Modeling and control of nonlinear oscillations in neuroscience and energy systems

Anatoly Zlotnik
(with Raphael Nagao, Jr-Shin Li, Istvan Kiss, and Walter Bomela)

Applied Mathematics & Plasma Physics
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Siam Annual Meeting, Portland, OR
July 9, 2018

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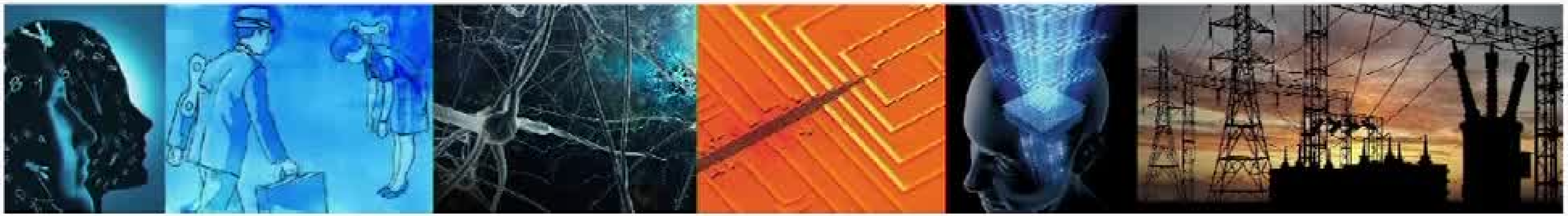
Oscillating Systems in Science and Engineering



- **Natural systems**: spiking of neurons, metabolic chemical reaction systems, circadian timekeepers, cardiac rhythms, auditory processing
- **Medical applications**: deep brain stimulation for Parkinson's disease and epilepsy, cardiac pacemaking, tinnitus treatment, protocols for jet lag recovery
- **Engineering applications**: semiconductor lasers, vibrating systems, power electronics, electrochemical reactors, neurocomputers, power grid design

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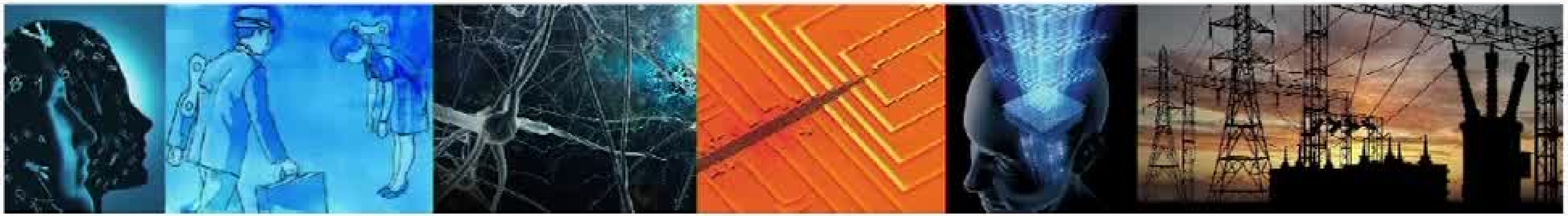

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Coupled Oscillator Models

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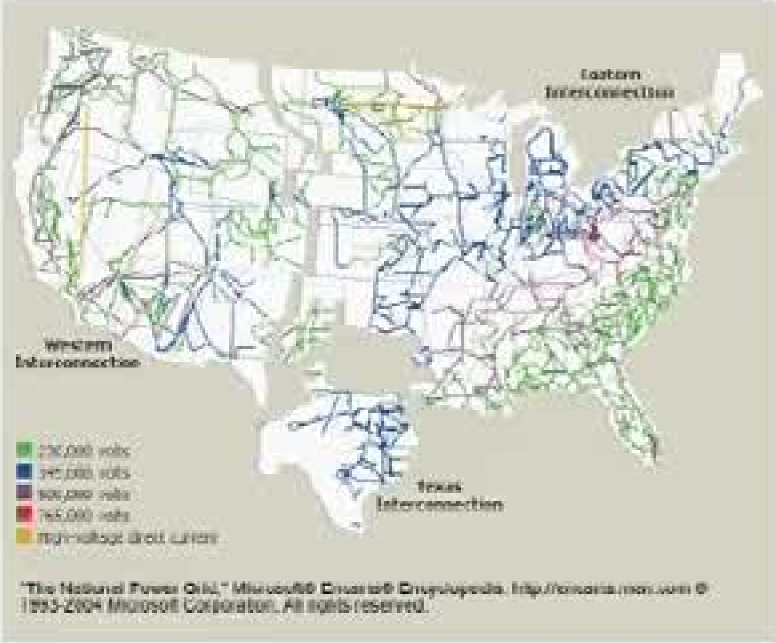
- Firefly synchronization (Buck 1937, Strogatz 1990), Kuramoto model (1975)

$$\dot{\varphi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_i - \varphi_j)$$

$$r = e^{i\theta} = \frac{1}{N}\sum_{j=1}^{N} e^{i\varphi_j}$$

- Stability analysis in power networks (Bergen & Hill 1981, Dörfler & Chertkov 2013)



$$D_i\dot{\varphi}_i + P_{1,i} = \sum_{j=1}^{N} a_{ij}\sin(\varphi_j - \varphi_i), \qquad \text{load}$$

$$M_i\ddot{\varphi}_i + D_i\dot{\varphi}_i = P_{m,i} + \sum_{j=1}^{N} a_{ij}\sin(\varphi_j - \varphi_i), \qquad \text{generator}$$

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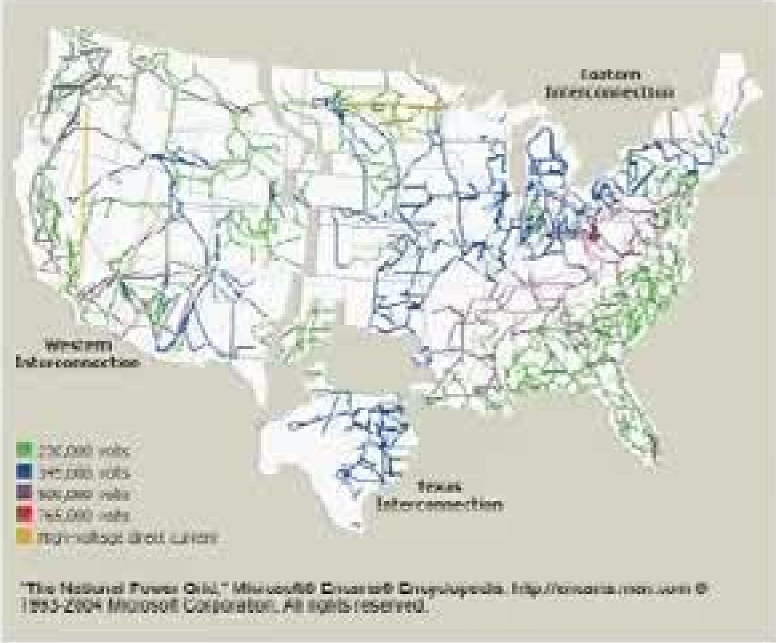

