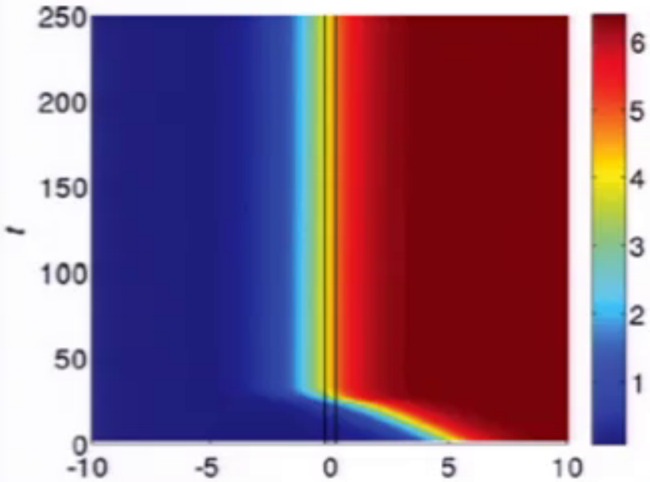
A geometric approach to stationary defect solutions in one space dimension

Arjen Doelman (Leiden U), Peter van Heijster (Queensland UT), Feng Xie (Donghua U)



Spatial heterogeneities

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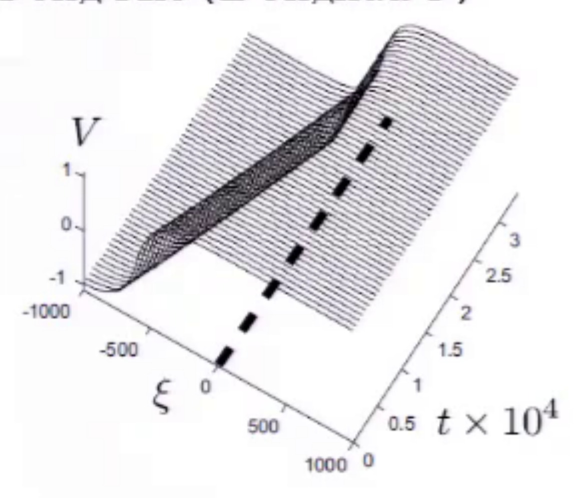
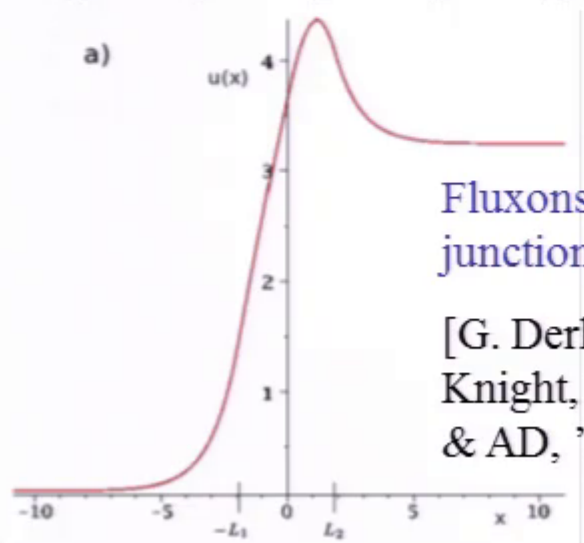


'Pinned' defect patterns

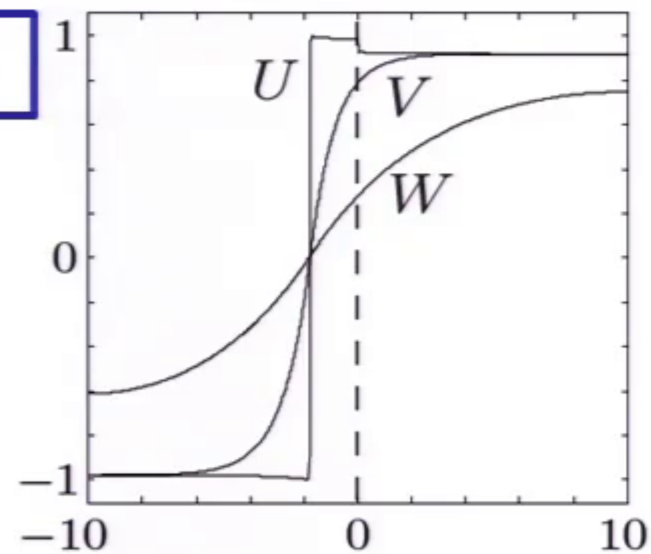


Fluxons in Josephson junctions

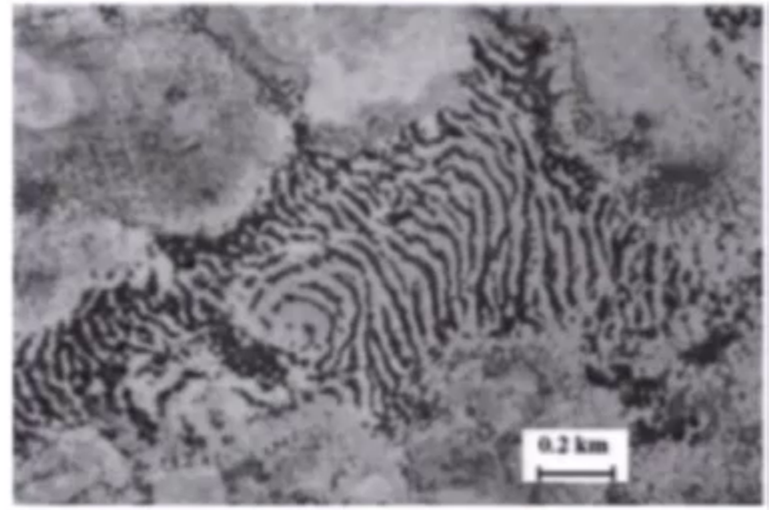
[G. Derks, C.J.K. Knight, H. Susanto & AD, '12,'13]

Fronts in a 3-component FitzHugh-Nagumo model

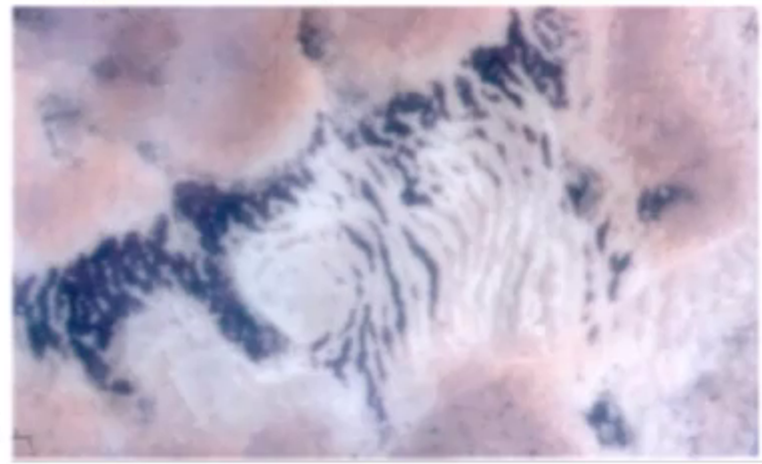
[P.J.A. van Heijster, T. Kaper, Y. Nishiura, K.-I. Ueda & AD, '11]



Structure of the talk



[Valentin '99] → Rademacher '10



- PDE with jump-type heterogeneity → non-autonomous, discontinuous ODE.
- Trivial, local & global defect solutions.
- Small effects: persistence of heteroclinic connections.
- In a $n = 3$-dimensional phase space: twist conditions or countably many defects.
- $n \geq 3$: general results on the existence of local defects.
- $n = 4$: kinks in the extended Fisher-KPP equation.
- $n = 6$: (multi-)fronts in a FitzHugh-Nagumo model.
- Discussion.

