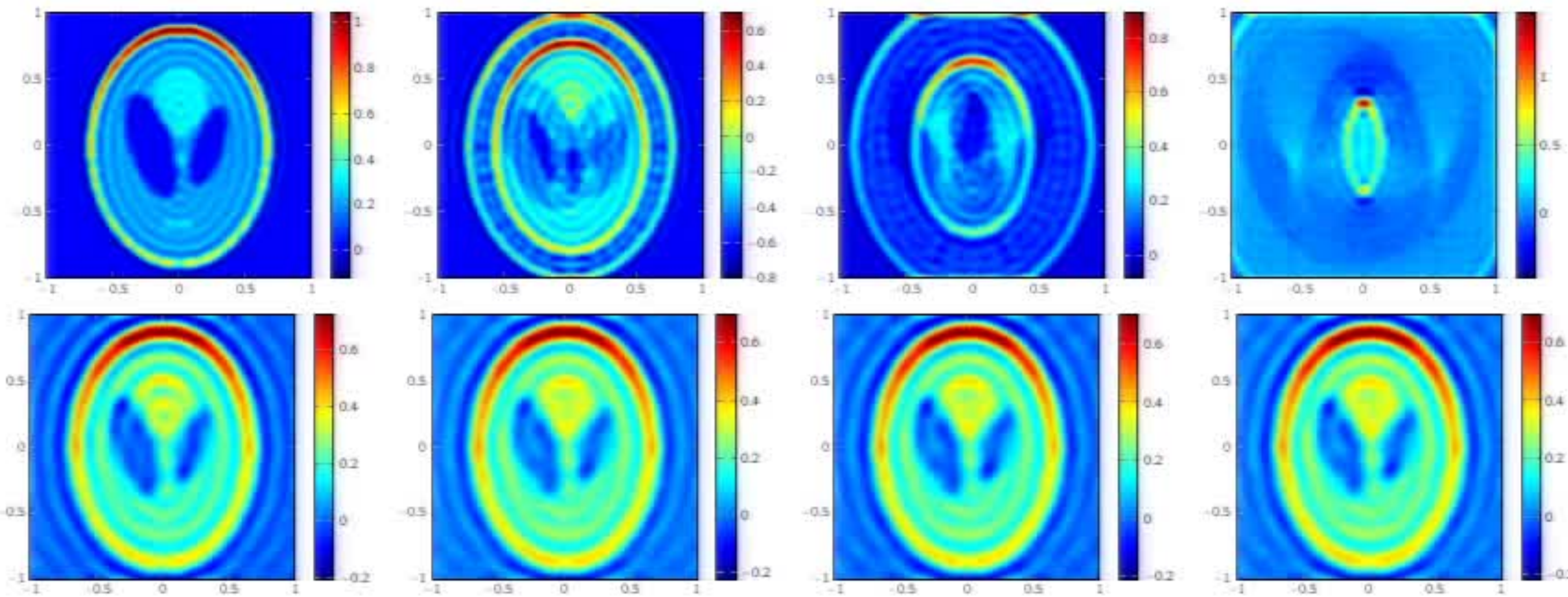

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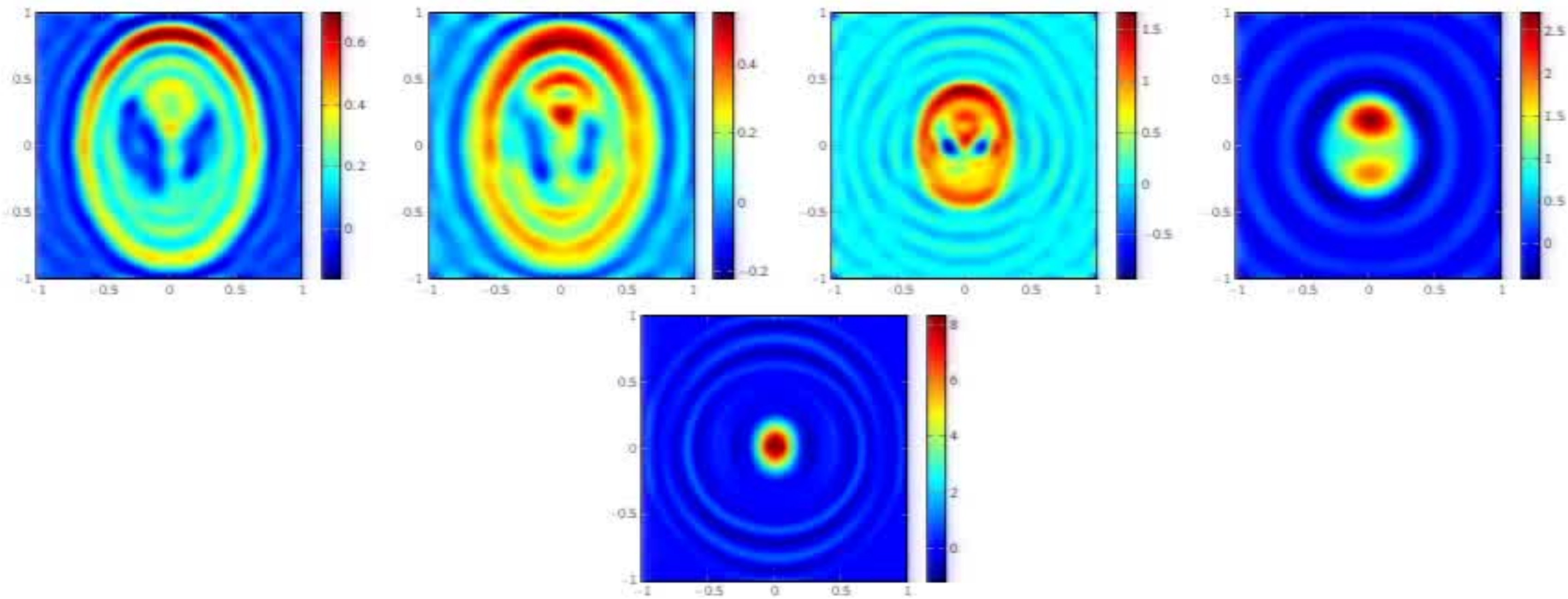

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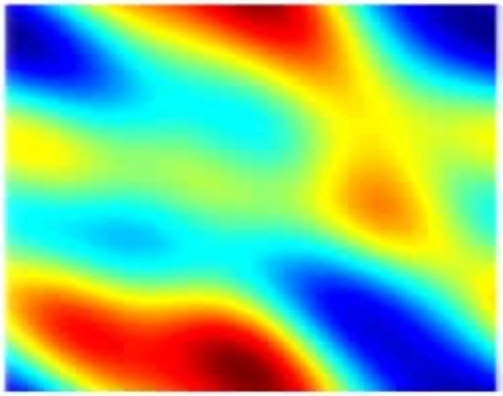
Non homogeneous medium

What happens in this medium ?



The phase differences cancellation principle says that we have to choose data pairs such that

$$\Delta_{i,j} = \omega_i T(x_i) - \omega_j T(x_j) - \left(\omega_i \frac{|x_i|}{c_0} - \omega_j \frac{|x_j|}{c_0}\right)$$

remains (bounded and small). And we have similar results than in the other cases.