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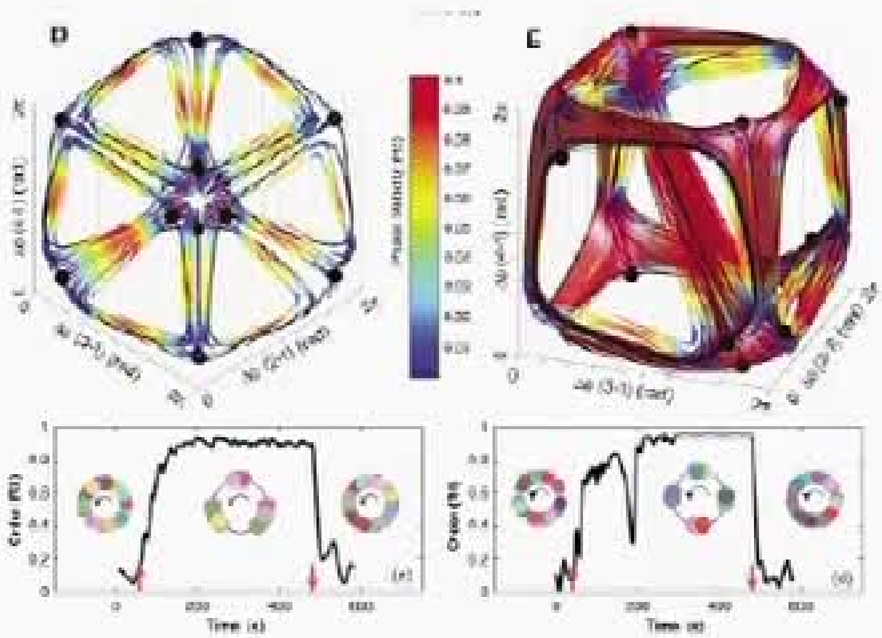

$$\dot{\varphi}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} H(\varphi_i - \varphi_j)$$

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$$p(t) = \frac{K}{N}\sum_{j=1}^{N} h(x_j(t)), \quad h(x) = \sum_{n=0}^{S} k_n x(t-\tau_n)^n$$

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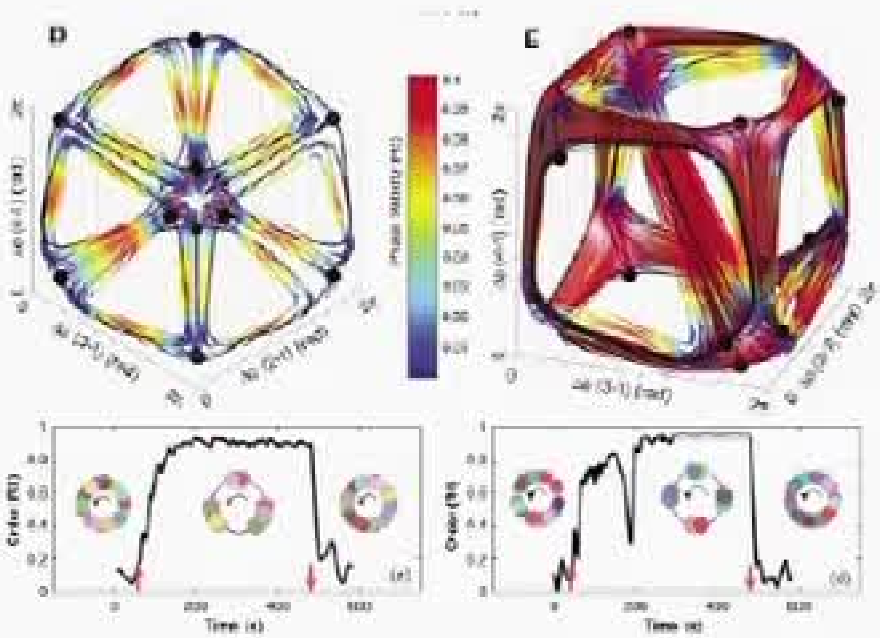

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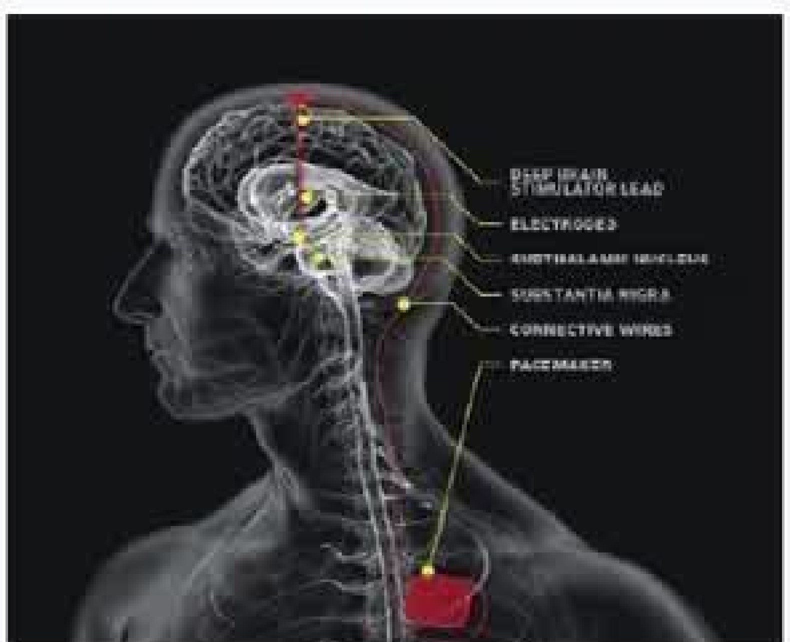

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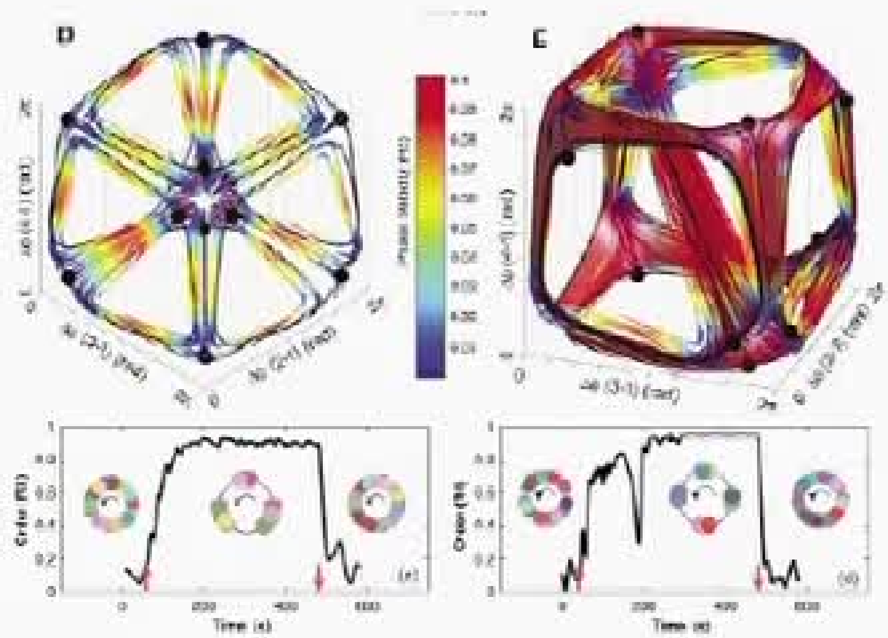

