Optimal-order multigrid preconditioners for linear systems arising in the semi-smooth Newton solution of certain PDE-constrained optimization problems

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Problem formulation

- Model problem:
  $\mathcal{K} : L^2(\Omega) \to L^2(\Omega)$ linear, compact, $y_d, a, b \in L^2(\Omega)$

Optimal control problem

$$\begin{aligned} \text{minimize} \quad & J_\beta(u) \stackrel{\text{def}}{=} \frac{1}{2}\|\mathcal{K}u - y_d\|^2 + \frac{\beta}{2}\|u\|^2 \\ \text{subject to:} \quad & u \in L^2(\Omega), \;\; a \le u \le b \end{aligned} \tag{1}$$

- Examples:
  1. $\mathcal{K} = -\Delta^{-1}$ - elliptic-constrained optimal control problem
  2. $\mathcal{K}$ is an explicit integral operator (image deblurring)

Outline

1. Motivation and problem formulation
2. Background
   - The unconstrained problem
   - The constrained problem
3. Piecewise constant controls
   - Algorithm design
   - Numerical results
4. Continuous piecewise linear constraints

The coarse space

- New coarse inactive index set:

$$\mathcal{I}_{2h} \stackrel{\text{def}}{=} \{i \in \{1, \dots, N_{2h}\} \ : \ \mu(\text{supp}(\varphi_i^{2h}) \cap \Omega_h^{\text{in}}) > 0\} \ .$$

- Coarse inactive space: $\mathcal{U}_{2h}^{\text{in}} = \text{span}\{\varphi_i^{2h} \ : \ i \in \mathcal{I}_{2h}\}$
- Coarse-level footprint: $\Omega_{2h}^{\text{in}} = \bigcup_{i \in \mathcal{I}_{2h}} \text{supp}(\varphi_i^{2h})$.
- Remarks:
  1. $\Omega_h^{\text{in}} \subseteq \Omega_{2h}^{\text{in}}$. (new)
  2. It may be that $\mathcal{U}_{2h}^{\text{in}} \not\subseteq \mathcal{U}_h^{\text{in}}$. (new)
  3. We have $\mathcal{U}_{2h}^{\text{in}} \subseteq \mathcal{U}_h^{\text{in}}$ iff $\Omega_h^{\text{in}} = \Omega_{2h}^{\text{in}}$.

Numerical example: elliptic-constrained problem

- PDE-constrained optimal control problem

$$\begin{aligned} \text{minimize} \quad & \tfrac{1}{2}\|y - y_d\|^2 + \tfrac{\beta}{2}\|u\|^2 \\ \text{subject to:} \quad & -\Delta y = u \\ & a \le u \le b \end{aligned}$$

Elliptic control problem

Target control u_d vs. optimal control u_{min} ($\beta = 10^{-6}$)