SIAM Conference on Applications of Dynamical Systems (DS19)

A Topological Study of Spatio-Temporal Pattern Formation

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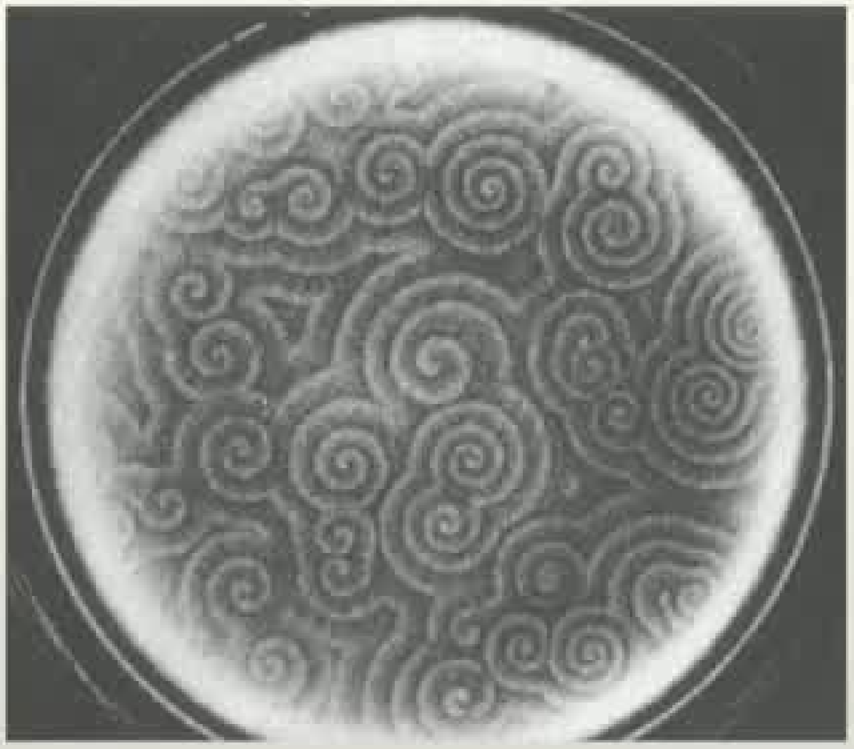
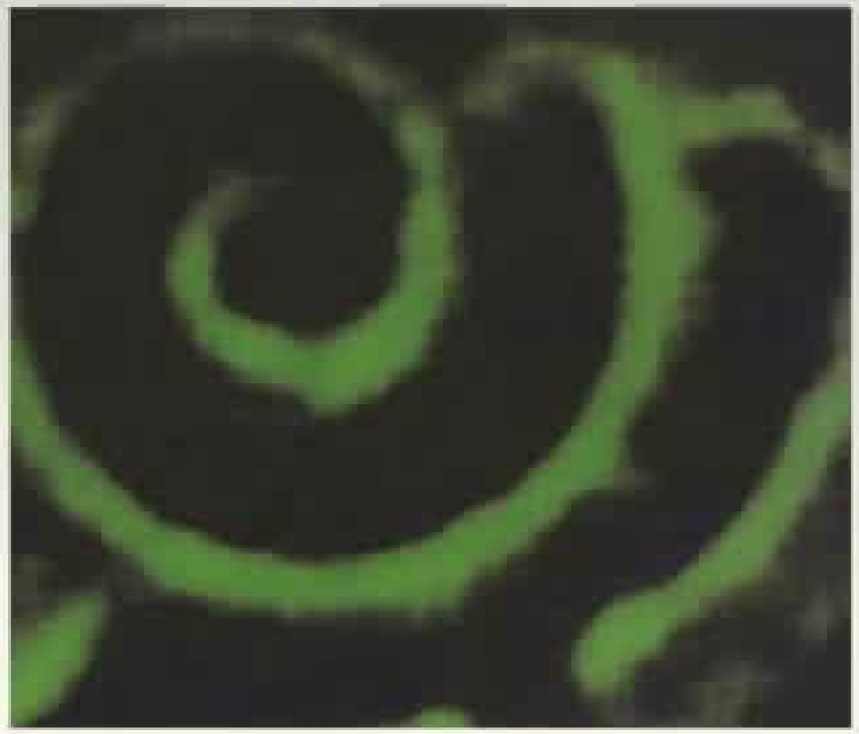
Research Goals: Topology-Based Study of Pattern Formation

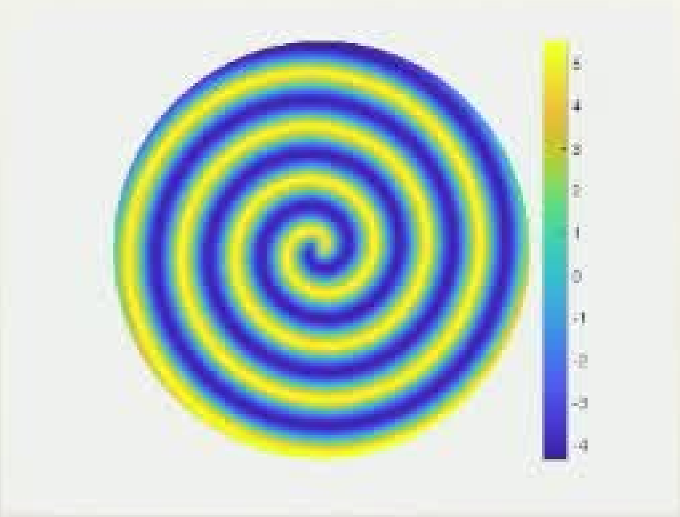
- Given a dynamical system corresponding to the spatio-temporal evolution of some pattern formation, we aim to develop methods that use the topological features of model solutions to learn about the dynamics of underlying the pattern formation
- Approach:
  - Use topological features to identify pattern defects or irregularities
  - Study dynamics of betti numbers corresponding to evolution of pattern formation
  - Test robustness of topological summaries in noisy systems
- Applications:
  - Spiral waves (reaction diffusion system)
  - Zebrafish (agent-based model)