Numerical Simulations

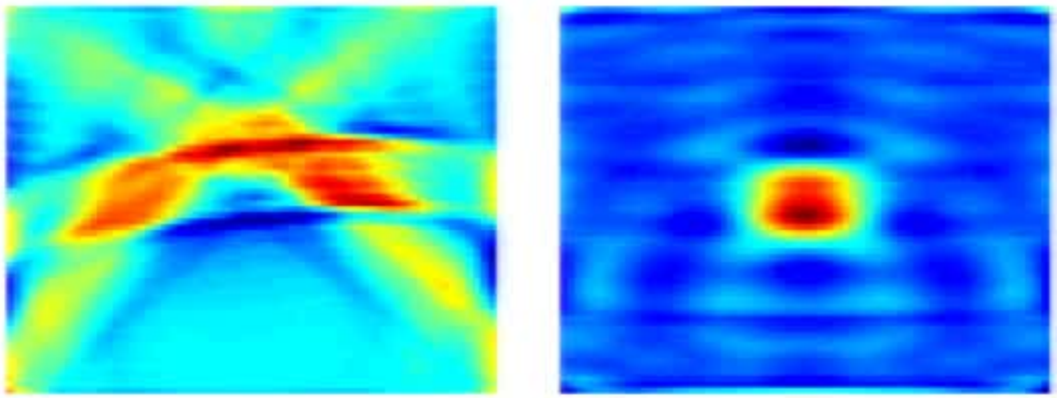


Figure: Reconstruction of a source square, (1) direct Least Squares, (2) interferometric method.

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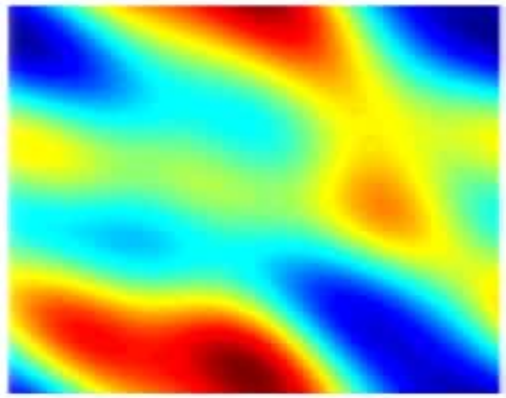

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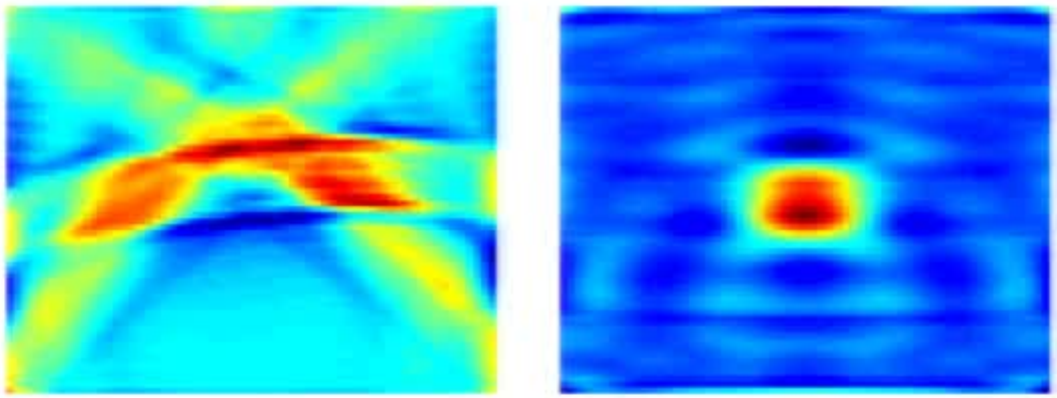

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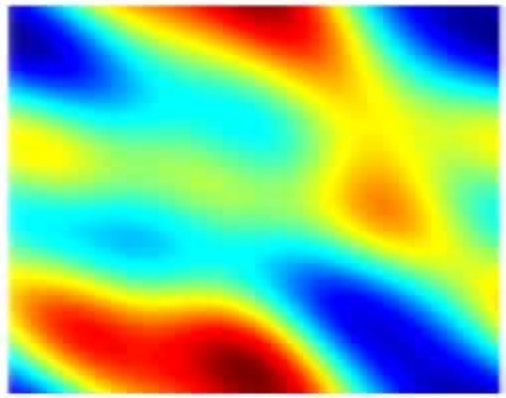