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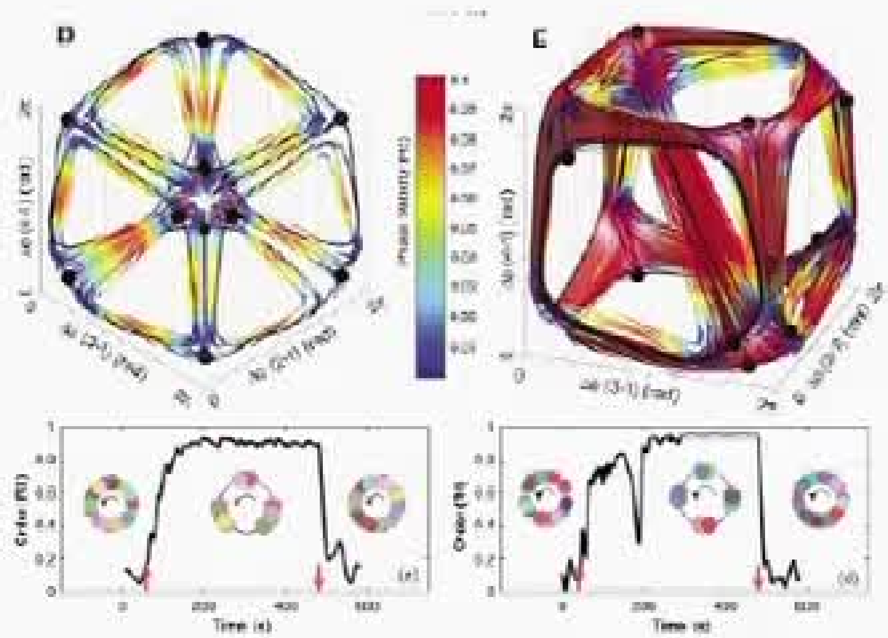

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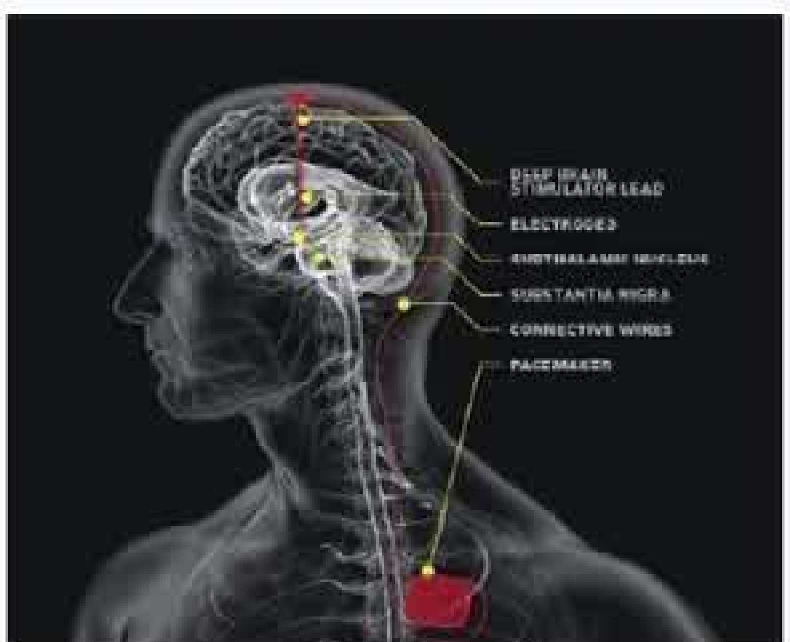

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Synchronization Engineering

- ***The design of coherent, spatially organized patterns in collections of heterogenous, nonlinear oscillators under the action of weak periodic inputs***
- **A Challenging Problem:**
  Phase assignment: stable pattern in phases of entrained oscillators relative to the forcing phase
  Robustness: stability of phase structure with minimal sensitivity to model parameter variation and unknown disturbances
  No feedback: state information is unavailable or intractable
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Asymptotic Response to a Weak Periodic Input

- Input $u(t) = v(\Omega t)$ where $\theta = \Omega t$ is the ***forcing phase*** and v is 2π-periodic
- When does $u(t) = v(\Omega t)$ entrain an oscillator $\dot{\psi} = \omega + Z(\psi)u$?
- Entrainment occurs when $\phi(t) = \psi(t) - \Omega t \to c$
- ***Phase drift dynamics***: $\dot{\phi}(t) = \Delta\omega + Z(\phi + \Omega t)u$
- Call $\Delta\omega = \omega - \Omega$ the ***frequency detuning***.
- Define an ***averaging operator*** by

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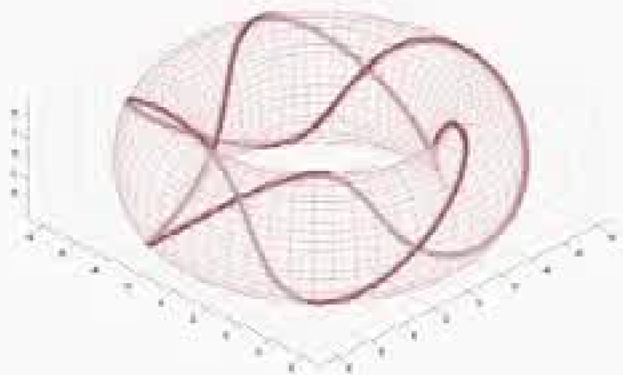
- Make the ***weak forcing assumption*** $u = \varepsilon u_1$ with $\varepsilon << 1$ where $\langle u_1^2 \rangle = 1$.
- The weak ergodic theorem for measure-preserving dynamical systems on the torus and formal averaging eliminate dependence on time
- **Kuramoto:** If forcing is weak, $\phi(t)$ is nearly constant over a single period $[0, T]$

$$\dot{\phi} = \Delta\omega + Z(\phi + \Omega t)f(\Omega t) \approx \Delta\omega + \frac{1}{T}\int_0^T Z(\phi + \Omega t)f(\Omega t)\mathrm{d}t = \Delta\omega + \langle Z(\theta + \phi)v(\theta)\rangle$$

- Examine ***averaged phase drift dynamics***

$$\dot{\varphi} = \Delta\omega + \Lambda_v(\varphi)$$

- $\Lambda_v(\varphi) = \langle Z(\theta + \varphi)v(\theta)\rangle$ is the ***interaction function***



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Example: Hodgkin-Huxley Spiking Neuron Model

Example: Hodgkin-Huxley Equations (1952)