Spiral Waves

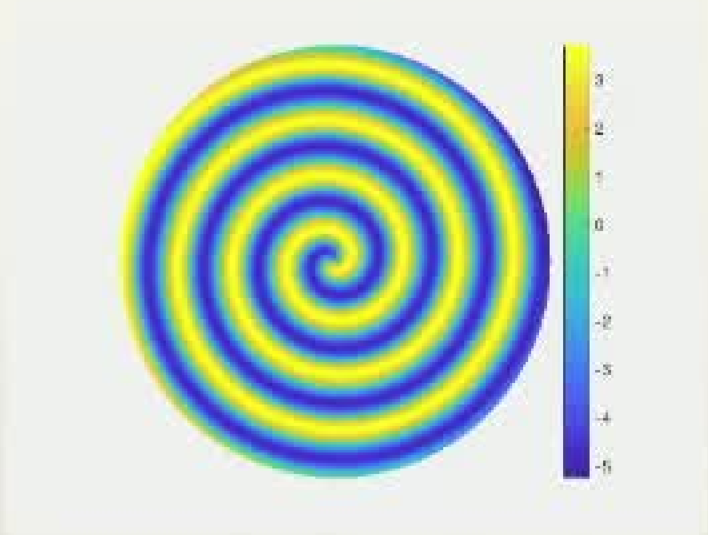


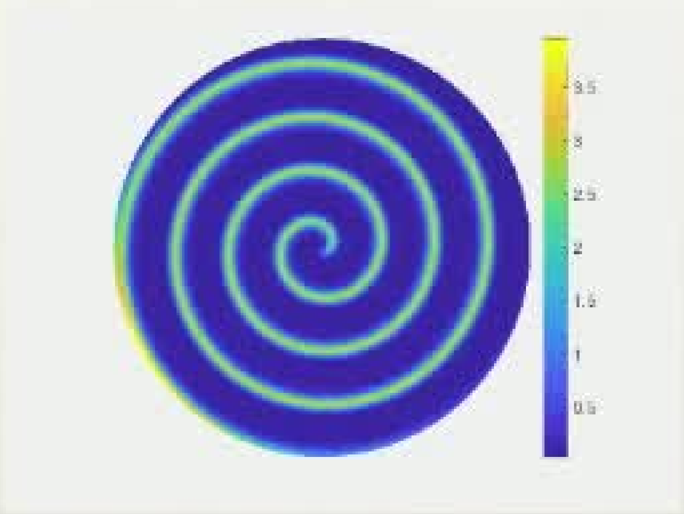
The Rössler Model



U-field, c = 3.0



V-field, c =3.0

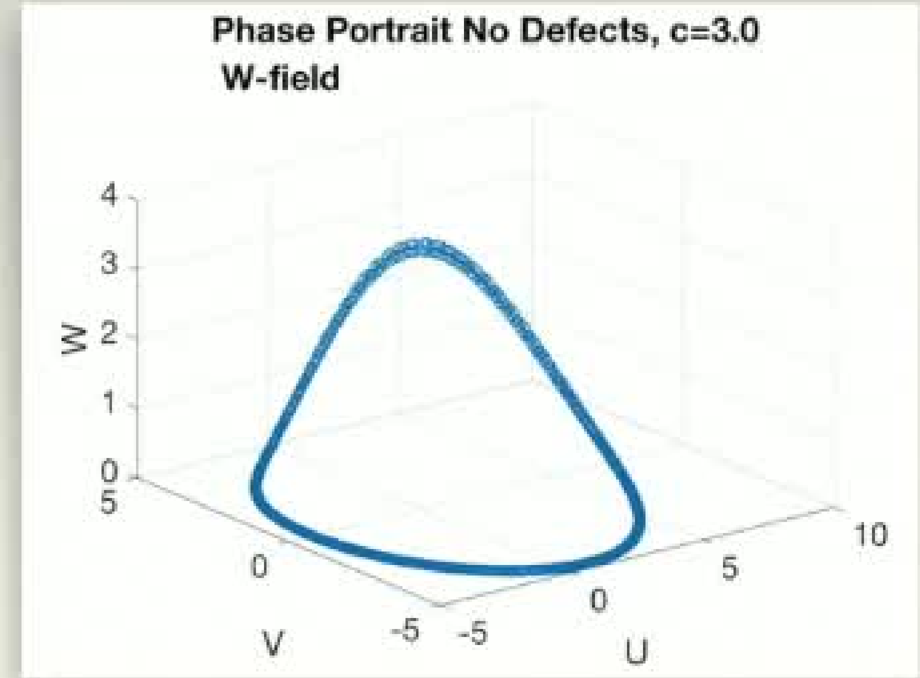


W-field, c= 3.0

$$U_t = 0.4\Delta U - V - W$$

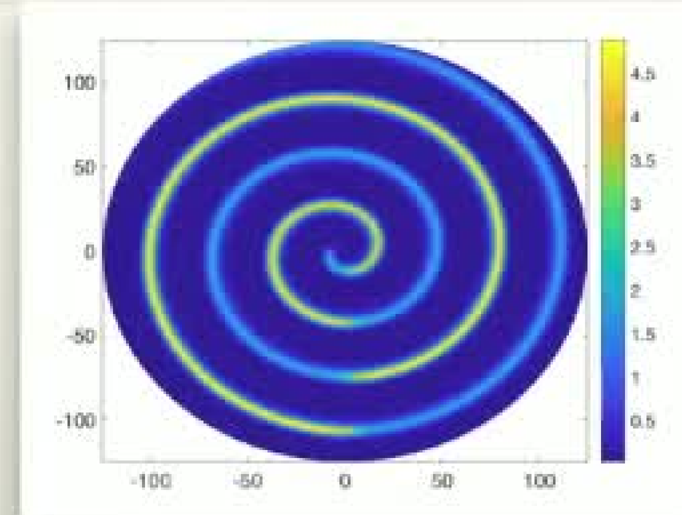
$$V_t = 0.4\Delta V + U + 0.2V$$

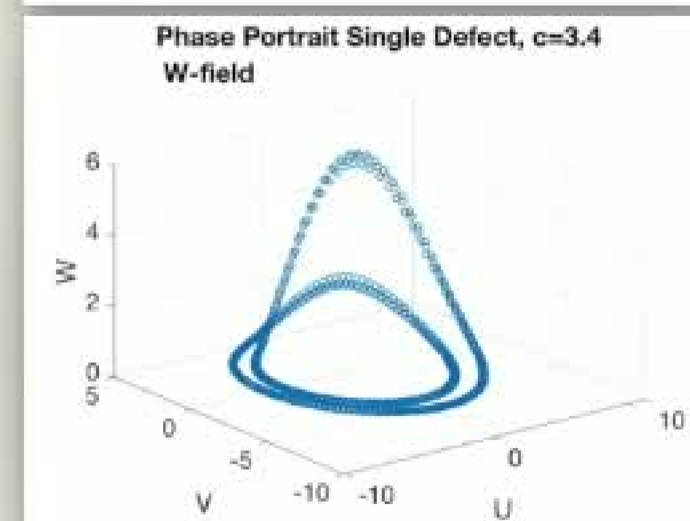
$$W_t = 0.4\Delta W + W(U - c) + 0.2$$

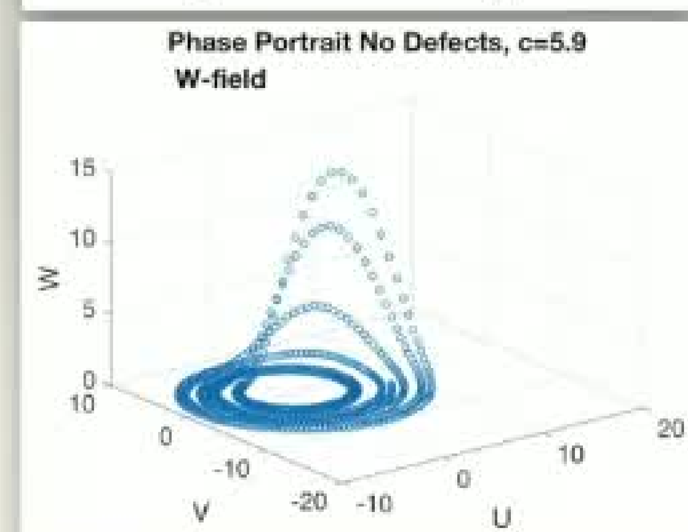


The Rössler Model

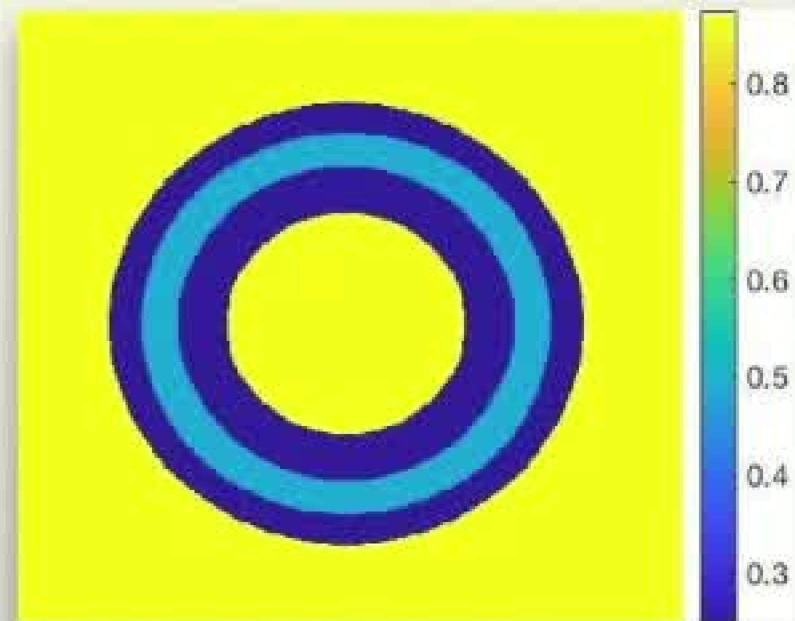
- Period-doubling bifurcation occurs around $c = c^* \approx 3.03$
- Bifurcation causes period-doubling instabilities and "line defects"
- Chaotic behavior is observed with higher c values
- Spectral analysis is traditionally used to study these bifurcations
- Can we instead use persistent homology to identify, classify, and/or analyze the spatio-temporal patterns observed under different parameter regimes and noise models for this dynamical system?