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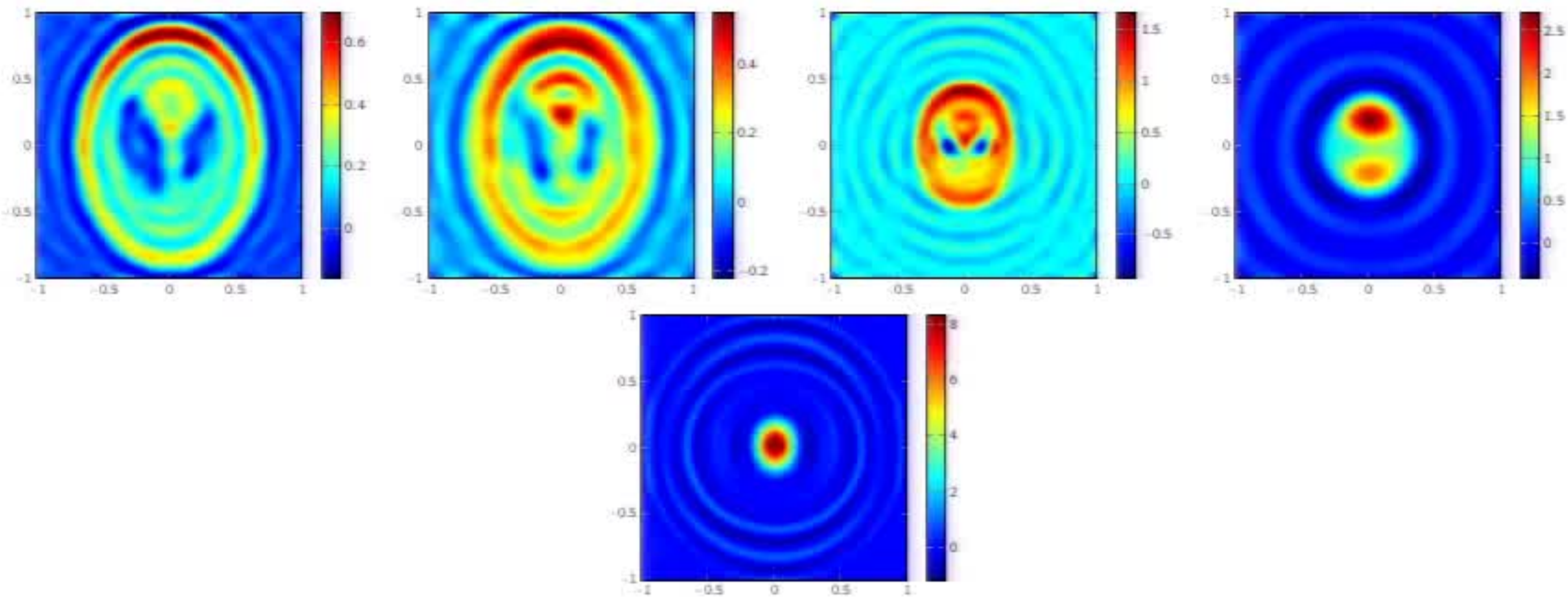

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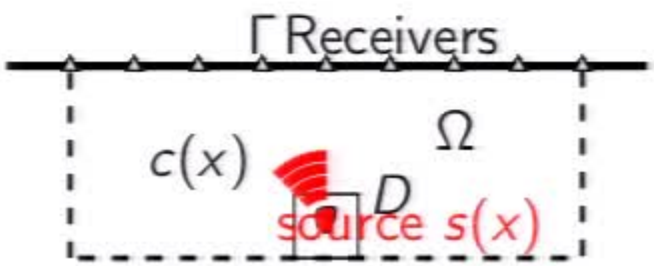