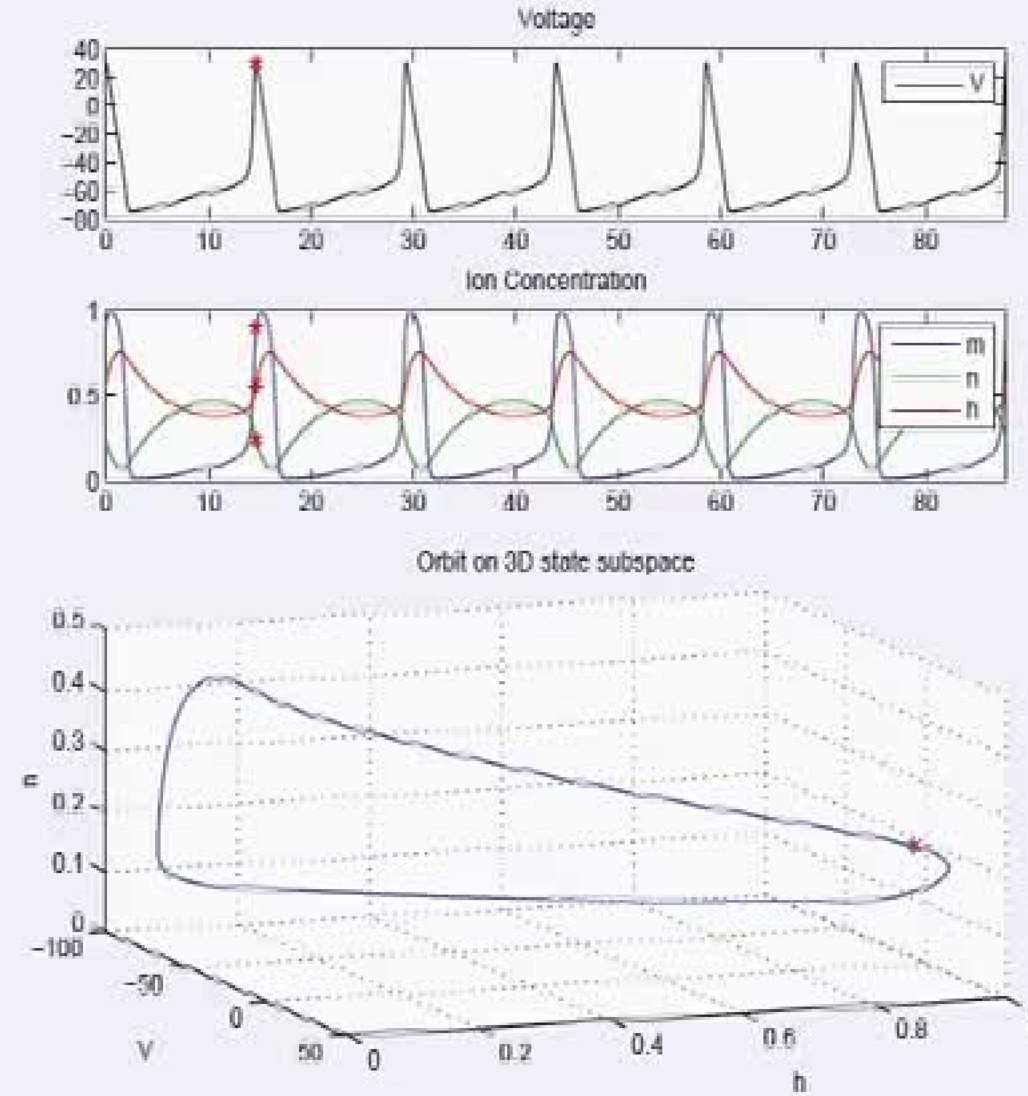
Model of action potential propagation in a squid giant axon:

$$\begin{aligned} c\dot{V} &= I_b + I(t) - \overline{g}_{Na} h(V - V_{Na}) m^3 - \overline{g}_K (V - V_k) n^4 \\ &\quad \ldots - \overline{g}_L (V - V_L) \\ \dot{m} &= a_m(V)(1 - m) - b_m(V)m, \\ \dot{h} &= a_h(V)(1 - h) - b_h(V)h, \\ \dot{n} &= a_n(V)(1 - n) - b_n(V)n, \end{aligned}$$

$$\begin{aligned} a_m(V) &= 0.1(V + 40)/(1 - \exp(-(V + 40)/10)), \\ b_m(V) &= 4\exp(-(V + 65)/18), \\ a_h(V) &= 0.07\exp(-(V + 65)/20), \\ b_h(V) &= 1/(1 + \exp(-(V + 35)/10)), \\ a_n(V) &= 0.01(V + 55)/(1 - \exp(-(V + 55)/10)), \\ b_n(V) &= 0.125\exp(-(V + 65)/80). \end{aligned}$$

$V_{Na} = 50\,mV,\, V_K = -77\,mV,\, V_L = -54.4\,mV,$
$\overline{g}_{Na} = 120\,mS/cm^2, \overline{g}_K = 36\,mS/cm^2,$
$\overline{g}_L = 0.3\,mS/cm^2,\, I_b = 10\,\mu A/cm^2,\, c = 1\,\mu F/cm^2.$

Here $x = (V, m, h, n)$ and $u = I$ are the state and control, respectively.



Minimum energy entrainment of a single oscillator

- Entrain an oscillator $\dot{\psi} = \omega + Z(\psi)u$ to a frequency Ω with minimum energy $\langle v^2 \rangle$
- Requires $\Delta\omega + \Lambda(\varphi_+) \geq 0$ and $\Delta\omega + \Lambda(\varphi_-) \leq 0$, so the active constraint is $\Delta\omega + \Lambda(\varphi_+) = 0$ if $\Omega > \omega$, or $\Delta\omega + \Lambda(\varphi_-) = 0$ if $\Omega < \omega$.
- ***Adjoin frequency constraint to minimum energy objective*** by Lagrange multiplier λ:

$$\min \mathcal{J}[v] = \langle v^2 \rangle - \lambda(\Delta\omega + \Lambda(\varphi_\pm)) = \frac{1}{2\pi}\int_0^{2\pi} [v(\theta)(v(\theta) - \lambda Z(\theta + \varphi_\pm)) - \lambda\Delta\omega]\mathrm{d}\theta$$

- Solve ***Euler-Lagrange equation*** and use the constraint to solve for λ.

Minimum energy control to entrain a single oscillator

$$v(\theta) = -\frac{\Delta\omega}{\langle Z^2 \rangle} Z(\theta)$$

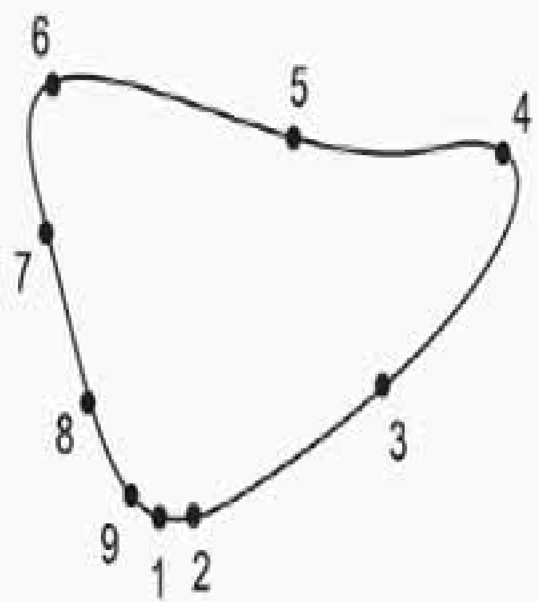
- This control is a ***re-scaling of the phase response curve***