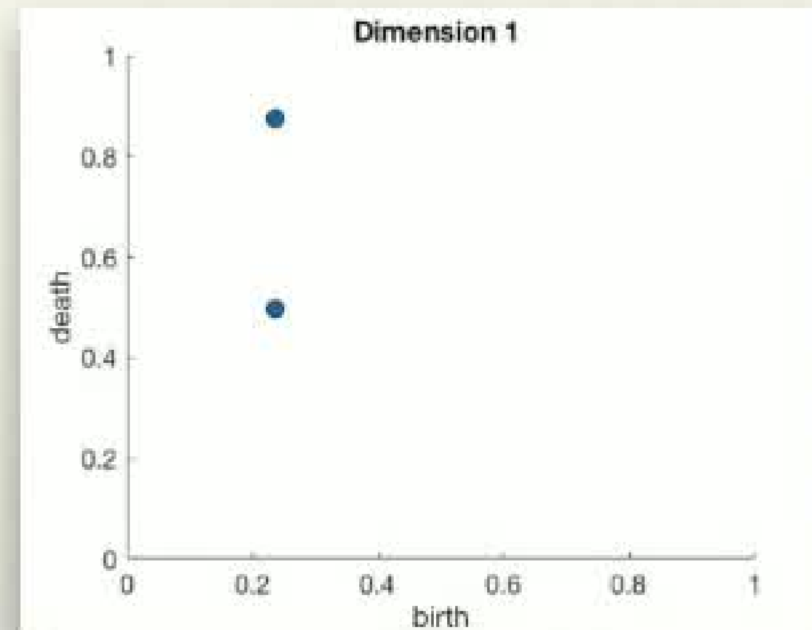


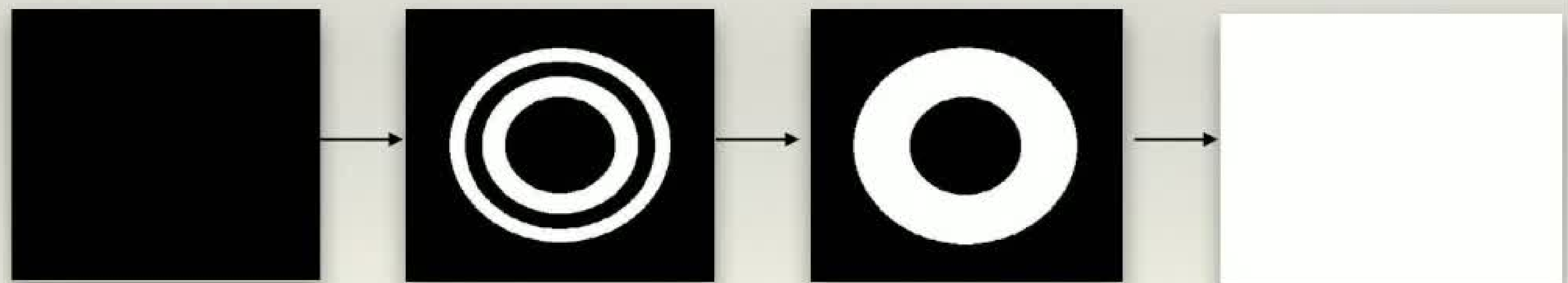




Sublevel Set Persistent Homology



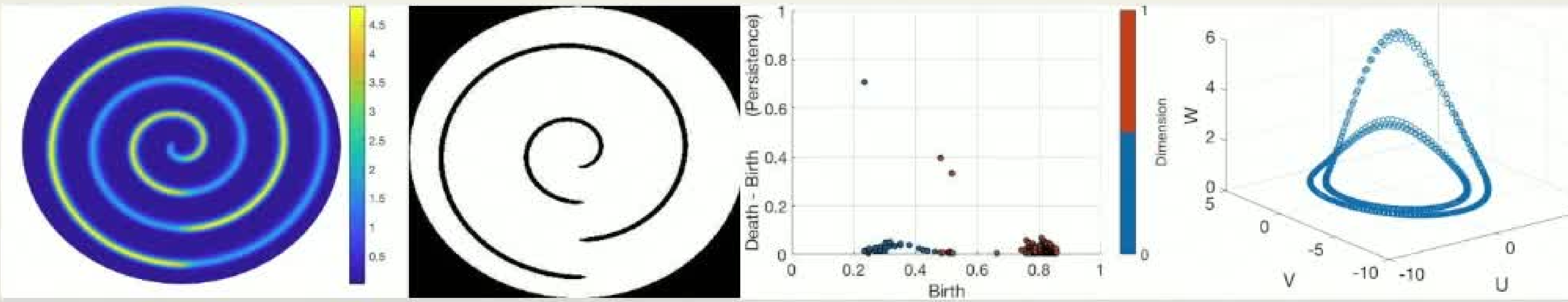


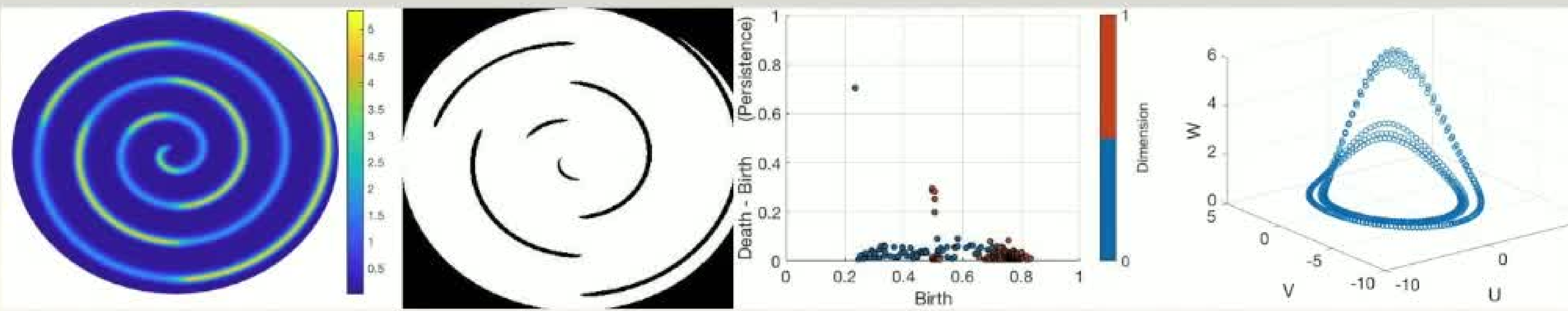


$W^{-1}((-\infty, 0.1])$	$W^{-1}((-\infty, 0.3])$	$W^{-1}((-\infty, 0.5])$	$W^{-1}((-\infty, 1])$
$\beta_1 = 0$ holes	$\beta_1 = 2$ holes	$\beta_1 = 1$ holes	$\beta_1 = 0$ holes

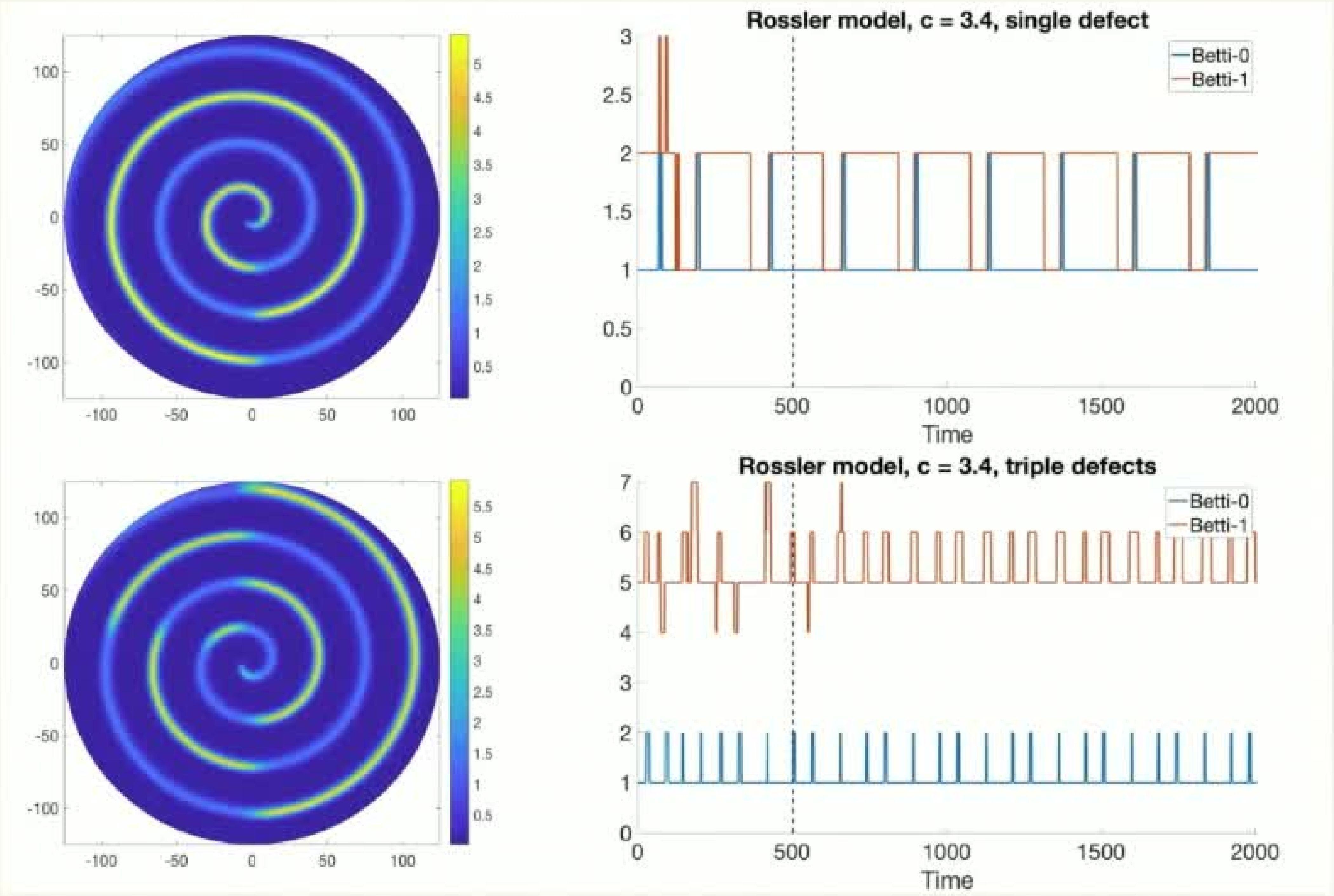
Sublevel Set Persistence Captures the Number of Line Defects in the Rössler Model