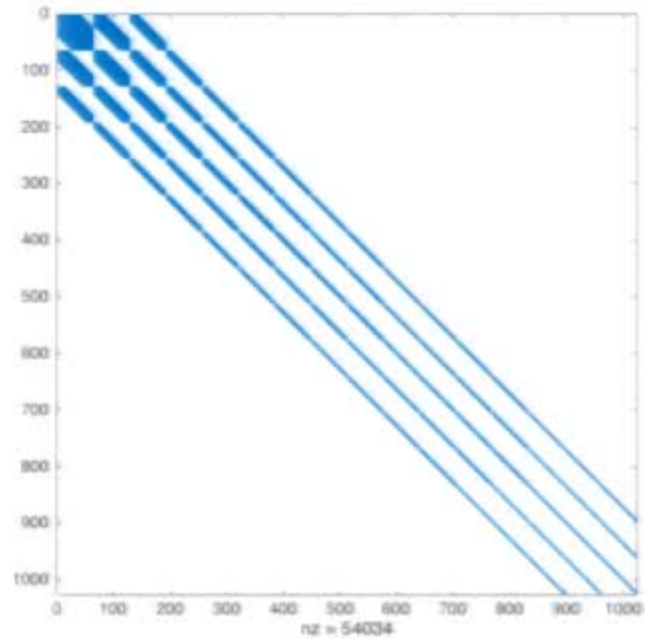

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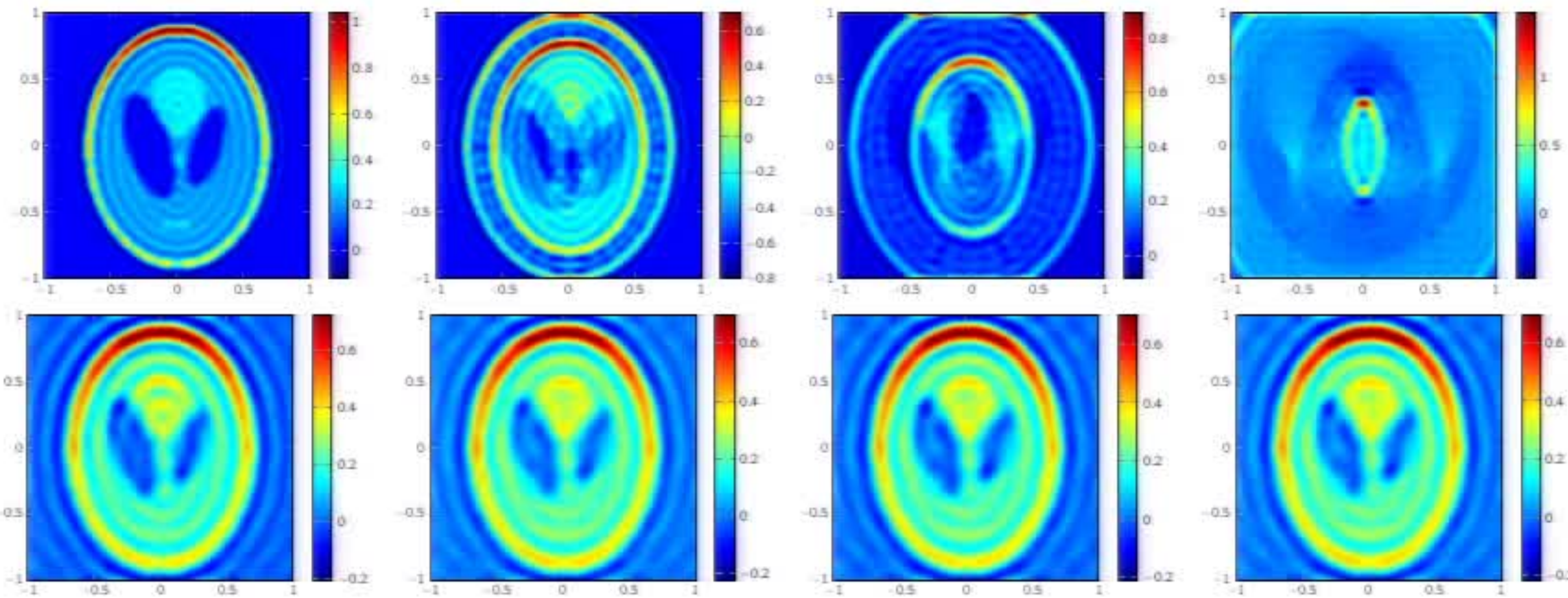