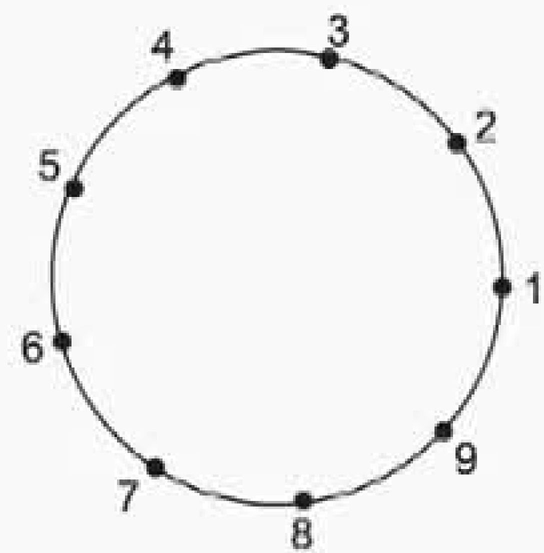
Oscillating Systems and Phase Reduction

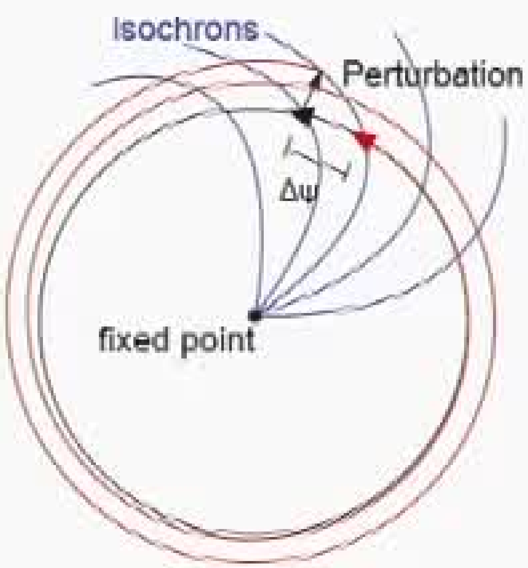
- In general, oscillation can refer to any repetitive activity.
- **Oscillator**: A smooth dynamical system

$$\dot{x} = f(x, u)$$

  where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}$ are the state and control
- An attractive, non-constant limit cycle $\gamma(t) = \gamma(t + T)$ satisfying $\dot{\gamma} = f(\gamma, 0)$ on the periodic orbit $\Gamma = \{y \in \mathbb{R}^n \ : \ y = \gamma(t), \ t \in [0, T)\}$.







State space model	Phase model
$\dot{x} = f(x, u)$ State: $x \in \mathbb{R}^n$	$\dot{\psi} = \omega + Z(\psi)u$ phase: $\psi \in [0, 2\pi)$
control: $u \in \mathbb{R}$	

- $\psi \equiv$ aymptotic phase
- $\omega \equiv$ natural frequency
- $Z \equiv$ ***phase response curve*** **(PRC)**: infinitesimal sensitivity to input u at a given phase.

Asymptotic Response to a Weak Periodic Input

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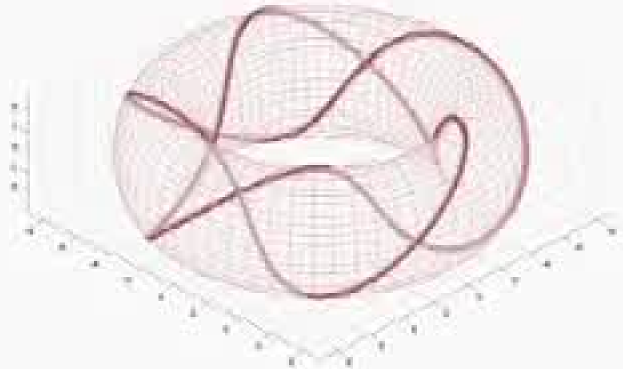
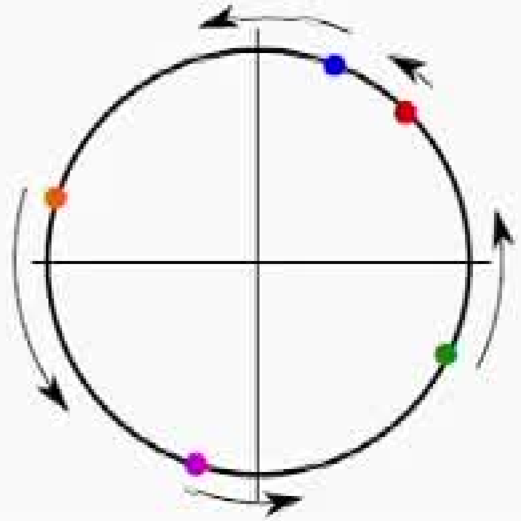
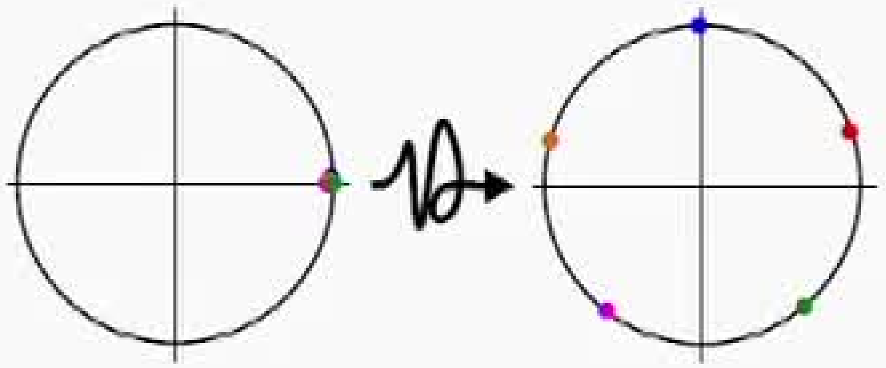


Phase Patterns

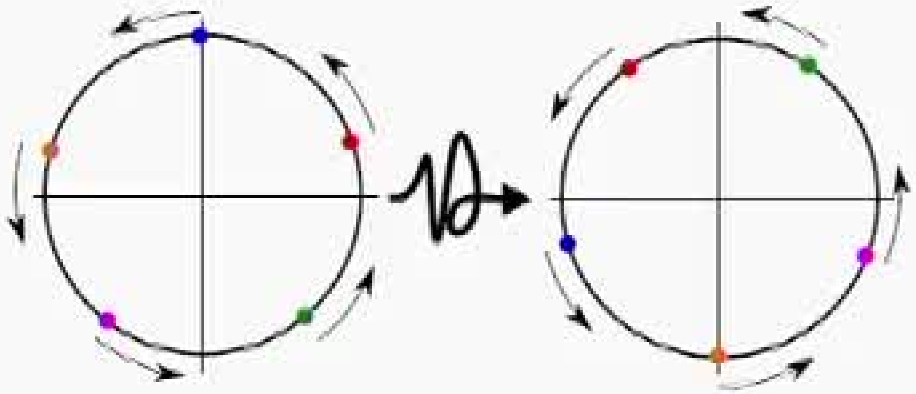
- Collection of phase oscillators $\{\dot{\psi}_i = \omega_i + Z(\psi)u\}$ for $i = 1, \ldots, P$, with $\omega_i \neq \omega_j$ for $i \neq j$.