W-field solution | Sublevel set | Transformed PD | Phase Portrait





Sublevel Set Persistence Captures Periodicity and Period-Doubling Bifurcation in the Rössler Model



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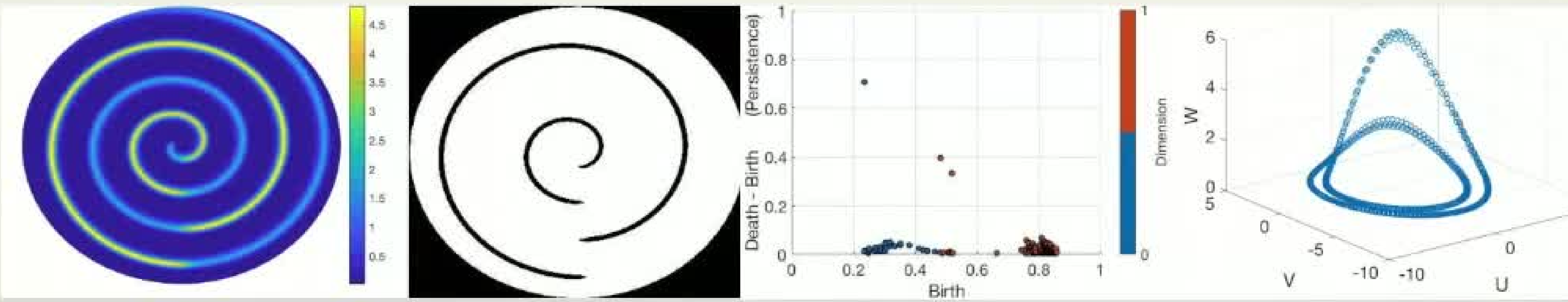

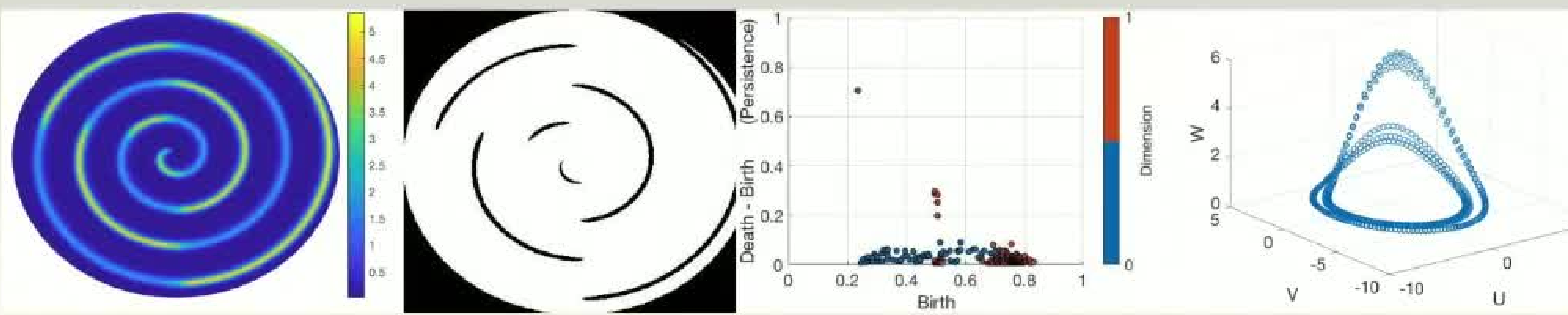

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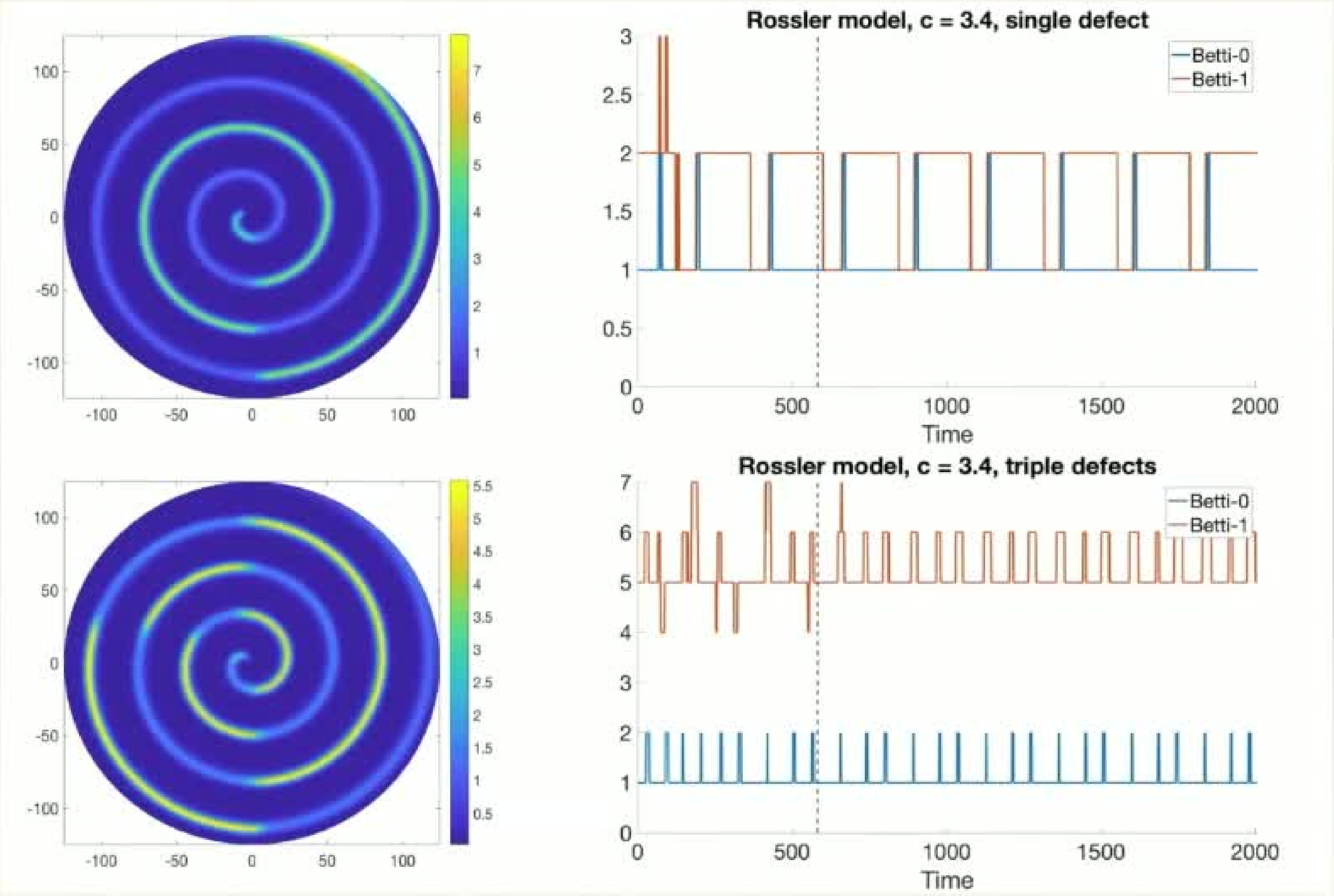
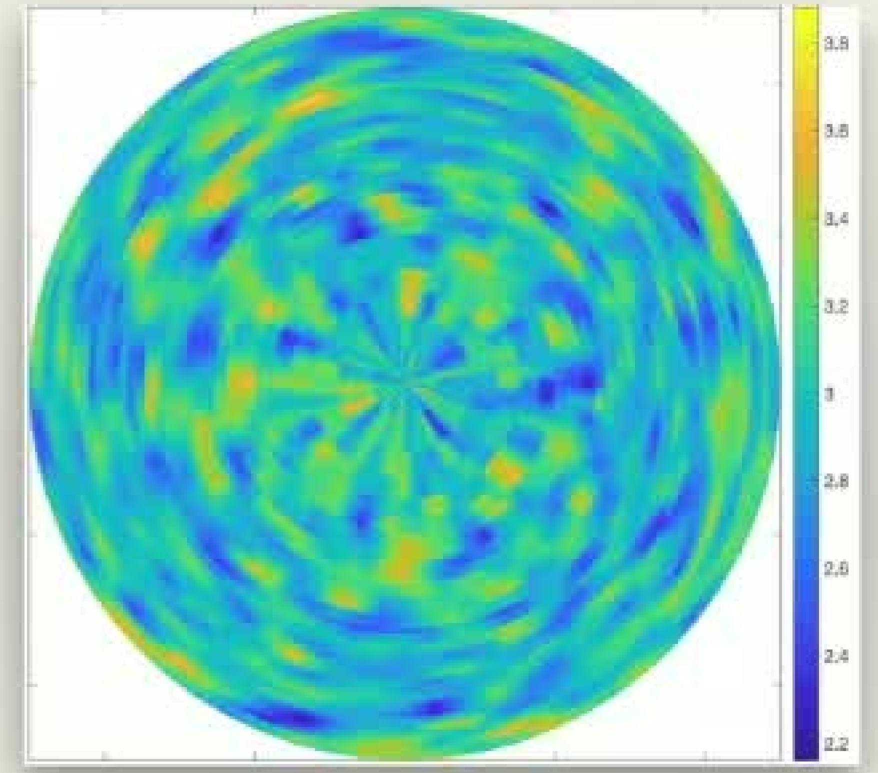
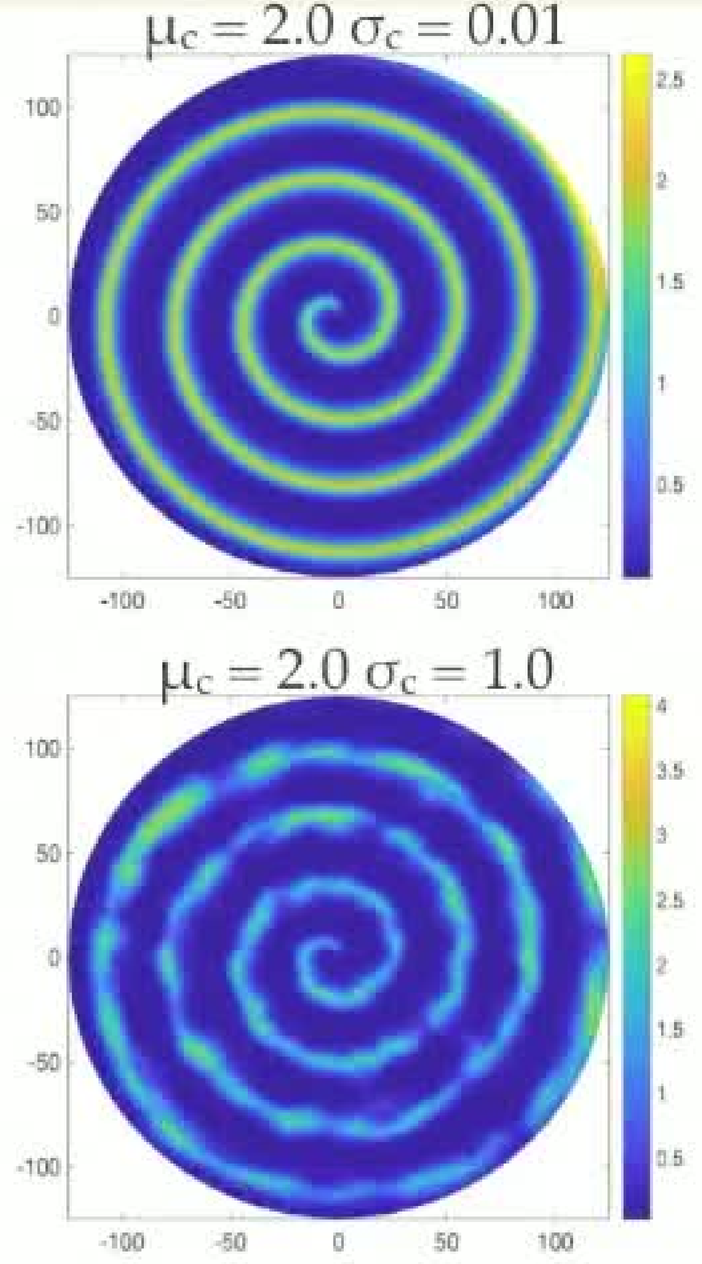