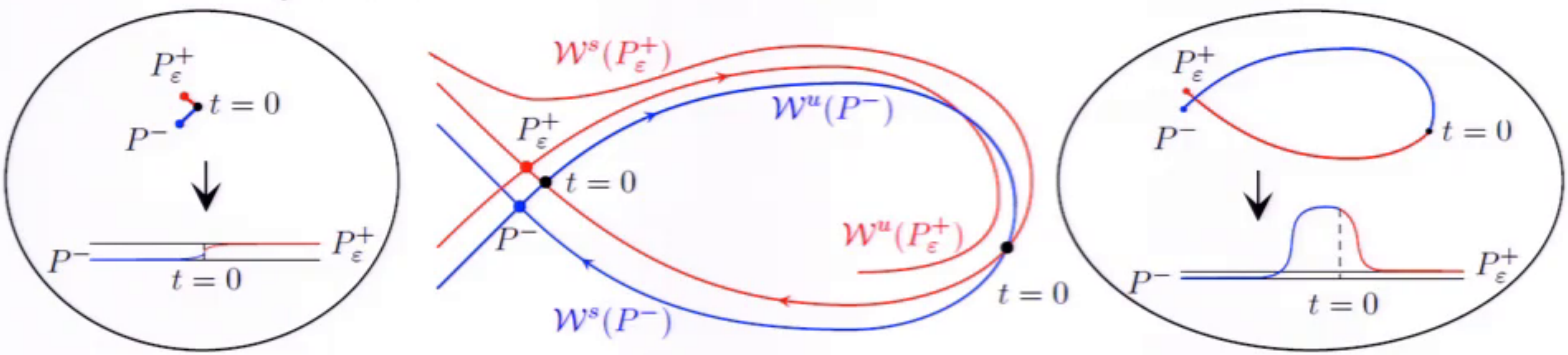
Some definitions

PDE model with small jump-type spatial heterogeneity, in one spatial dimension,

$$\frac{\partial U}{\partial t} = \mathcal{L}U + \mathcal{N}(U) + \begin{cases} 0, & x \leq 0, \\ \varepsilon \mathcal{G}(U), & x > 0, \end{cases} \quad U(x,t) : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^N$$

Existence of stationary patterns $\to$ spatial ODE $(x \leftrightarrow t)$,

$$\dot{u} = \begin{cases} f(u), & t \leq 0, \\ f(u) + \varepsilon g(u), & t > 0, \end{cases} \quad u(t) : \mathbb{R} \to \mathbb{R}^n.$$



N=1/n=2, Fisher-KPP-type

Trivial, local & global defect solutions

A defect solution = a heteroclinic $\Gamma_\varepsilon(t)$ from $P^- \to P^+_\varepsilon$ =

- a trivial defect solution if $P^- = P^+$ and

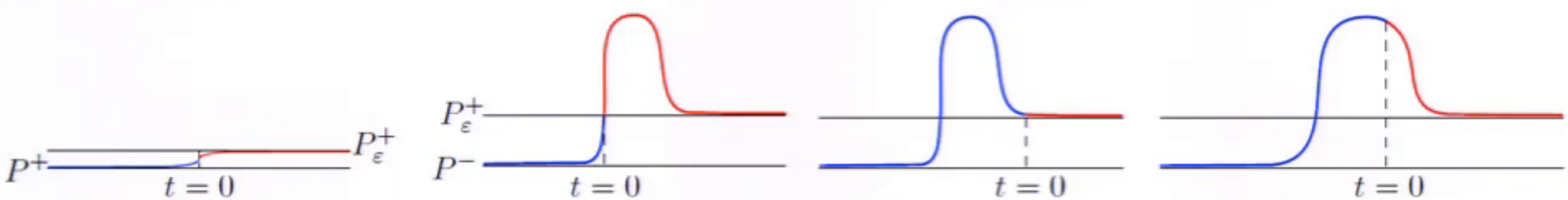
$$\lim_{\varepsilon\to 0} \|\Gamma_\varepsilon(t) - P^+_\varepsilon\|_\infty = 0;$$

- a local defect solution if either

$$\lim_{\varepsilon\to 0} \|\Gamma_\varepsilon(t) - P^+_\varepsilon\|_{\infty,\mathbb{R}^+} = 0, \qquad \text{or} \qquad \lim_{\varepsilon\to 0} \|\Gamma_\varepsilon(t) - P^-\|_{\infty,\mathbb{R}^-} = 0,$$

- a global defect solution if

$$\lim_{\varepsilon\to 0} \|\Gamma_\varepsilon(t) - P^+_\varepsilon\|_{\infty,\mathbb{R}^+} > 0, \qquad \text{and} \qquad \lim_{\varepsilon\to 0} \|\Gamma_\varepsilon(t) - P^-\|_{\infty,\mathbb{R}^-} > 0.$$

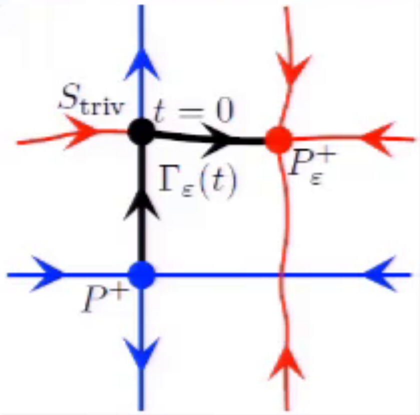


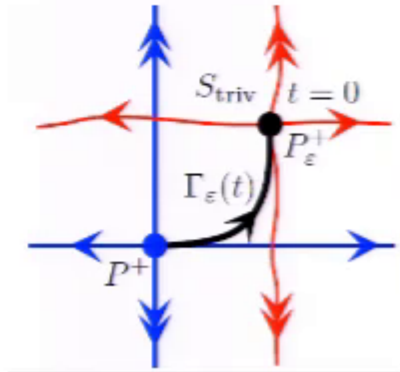
trivial defect — local near P^- — local near P^+_ε — global defect

Trivial and global defects

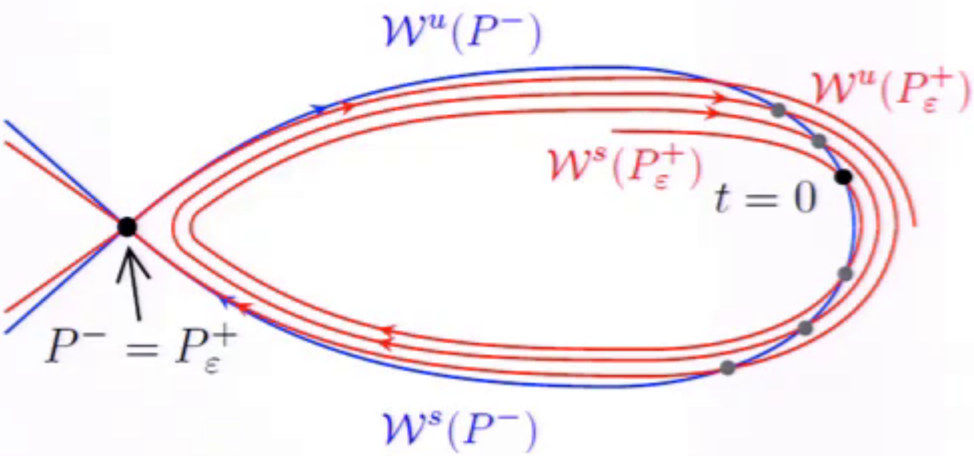
- Trivial defects: 'too easy'