- ***Objective 1***: Establish a fixed phase relationship. If $\psi_i(0)$ is unknown for all $i = 1, \ldots, N$, steer the oscillators to $\psi_i(T) = \phi_i^\infty$ where ϕ_i^∞ are desired phases for $i = 1, \ldots, P$



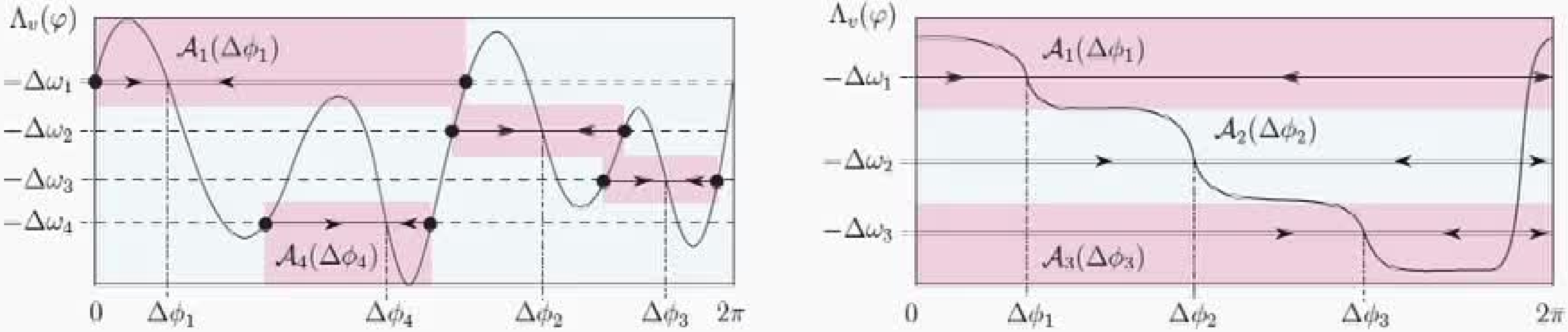
- ***Objective 2***: Maintain a fixed phase relationship. After Objective 1 is established, produce a repeated input that ensures $\psi_i(kT) = \psi_i$ for $i = 1, \ldots, N$ for all $k \in \mathbb{N}$.



Interaction Function Design

- Entrain $\mathcal{F} = \{\dot{\psi}_i = \omega_i + Z(\psi_i)u,\ i = 1, \ldots, P\}$ using $u(t) = v(\Omega t)$
- $\phi_i(t) = \psi_i(t) - \Omega t$ are relative to forcing phase $\theta = \Omega t$
- ***Goal***: Assign $\phi_i(t) = \Delta\phi_i$
- ***Averaging***: $\langle \mathcal{F} \rangle = \{\dot{\varphi}_i = \Delta\omega_i + \Lambda_v(\varphi_i),\ i = 1, \ldots, P\}$, $\varphi_i^\infty = \lim_{t\to\infty} \varphi_i(t)$
- ***Attractivity***: $\mathcal{A}_i(\varphi_i^\infty) \subset [0, 2\pi)$ is the region that attracts to φ_i^∞
- ***Design Criteria***: For $i = 1, \ldots, P$,

$$(i)\ \Lambda_v(\Delta\phi_i) = -\Delta\omega_i, \quad (ii)\ \Lambda_v'(\Delta\phi_i) < 0, \quad (iii)\ \mathcal{A}_i(\Delta\phi_i) = [0, 2\pi)$$



- ***Key Idea***: Design interaction function, reverse-engineer control using PRC

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Coherent Phase Pattern Control Waveform

- Represent Z and v using truncated Fourier series expansions

$$Z(\theta) \approx Z^r(\theta) = \frac{a_0}{2} + \sum_{n=1}^{r} [a_n \cos(n\theta) + b_n \sin(n\theta)],$$

$$v(\theta) \approx v^r(\theta) = \frac{c_0}{2} + \sum_{n=1}^{r} [c_n \cos(n\theta) + d_n \sin(n\theta)],$$

- Series for $\Lambda_v(\varphi) = \langle Z(\theta+\varphi)v(\theta)\rangle$ obtained using trigonometric angle sum identities and the orthogonality of the Fourier basis

$$\Lambda_v^r(\varphi) = \frac{f_0}{2} + \frac{1}{2}\sum_{n=1}^{r} f_n \cos(n\varphi) + \frac{1}{2}\sum_{n=1}^{r} g_n \sin(n\varphi),$$

where $f_0 = \frac{a_0 c_0}{2}, \quad f_n = a_n c_n + b_n d_n, \quad g_n = b_n c_n - a_n d_n.$

- Coefficients of truncated Fourier series for the control waveform $v^r(\theta)$ are

$$c_0 = 4\frac{f_0}{a_0}\chi_{[a_0 \neq 0]}, \quad c_n = 2\frac{(f_n a_n + b_n g_n)\chi_{[a_n^2+b_n^2 \neq 0]}}{a_n^2 + b_n^2}, \quad d_n = 2\frac{(f_n b_n - a_n g_n)\chi_{[a_n^2+b_n^2 \neq 0]}}{a_n^2 + b_n^2},$$

where $\chi_A = 1$ if A is true, and $\chi_A = 0$ otherwise.

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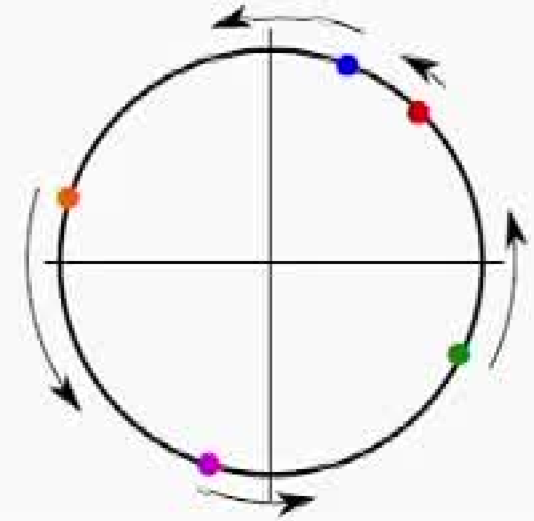

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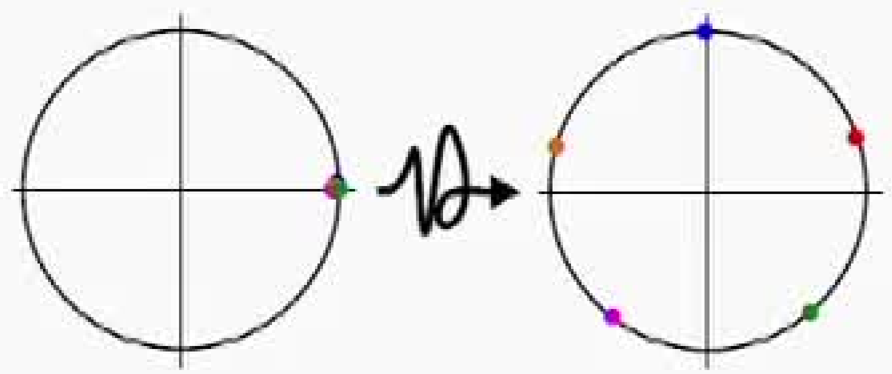
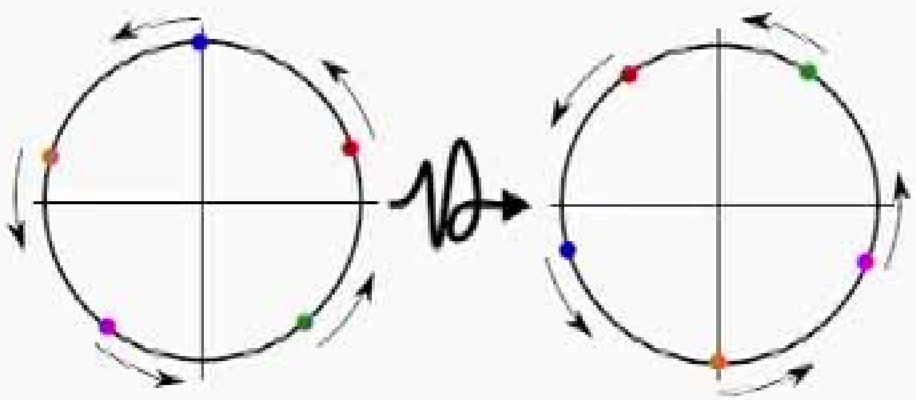


- ***Objective 2***: Maintain a fixed phase relationship. After Objective 1 is established, produce a repeated input that ensures $\psi_i(kT) = \psi_i$ for $i = 1, \dots, N$ for all $k \in \mathbb{N}$.