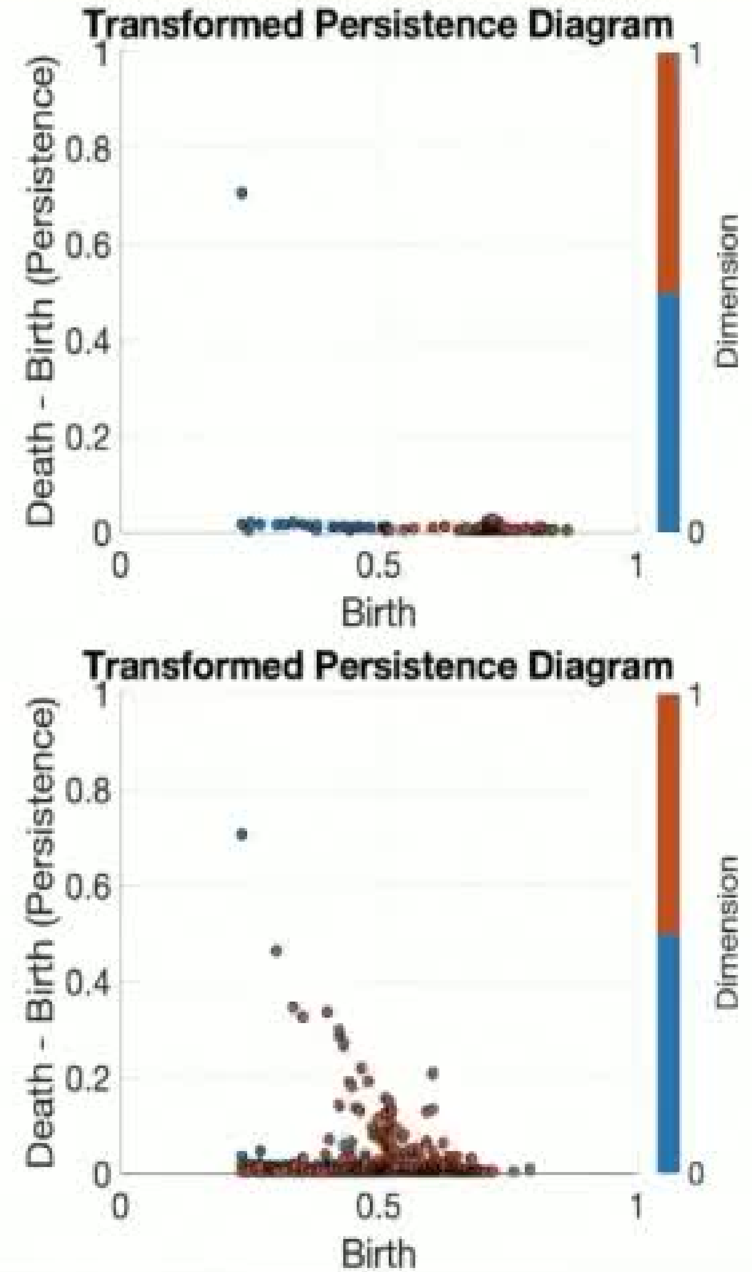
Adding Spatial C-Noise to the Rössler Model

Sample c-noise on polar disk



- For each grid point, choose $c \sim N(\mu_c, \sigma_c)$





Zebrafish

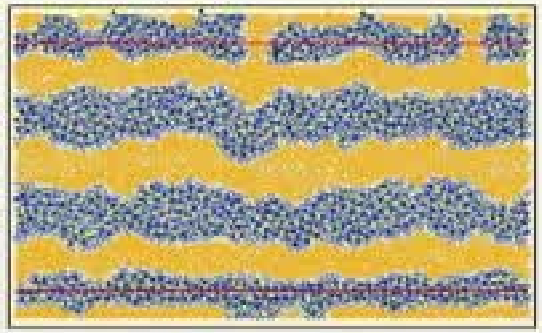
In collaboration with A. Volkening and B. Sandstede

- Utilize topological measures to automatically quantify and classify collective behavior pattern formation
- Apply quantification method to study robustness and variability of zebrafish patterns
- Investigate how genetic changes translate into changes in cellular interactions and ultimately pattern features

Pattern variability in zebrafish and zebrafish mutations from the Max-Planck-Institut für Entwicklungsbiologie.