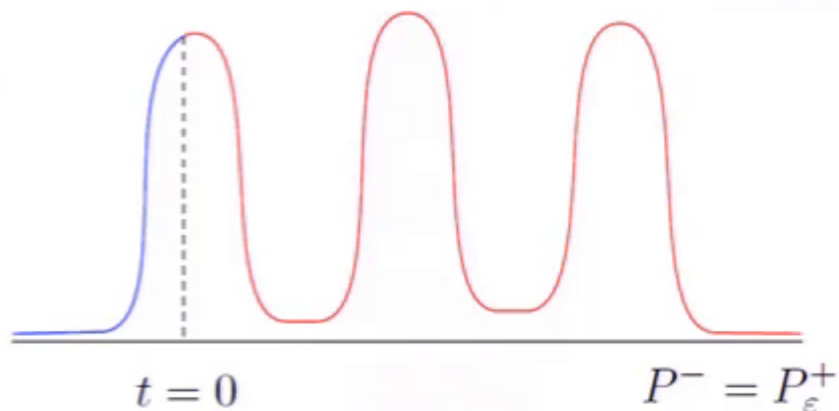
$\Rightarrow$ Unique existence result under generic conditions.





- Global defects, $n \geq 3$: 'too hard' (in general)





- Singularly perturbed systems ([vHeijster et al.],$n = 6$)

Basic assumptions

- There exists an unperturbed heteroclinic connection $\Gamma(t) \in \mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)$.

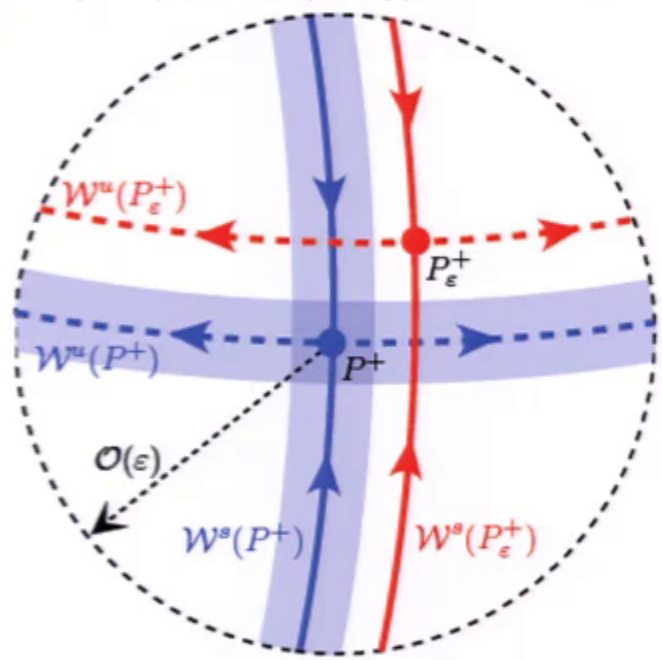
Note: Γ corresponds to a (stationary) 'localized structure' in the underlying homogeneous PDE.

- Γ is minimally non-transversal:

– if $\dim(\mathcal{W}^u(P^-)) + \dim(\mathcal{W}^s(P^+)) \leq n$, then $\dim(\mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)) = 1$

– if $\dim(\mathcal{W}^u(P^-)) + \dim(\mathcal{W}^s(P^+)) = m > n$, then $\dim(\mathcal{W}^u(P^-) \cap \mathcal{W}^s(P^+)) = m - n$.

- $P^{\pm}$ are hyperbolic fixed points with only simple eigenvalues.
- $g(u)$ is a generic perturbation.



Dimensions

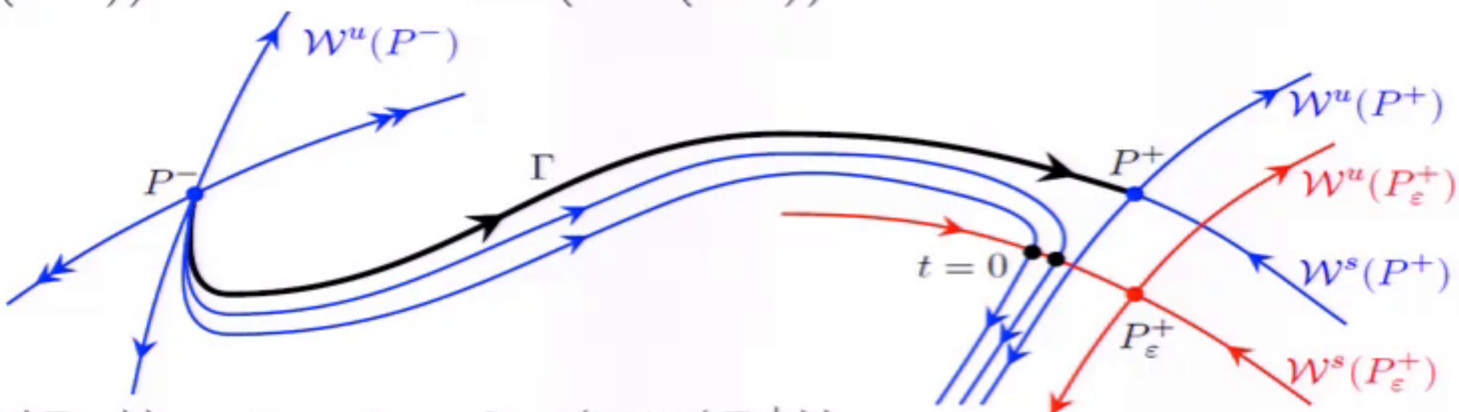
- $\dim(\mathcal{W}^u(P^-)) < \dim(\mathcal{W}^u(P^+)) = n - \dim(\mathcal{W}^s(P^+))$:

(generically) no local defects.