Hodgkin-Huxley Spiking Neuron Model

Example: Hodgkin-Huxley Equations (1952)

Model of action potential propagation in a squid giant axon:

$$
\begin{aligned}
c\dot{V} &= I_b + I(t) - \overline{g}_{Na}h(V - V_{Na})m^3 - \overline{g}_K(V - V_k)n^4 \\
&\quad \ldots - \overline{g}_L(V - V_L) \\
\dot{m} &= a_m(V)(1 - m) - b_m(V)m, \\
\dot{h} &= a_h(V)(1 - h) - b_h(V)h, \\
\dot{n} &= a_n(V)(1 - n) - b_n(V)n,
\end{aligned}
$$

$$
\begin{aligned}
a_m(V) &= 0.1(V + 40)/(1 - \exp(-(V + 40)/10)), \\
b_m(V) &= 4\exp(-(V + 65)/18), \\
a_h(V) &= 0.07\exp(-(V + 65)/20), \\
b_h(V) &= 1/(1 + \exp(-(V + 35)/10)), \\
a_n(V) &= 0.01(V + 55)/(1 - \exp(-(V + 55)/10)), \\
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$V_{Na} = 50\,mV, V_K = -77\,mV, V_L = -54.4\,mV,$
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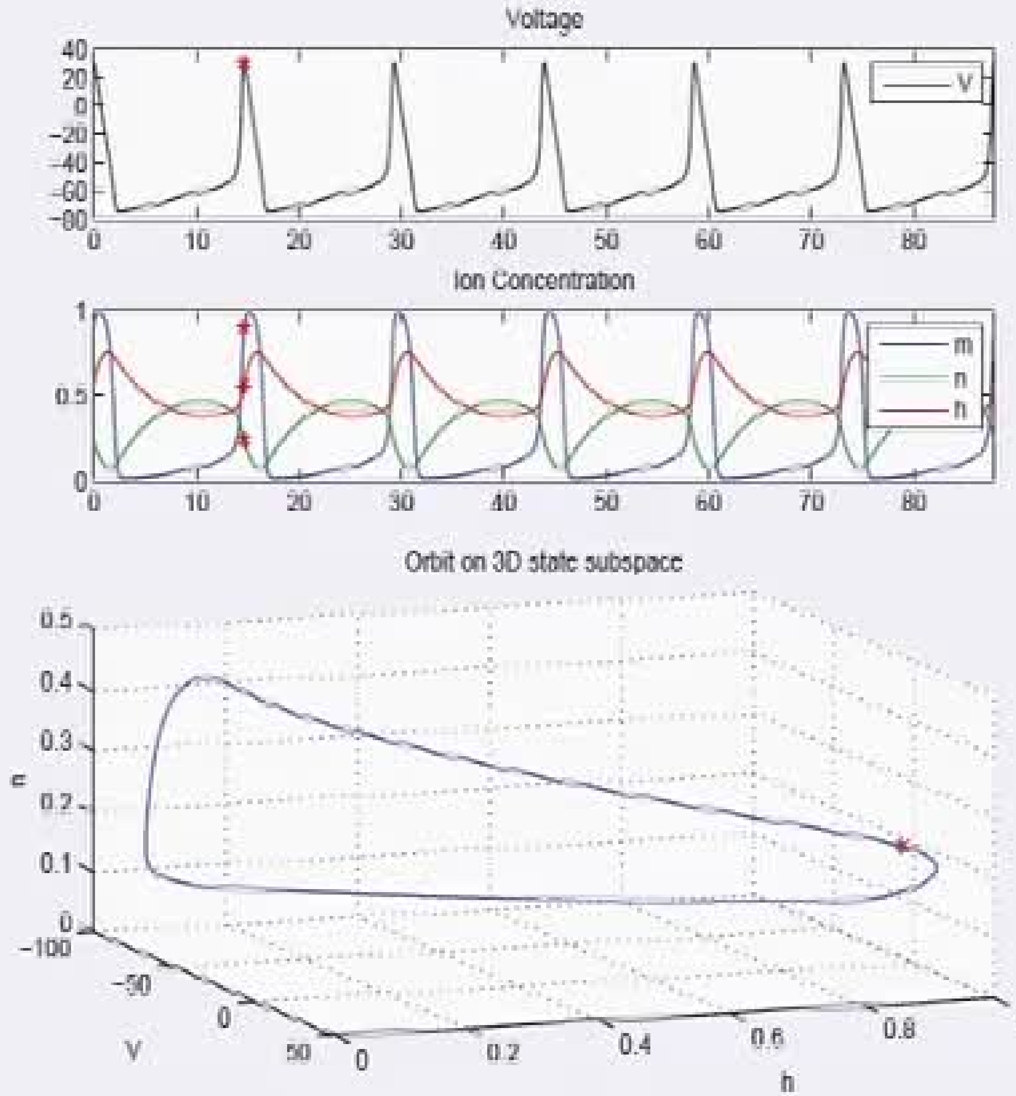