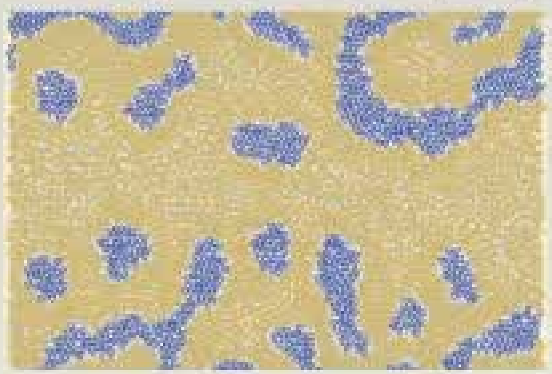
Wild Type Pattern Formation

- Wild Type patterns are characterized by interstripes (yellow) and stripes (black) that are about 300-500 μm wide

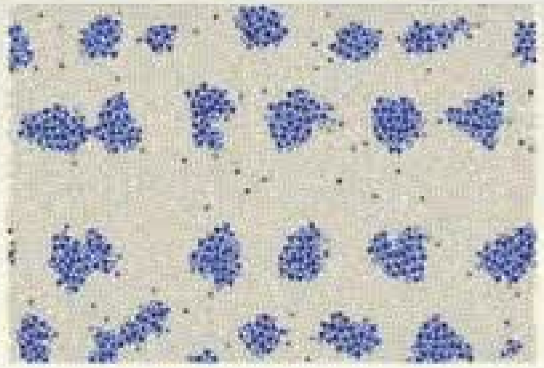


- Early stage mutations:
  - Shady: no blue/silver iridophores
    - Phenotype: round black spots aligned in stripes

  - Nacre: no black melanophores
    - Phenotype: large gold region with scattered blue regions



  - Pfeffer: no yellow/gold xanthophores
    - Phenotype: messy spots of blue and black cells with peppered black melanophores across the skin



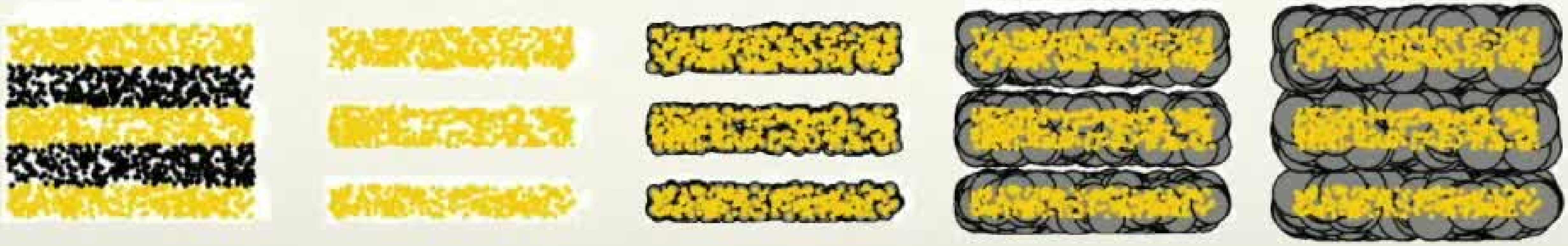
Modeling Pattern Development of Zebrafish

- A. Volkening and B. Sandstede developed an agent-based model for zebrafish pattern development*
- Cells exhibit short- (30-90 μm) and long- (90-250 μm) range length scales which dictate migration, birth, death, and other dynamics
- Black and yellow stripes develop on zebrafish as pigment cells self-organize into distinct light and dark regions

Zebrafish stripe development from the Max-Planck-Institut für Entwicklungsbiologie.

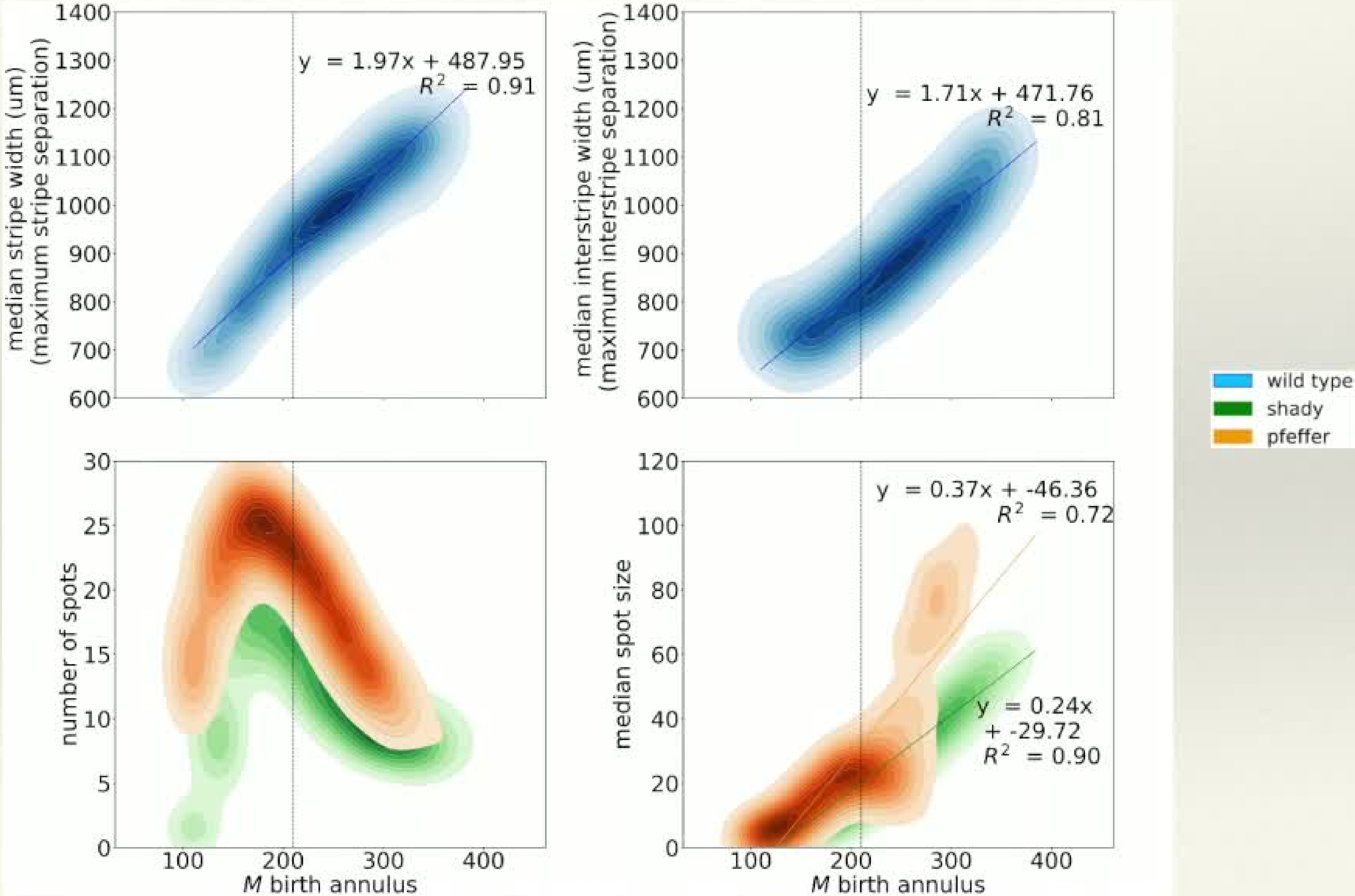
*Volkening A, Sandstede B (2018) Iridophores as a source of robustness in zebrafish stripes and variability in Danio patterns. *Nature Communications* 9(3231).

TDA for Zebrafish Pattern Classification



- Separate different cell types at final time point and treat as separate data sets
- Use standard persistent homology to compute the dimension 0 and dimension 1 homology of each stripe or spot type
  - Betti numbers count the number of spots and stripes, and capture when stripe breaks occur
  - Death times of significant dimension 1 features act as a proxy for stripe radius
- Combine TDA-based methods with machine learning and direct calculation to quantify spot and stripe pattern features

Results: Measuring pattern variation as we vary long-range melanophore differentiation signals



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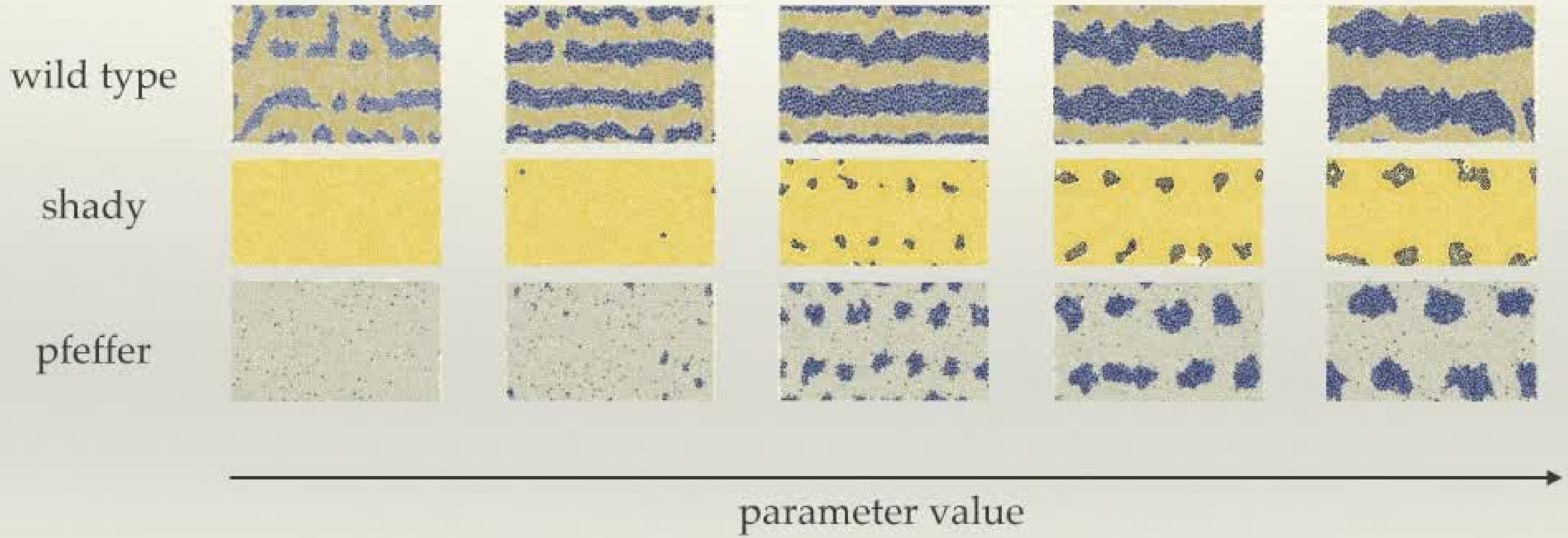