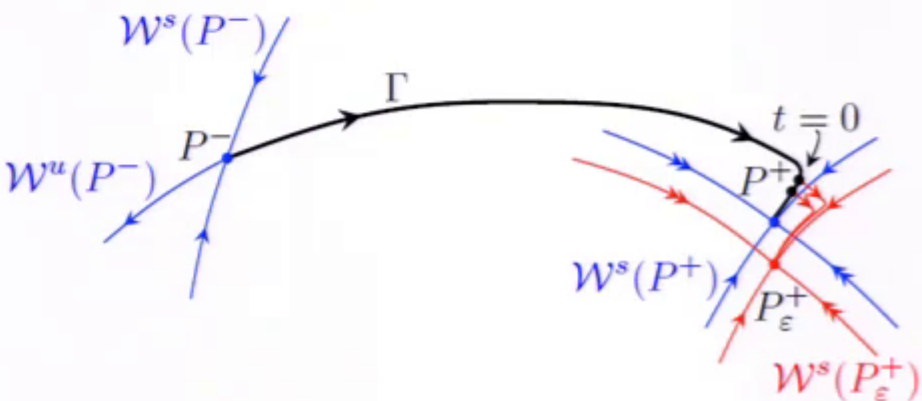
- $\dim(\mathcal{W}^u(P^-)) > \dim(\mathcal{W}^u(P^+)) = n - \dim(\mathcal{W}^s(P^+))$:

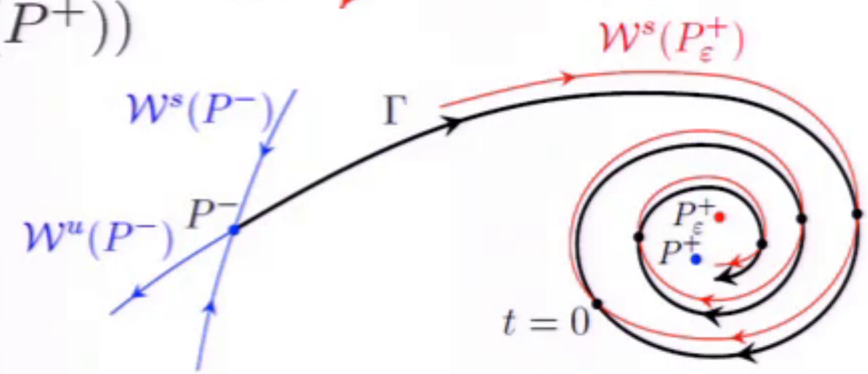
continuous families of local defects.

- $\dim(\mathcal{W}^u(P^-)) = 2 > 1 = \dim(\mathcal{W}^u(P^+))$



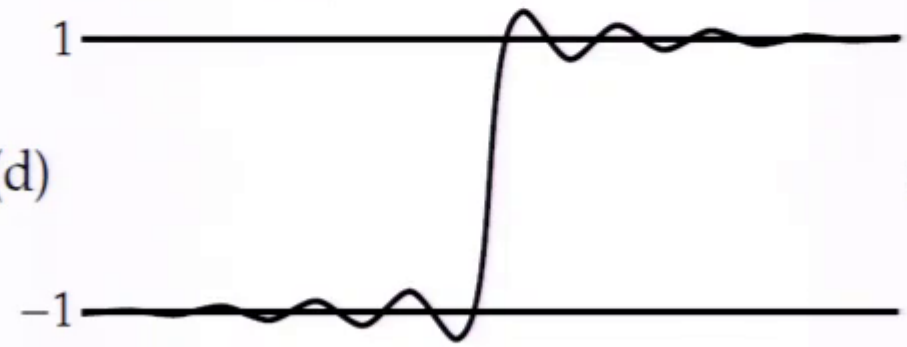
- $\dim(\mathcal{W}^u(P^-)) = 1 > 0 = \dim(\mathcal{W}^u(P^+))$

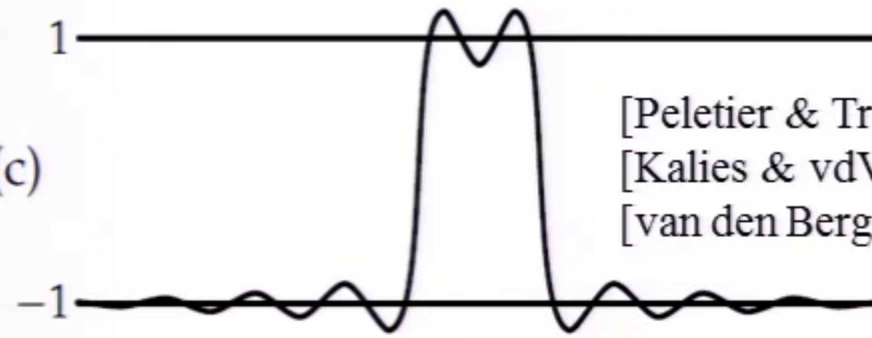




Most interesting/relevant: $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+))$.

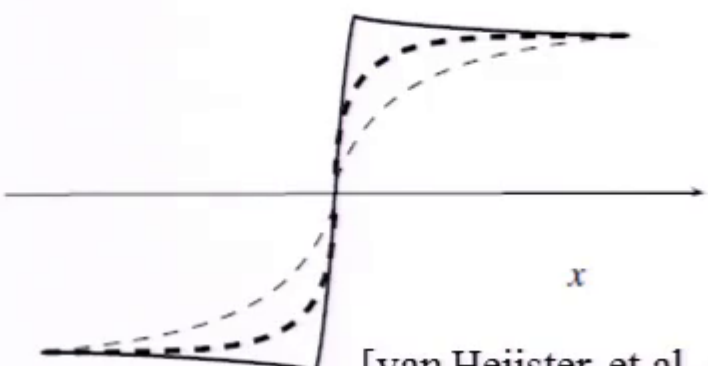
- $P^- = P^+$: Homoclinic orbits $\leftrightarrow$ pulses.
- Fronts/pulses in extended Fisher-KPP ($n = 4$).



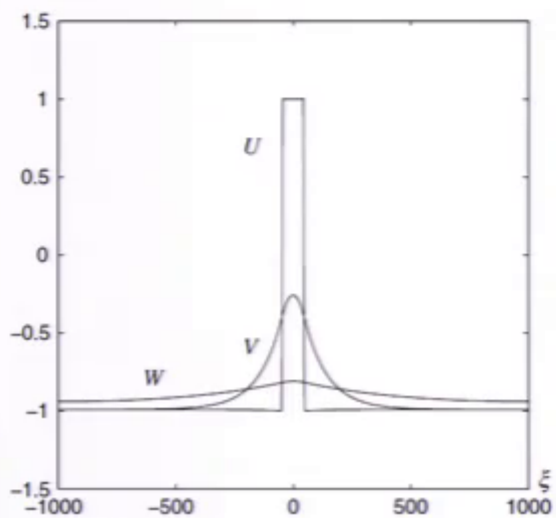


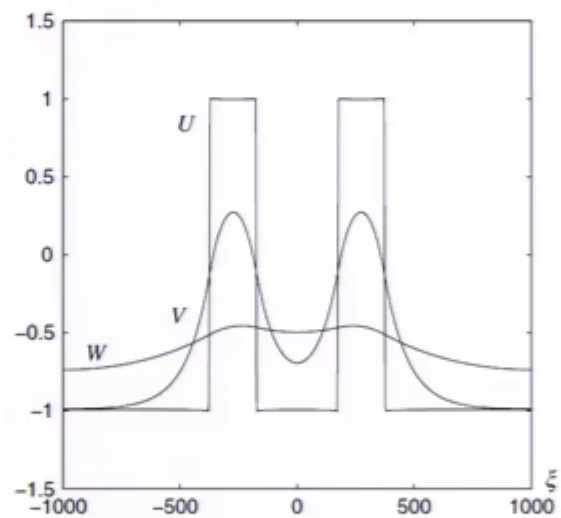
[Peletier & Troy, '01],
[Kalies & vdVorst, '96],
[van den Berg, '00]

- Fronts/pulses in FitzHugh-Nagumo model ($n = 6$).