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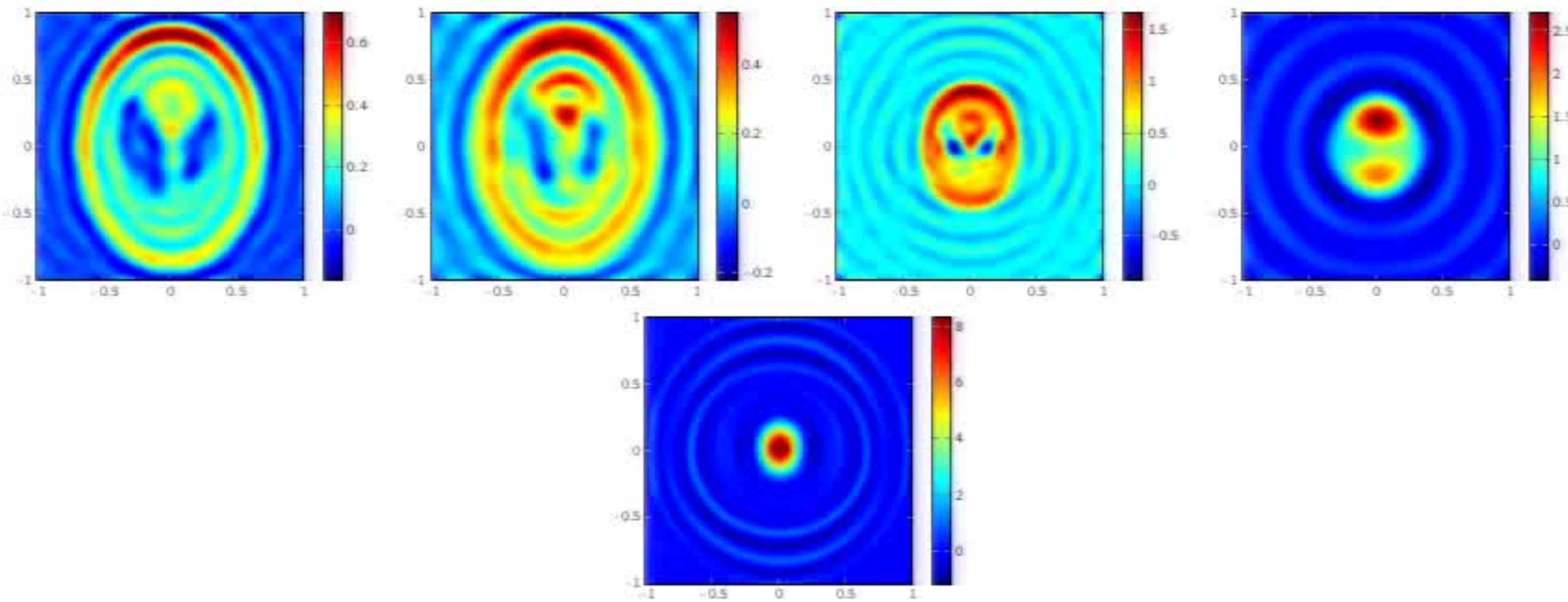

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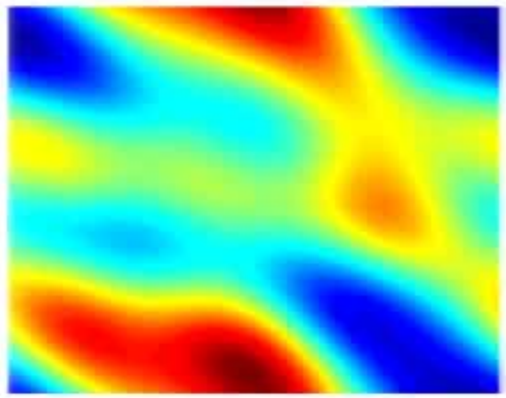

The phase differences cancellation principle says that we have to choose data pairs such that

$$\Delta_{i,j} = \omega_i T(x_i) - \omega_j T(x_j) - \left(\omega_i \frac{|x_i|}{c_0} - \omega_j \frac{|x_j|}{c_0}\right)$$

remains (bounded and small). And we have similar results than in the other cases.