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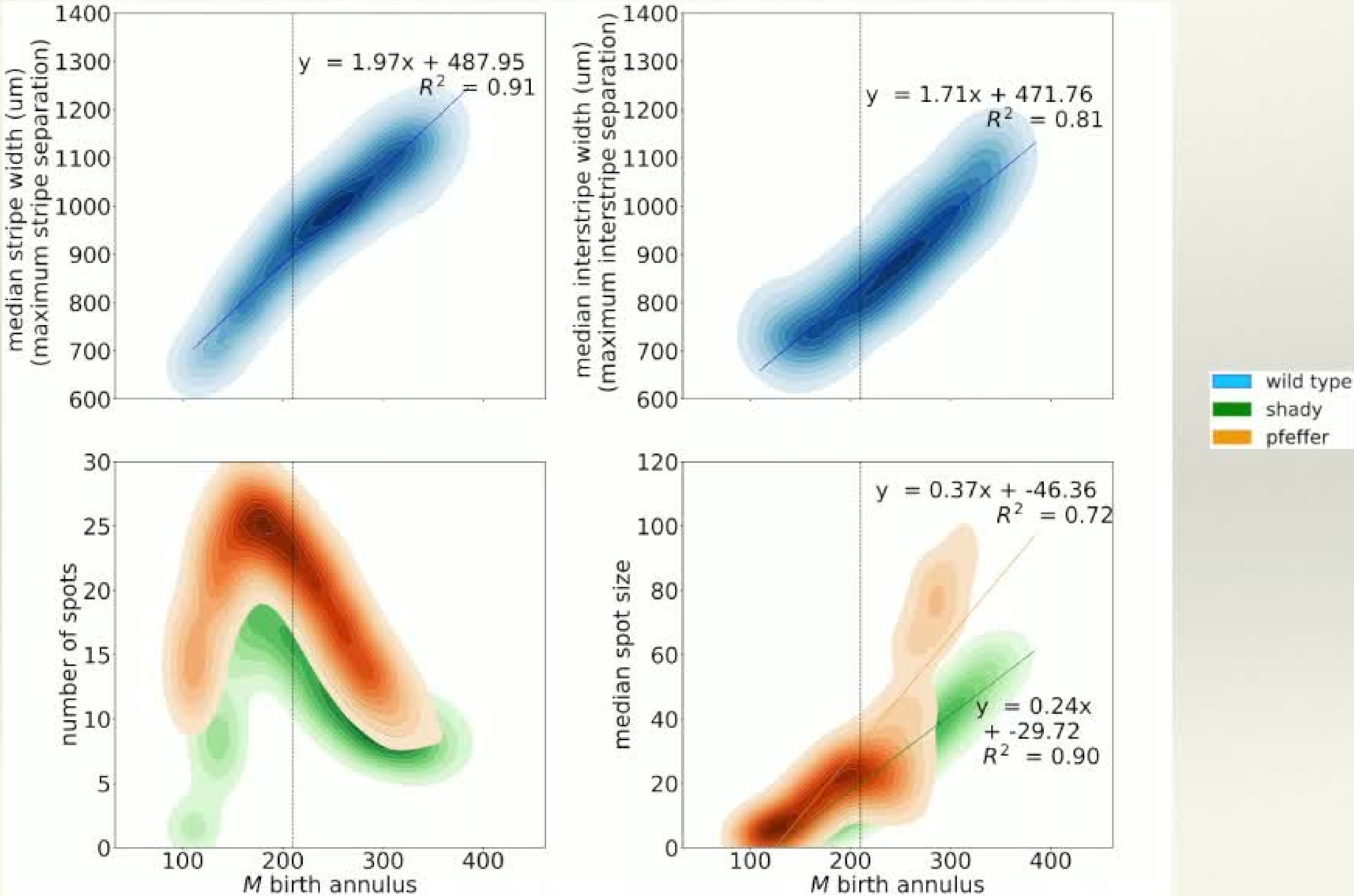

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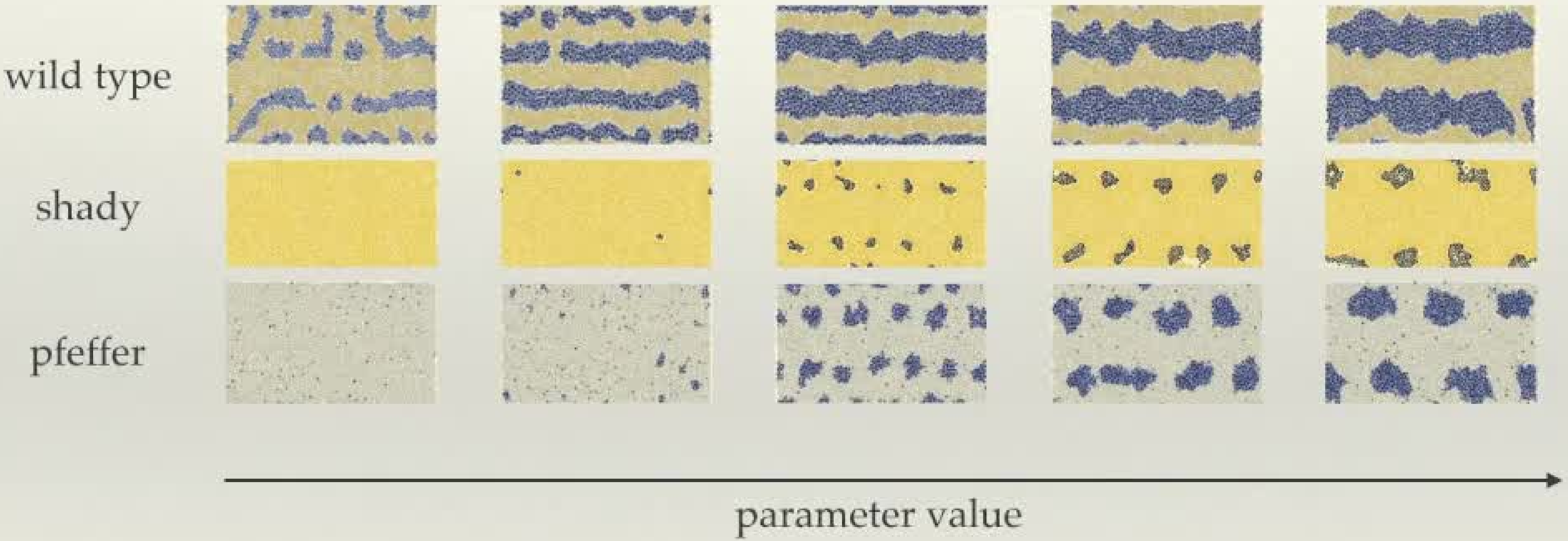