[van Heijster et al., '09]
[Chirilus-Bruckner et al.,'13]

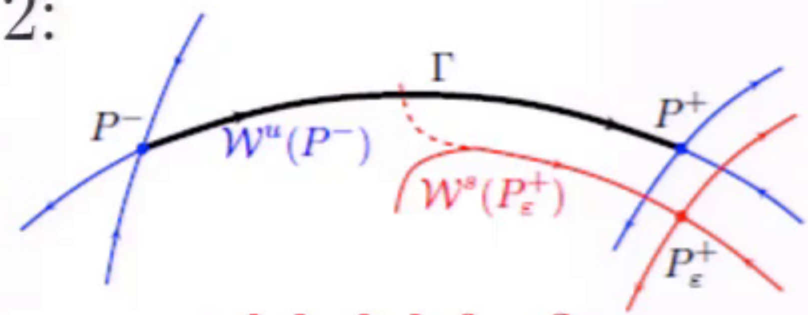




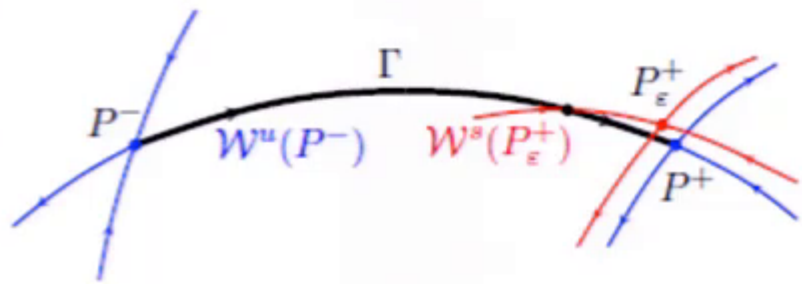
$\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 1$:

in general no local defects.

- $n = 2$:

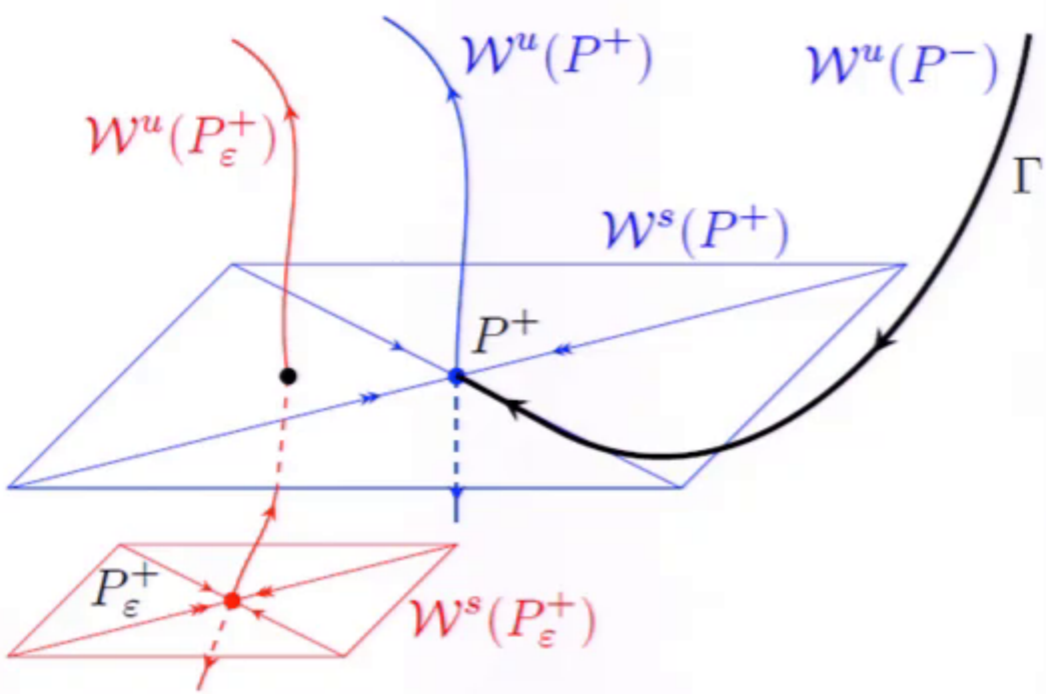


global defect?



a non-generic perturbation

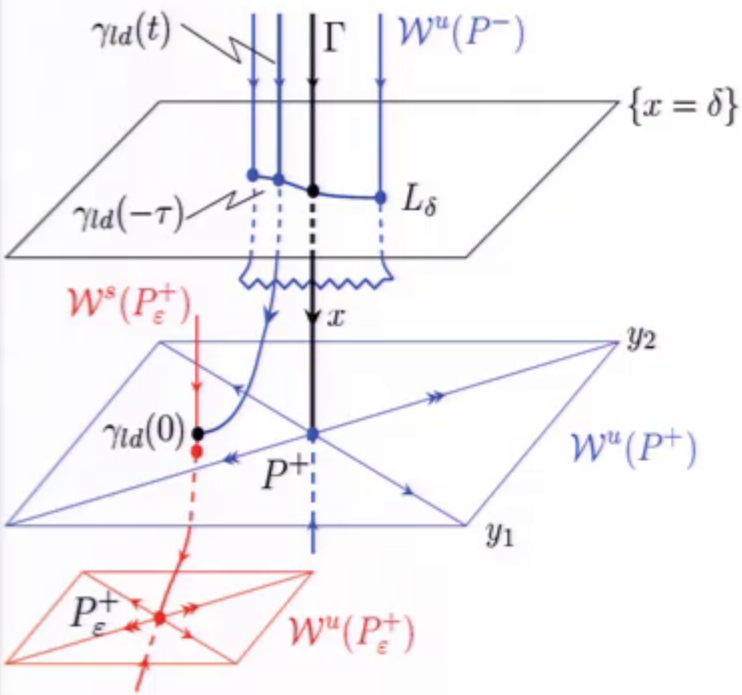
- $n = 3$:

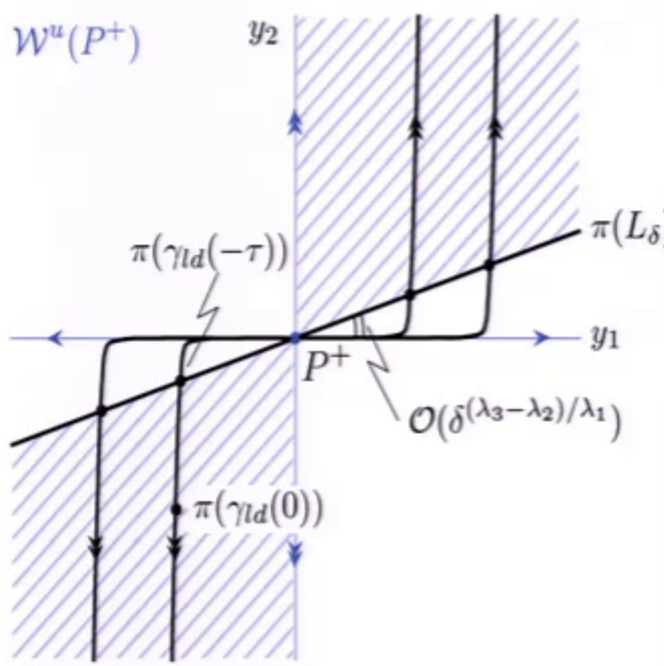


n=3, real eigenvalues: a twist condition

Assumptions on $P^{\pm}$, Γ & $g(u)$ + ε small enough $\Rightarrow$

Theorem. *If* $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 2$, P^+ *has two unstable real eigenvalues and some non-tangency conditions hold, then there exists a unique local defect solution near* P^+ *if and only if a twist condition holds.*