Numerical Simulations

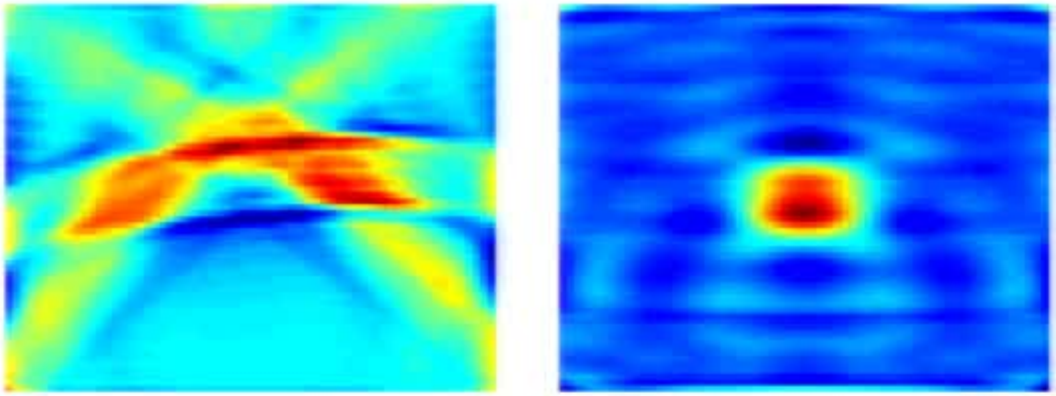

Figure: Reconstruction of a source square, (1) direct Least Squares, (2) interferometric method.

Relative error

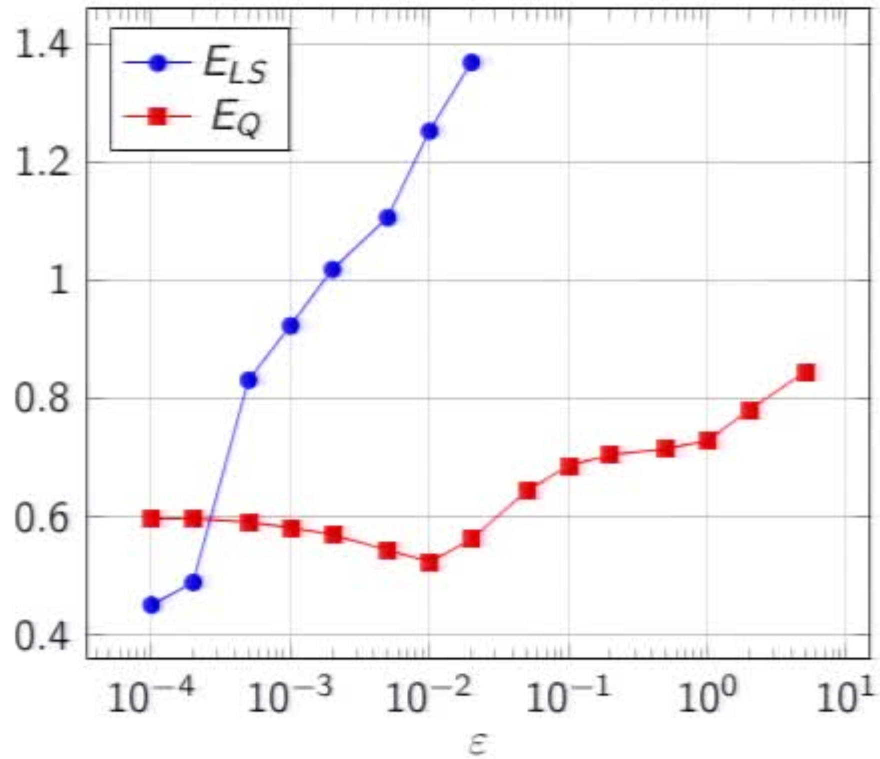

Figure: Relative L^2-error on the source reconstruction with respect to the slowness error ε.