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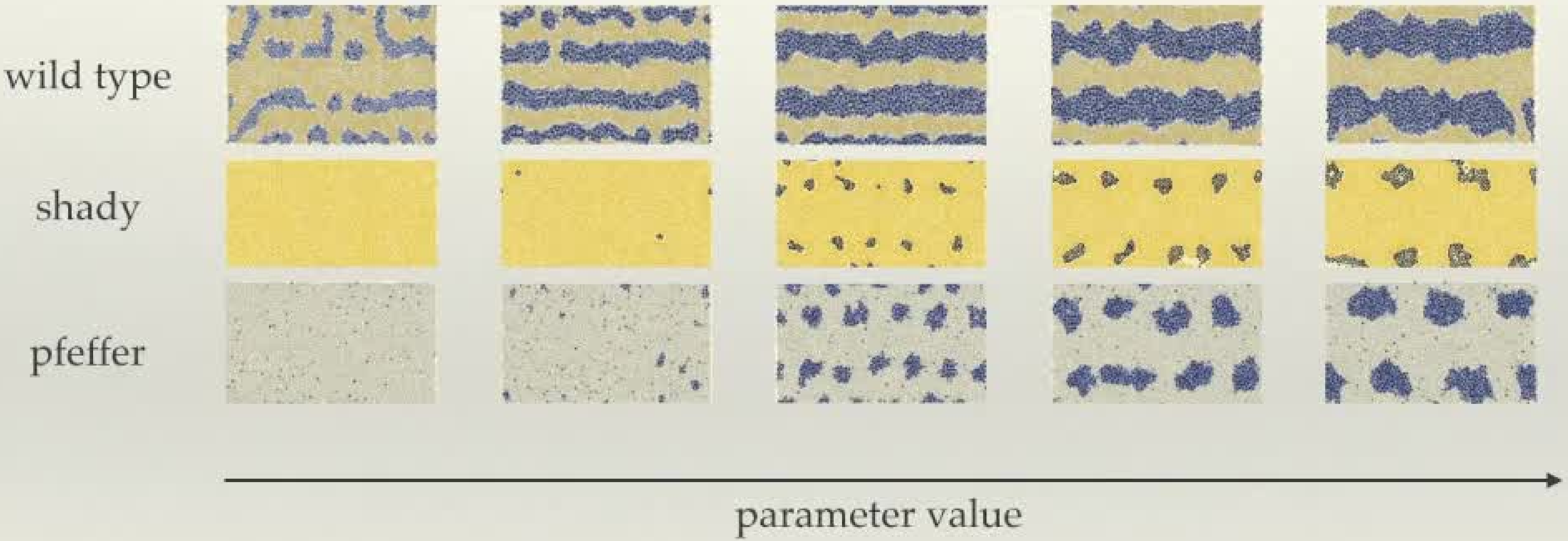

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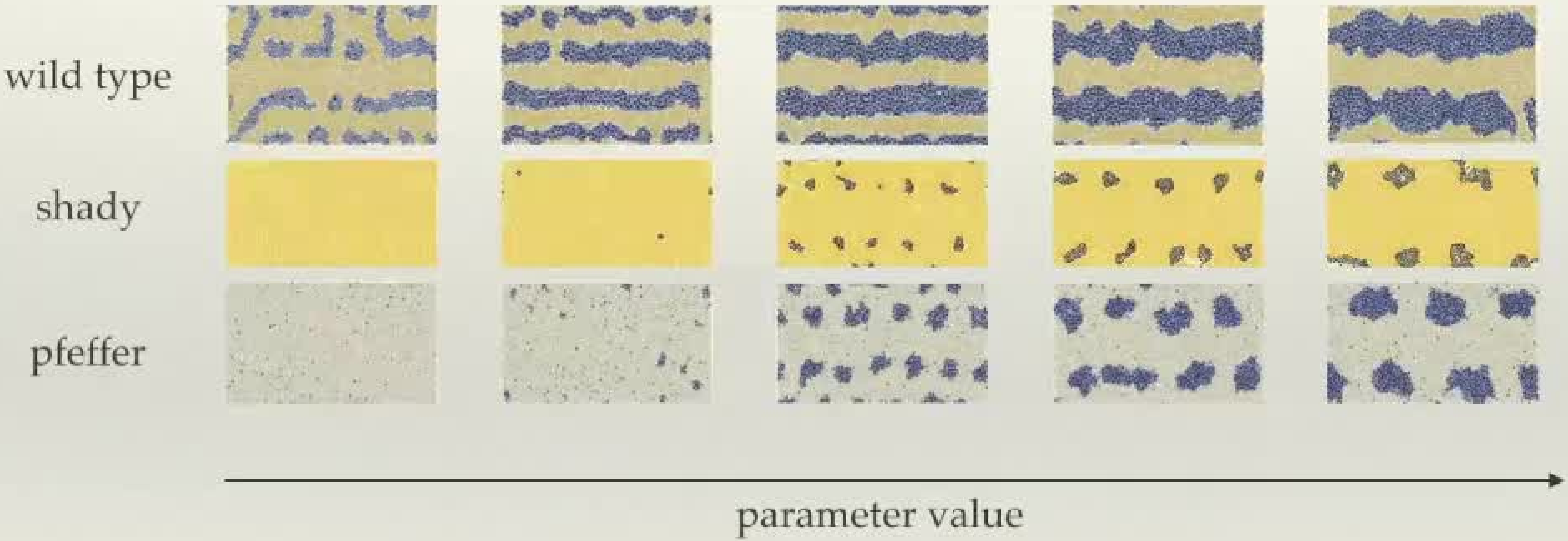

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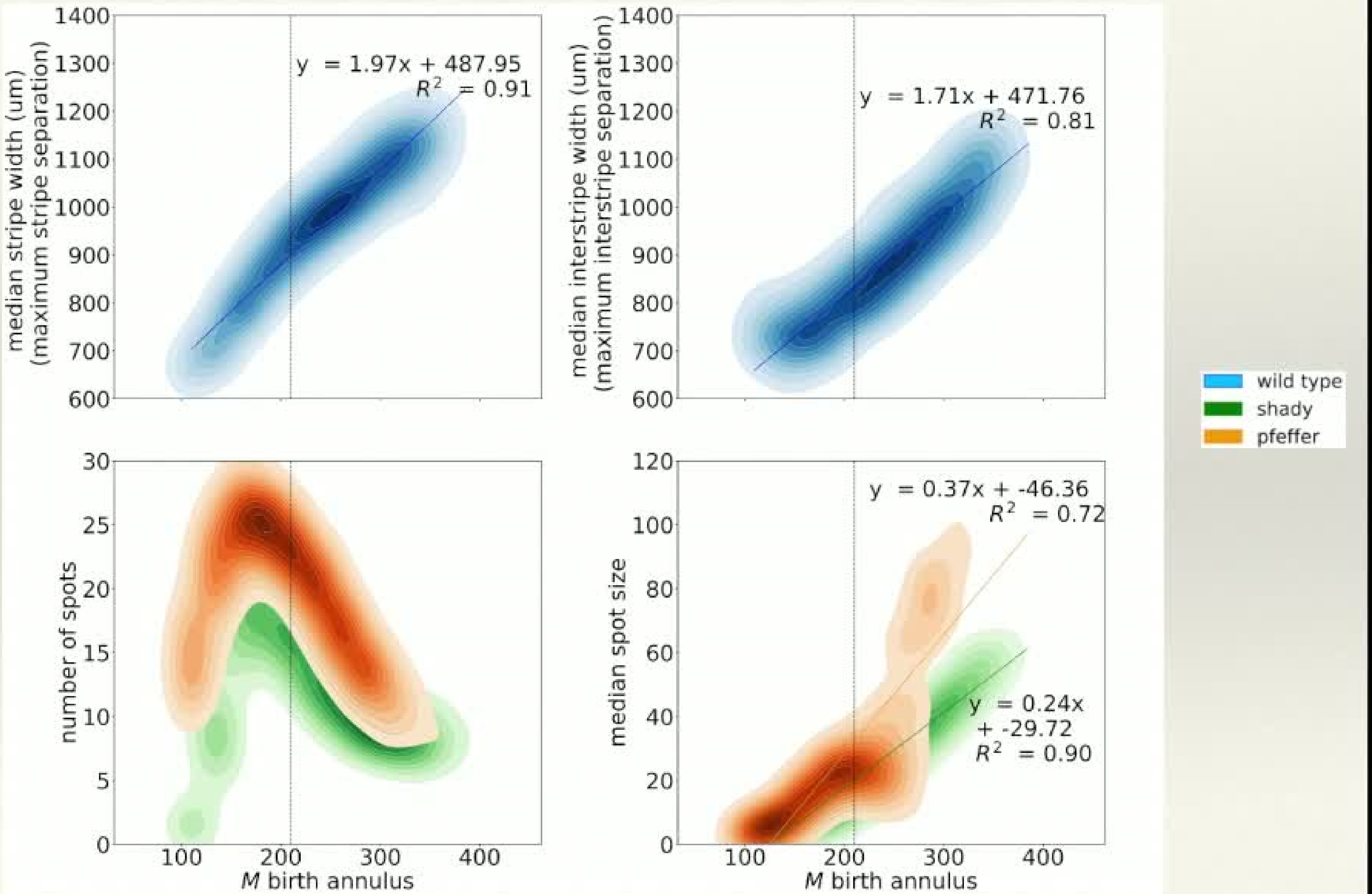

Summary: TDA for Pattern Formation

- Spiral wave dynamics:
  - Dimension 1 topological features become non-trivial in the presence of line defects and β_1 can approximate number of line defects present in the system
  - Periodicity in the dynamical system induces periodic movement of β_1
  - Persistence diagrams are stable with respect to low spatial c-noise where spiral remains connected, while high levels of spatial c-noise cause chaos in spiral wave dynamics and blow up in Wasserstein/Bottleneck distances
- Zebrafish patterns:
  - For wild type, the dimension 1 persistence coordinates capture number of stripes and distances between stripes
  - For early stage mutations, the dimension 0 persistence features help to characterize spots
  - Parameter testing provides insight into which cell interactions drive different pattern types