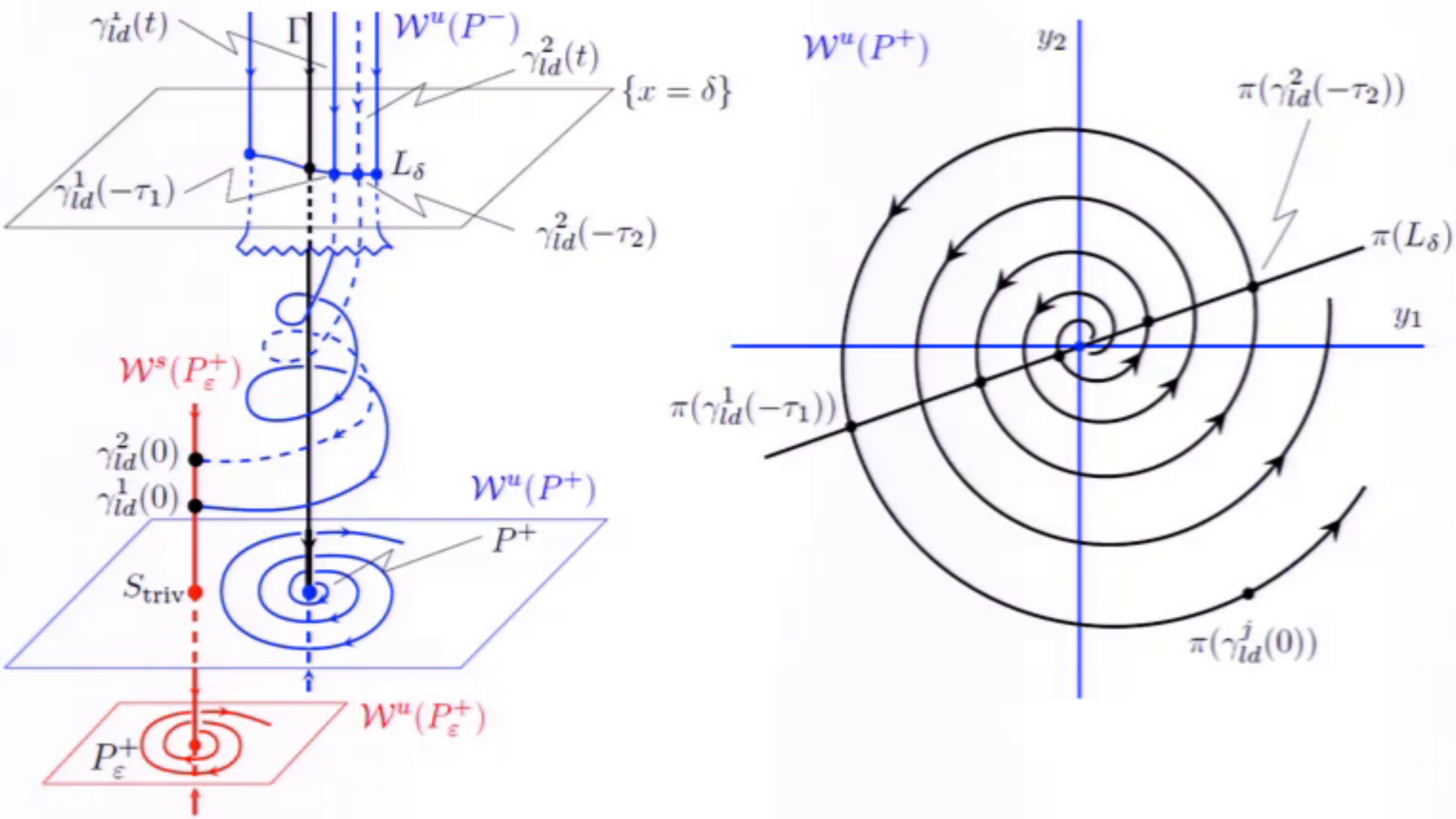


The cone-type twist condition encodes the global twist of $\mathcal{W}^u(P^-)$ around Γ as it travels from P^- to P^+.

n=3, complex eigenvalues: countably many defects

Assumptions on $P^{\pm}$, Γ & $g(u)$ + ε small enough $\Rightarrow$

Theorem. *If* $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = 2$ *and* P^+ *has a complex conjugate pair of unstable eigenvalues, then there exists countably many local defect solutions near* P^+.



Proofs.
Normal forms
+
flows dominated by linear approximations.

$n>3$: generalizations

Definition. The leading (unstable) eigenvalue(s) = the (unstable) eigenvalue(s) closest to the Im-axis.

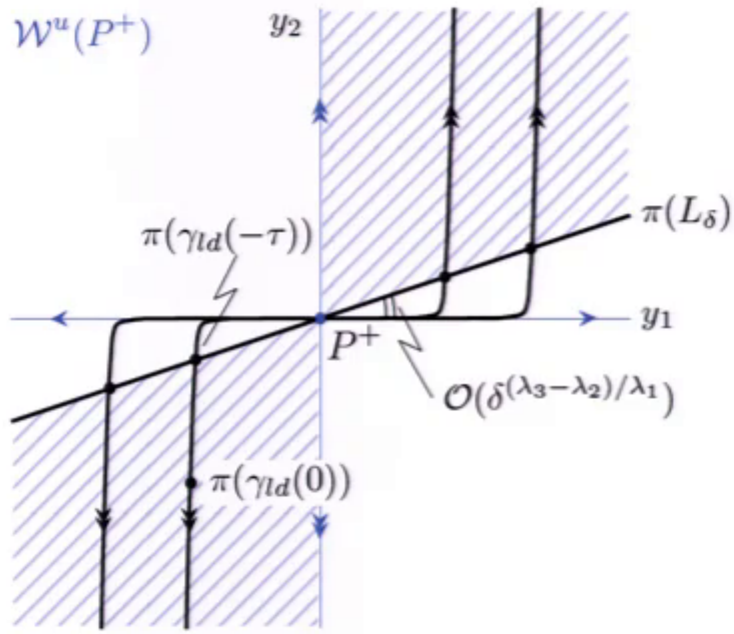
Assumptions on $P^{\pm}$, Γ & $g(u)$, ε small enough & $\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) \geq 2 \Rightarrow$

Theorem. *If P^+ has a real leading unstable eigenvalues, some non-tangency conditions and a twist condition hold, then there exists at least one local defect solution.*

Theorem. *If P^+ has a complex conjugate pair of leading unstable eigenvalues and some non-tangency conditions hold, then there are countably many local defect solutions.*