Numerical Simulations

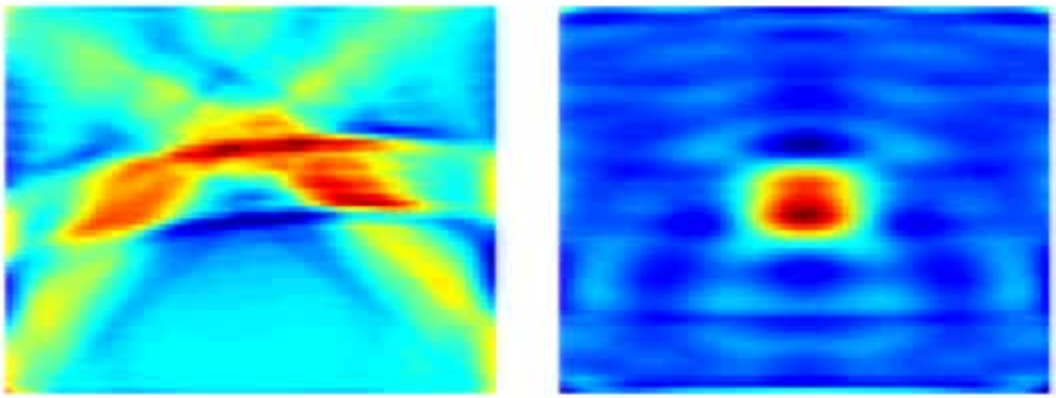

Figure: Reconstruction of a source square, (1) direct Least Squares, (2) interferometric method.

Relative error

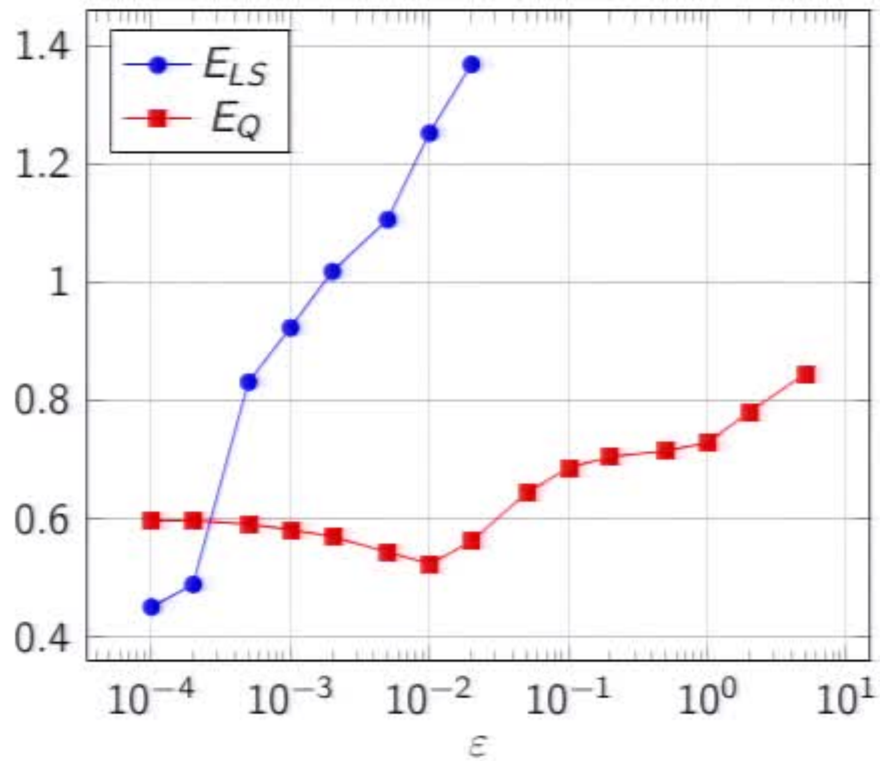

Figure: Relative L^2-error on the source reconstruction with respect to the slowness error ε.