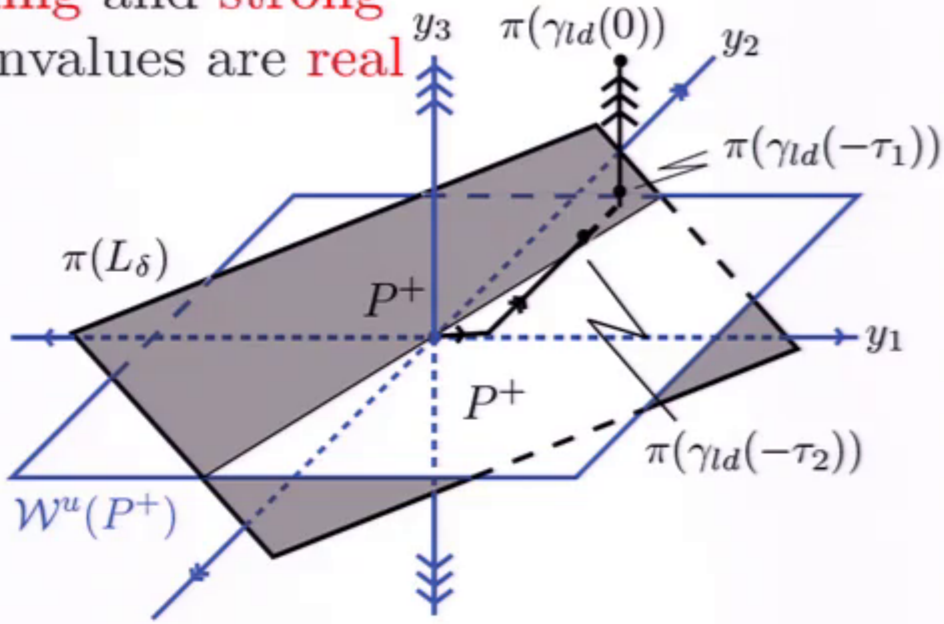
$n>3$, the impact of the strong eigenvalues

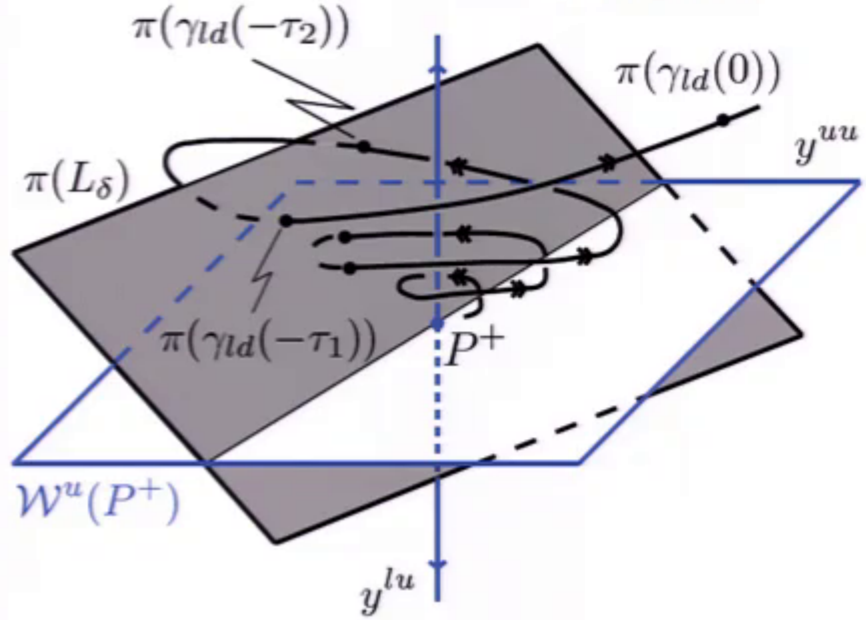
$\dim(\mathcal{W}^u(P^+)) = 2$:
leading and strong eigenvalues are real



$\dim(\mathcal{W}^u(P^+)) = 3$:
leading eigenvalue is real,
strong eigenvalues are complex

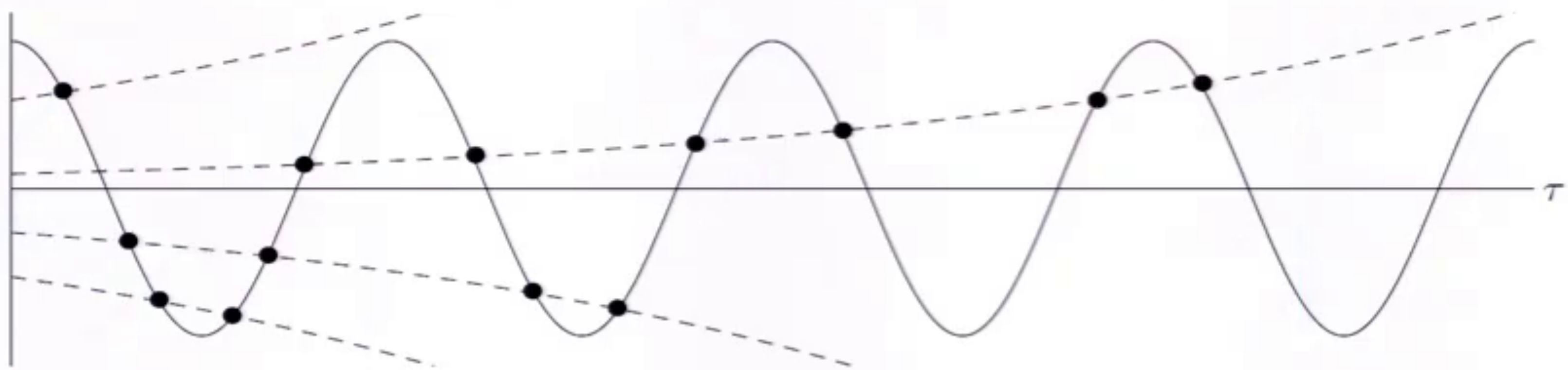
$\dim(\mathcal{W}^u(P^+)) = 3$:
leading and strong eigenvalues are real





Assumptions on $P^{\pm}$, Γ & $g(u)$, ε small enough &
$\dim(\mathcal{W}^u(P^-)) = \dim(\mathcal{W}^u(P^+)) = k \geq 2 \Rightarrow$

Theorem. P^+ has a real leading unstable eigenvalue.
A. *If P^+ has k real unstable eigenvalues (...), then there are open regions in 'parameter space' in which there are exactly $j = 0, 1, \ldots$ up to $k-1$ local defect solutions.*
B. *If P^+ has a complex conjugate pair of unstable eigenvalues (...), then there are open regions in 'parameter space' in which there are $j = 0, 1, \ldots$ up to K, where K is arbitrarily large, but finite, local defect solutions.*

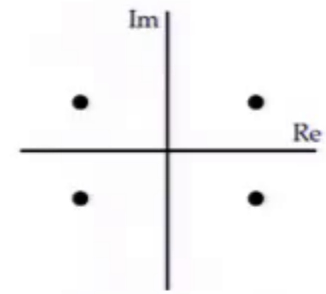


Example: the extended Fisher-KPP equation

Existence problem heterogeneous extended Fisher-KPP,

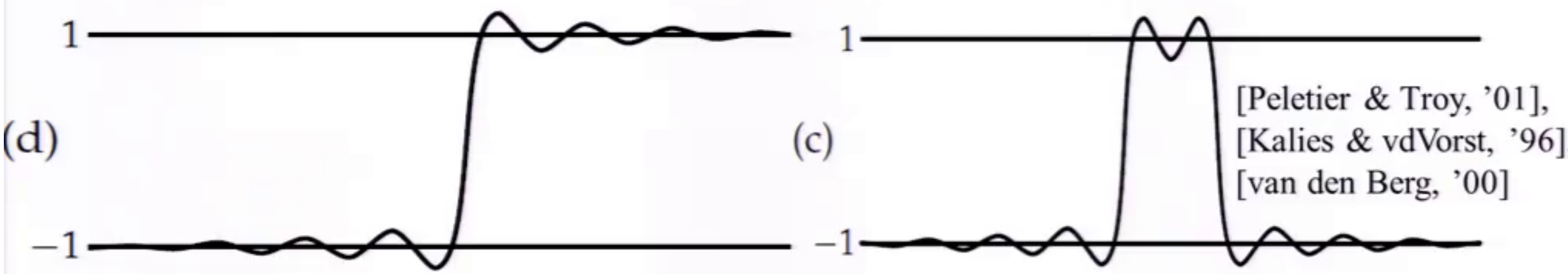
$$\frac{d^4u}{d\xi^4} + \beta\frac{d^2u}{d\xi^2} - u + u^3 = \begin{cases} 0, & \xi < 0, \\ \varepsilon g(u, u_\xi, u_{\xi\xi}, u_{\xi\xi\xi}), & \xi > 0, \end{cases}$$

$\beta \in (-\sqrt{8}, 0), \varepsilon = 0$: critical points $P^{\pm} = (\pm 1, 0, 0, 0)$ are saddle-foci.



Theorem. [Peletier & Troy, '95]
$\beta \in (-\sqrt{8}, 0)$: *existence 'kinks' in homogeneous limit.*

Theorem. $\beta \in (-\sqrt{8}, 0)$, $g(1,0,0,0) \neq 0$, ε *small enough: there are countably many local defects.*



Example: the 3-component FitzHugh-Nagumo model

$$\begin{cases} U_t &= \varepsilon^2 \Delta U + U - U^3 - \varepsilon(\alpha V + \beta W + \gamma(x)) \\ \tau V_t &= \Delta V + U - V \\ \theta W_t &= D^2 \Delta W + U - W\,, \end{cases} \qquad \gamma(x) = \begin{cases} \gamma_1 & x \leq 0 \\ \gamma_2 & x > 0 \end{cases}$$

[Purwins et al.], [Nishiura et al.], [van Heijster et al.]

Existence ODE is 6-dimensional and has critical points $P^{\pm} = (\pm 1, 0, \pm 1, 0, \pm 1, 0) + \mathcal{O}(\varepsilon)$ with,

$$\lambda_6^{\pm} < \lambda_5^{\pm} < \lambda_4^{\pm} < 0 < \lambda_3^{\pm} < \lambda_2^{\pm} < \lambda_1^{\pm}.$$

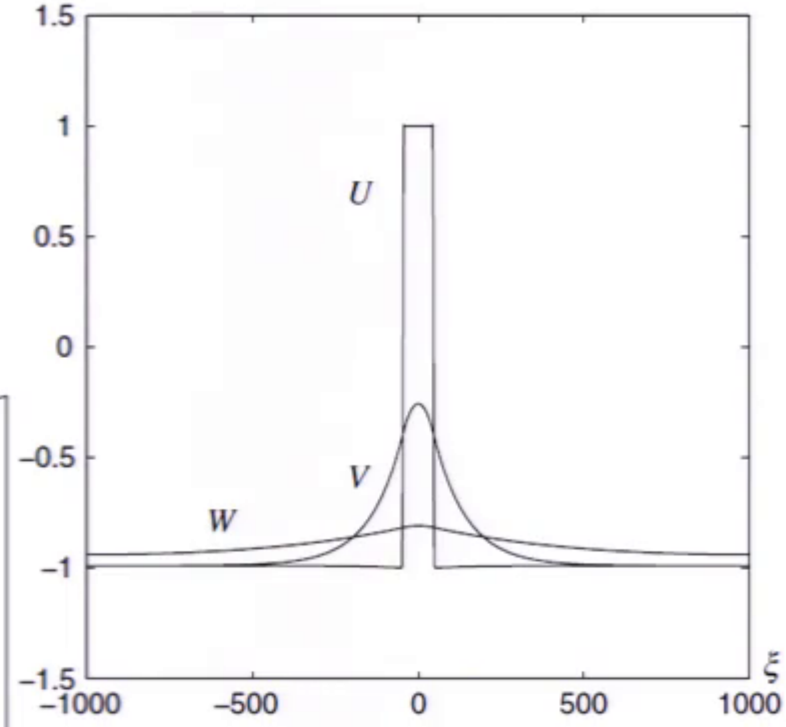
Theorem. [D, van Heijster, Kaper, '09]
Let $\gamma(x) \equiv \gamma_1$ and let $(\alpha, \beta, \gamma, D)$ be such that

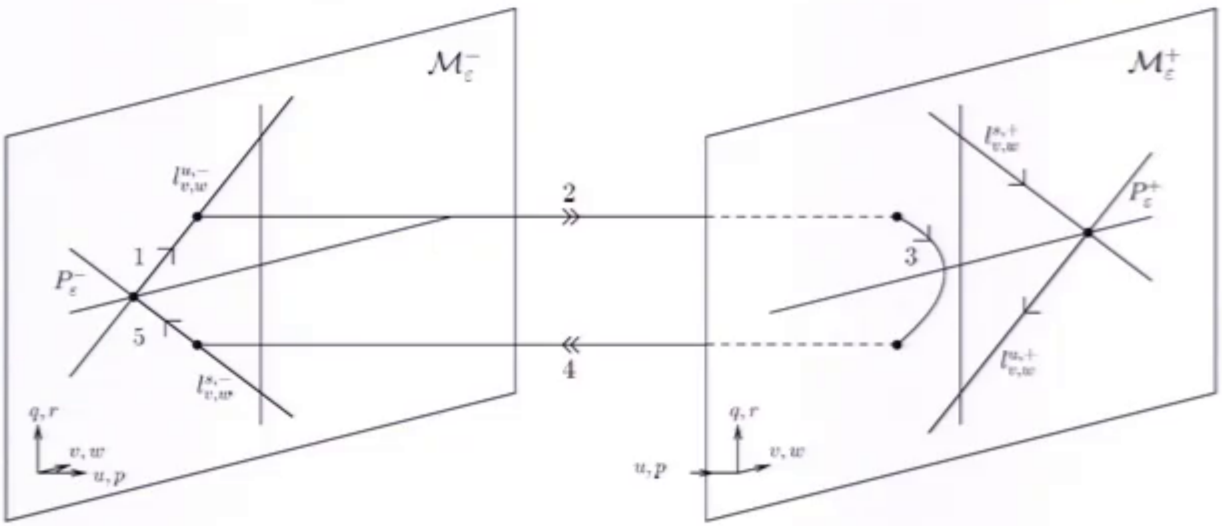
$$\alpha A + \beta A^{\frac{1}{D}} = \gamma_1$$

has K solutions with $A \in (0,1)$. If $K > 0$ and ε is small enough, then there are K homoclinic pulse, or 2-front, orbits $\Gamma_k(\xi)$ to P^- (and/or to P^+).

Question.

What is the *twist condition* *(that settles the existence of local defect solutions) and* *can it be satisfied?*





Theorem.

$\alpha, \beta > 0$ *(...): there is a local defect.*