Pattern Complexity

- Complexity of phase patterns is limited by information/complexity in the PRC
- Consider theoretical example of a PRC with sawtooth shape
- Interaction function $\Lambda_v^r(\varphi)$ designed using a $r = 20$ order Fourier series fit
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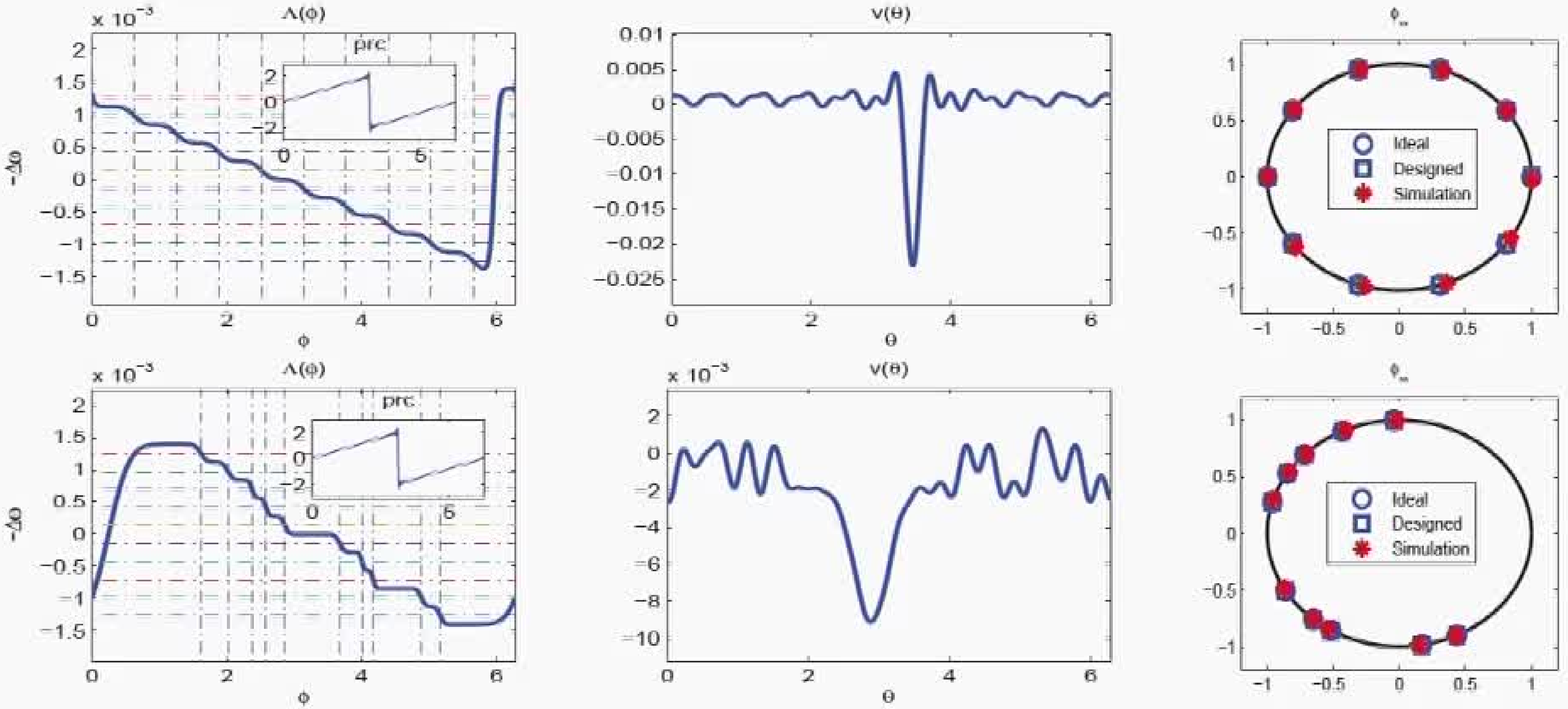
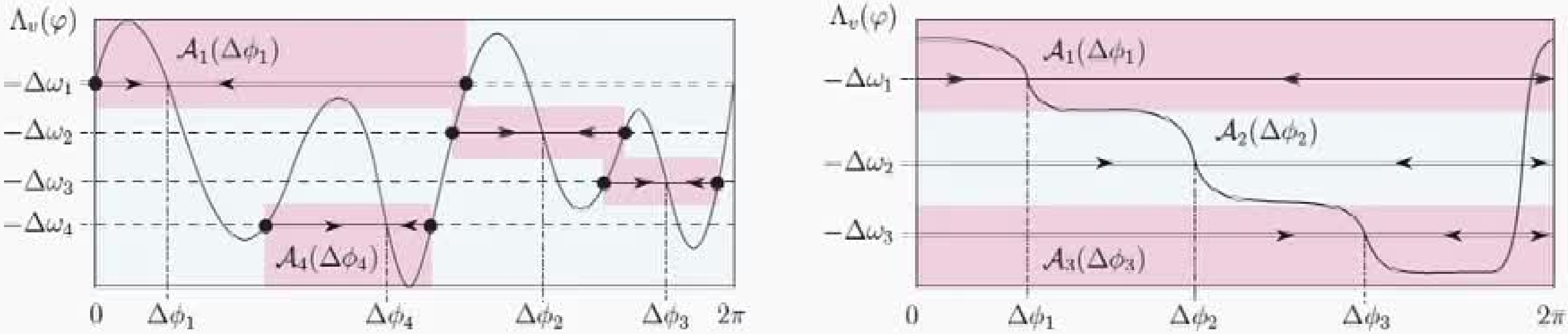


Interaction Function Design

- Entrain $\mathcal{F} = \{\dot{\psi}_i = \omega_i + Z(\psi_i)u,\ i = 1, \ldots, P\}$ using $u(t) = v(\Omega t)$
- $\phi_i(t) = \psi_i(t) - \Omega t$ are relative to forcing phase $\theta = \Omega t$
- ***Goal***: Assign $\phi_i(t) = \Delta\phi_i$
- ***Averaging***: $\langle \mathcal{F} \rangle = \{\dot{\varphi}_i = \Delta\omega_i + \Lambda_v(\varphi_i),\ i = 1, \ldots, P\}$, $\varphi_i^\infty = \lim_{t\to\infty} \varphi_i(t)$
- ***Attractivity***: $\mathcal{A}_i(\varphi_i^\infty) \subset [0, 2\pi)$ is the region that attracts to φ_i^∞
- ***Design Criteria***: For $i = 1, \ldots, P$,

$$(i)\ \Lambda_v(\Delta\phi_i) = -\Delta\omega_i, \quad (ii)\ \Lambda_v'(\Delta\phi_i) < 0, \quad (iii)\ \mathcal{A}_i(\Delta\phi_i) = [0, 2\pi)$$



- ***Key Idea***: Design interaction function, reverse-engineer control using PRC

Coherent Phase Pattern Control Waveform

- Represent Z and v using truncated Fourier series expansions

$$Z(\theta) \approx Z^r(\theta) = \frac{a_0}{2} + \sum_{n=1}^{r}[a_n \cos(n\theta) + b_n \sin(n\theta)],$$

$$v(\theta) \approx v^r(\theta) = \frac{c_0}{2} + \sum_{n=1}^{r}[c_n \cos(n\theta) + d_n \sin(n\theta)],$$

- Series for $\Lambda_v(\varphi) = \langle Z(\theta+\varphi)v(\theta)\rangle$ obtained using trigonometric angle sum identities and the orthogonality of the Fourier basis

$$\Lambda_v^r(\varphi) = \frac{f_0}{2} + \frac{1}{2}\sum_{n=1}^{r} f_n \cos(n\varphi) + \frac{1}{2}\sum_{n=1}^{r} g_n \sin(n\varphi),$$

where $f_0 = \dfrac{a_0 c_0}{2}, \quad f_n = a_n c_n + b_n d_n, \quad g_n = b_n c_n - a_n d_n.$

- Coefficients of truncated Fourier series for the control waveform $v^r(\theta)$ are

$$c_0 = 4\frac{f_0}{a_0}\chi_{[a_0 \neq 0]}, \quad c_n = 2\frac{(f_n a_n + b_n g_n)\chi_{[a_n^2+b_n^2 \neq 0]}}{a_n^2 + b_n^2}, \quad d_n = 2\frac{(f_n b_n - a_n g_n)\chi_{[a_n^2+b_n^2 \neq 0]}}{a_n^2 + b_n^2},$$

where $\chi_A = 1$ if A is true, and $\chi_A = 0$ otherwise.

Phase Selection with Electrochemical Reactions

- Electrochemical oscillation: dissolution of nickel electrodes in concentrated sulfuric acid solution

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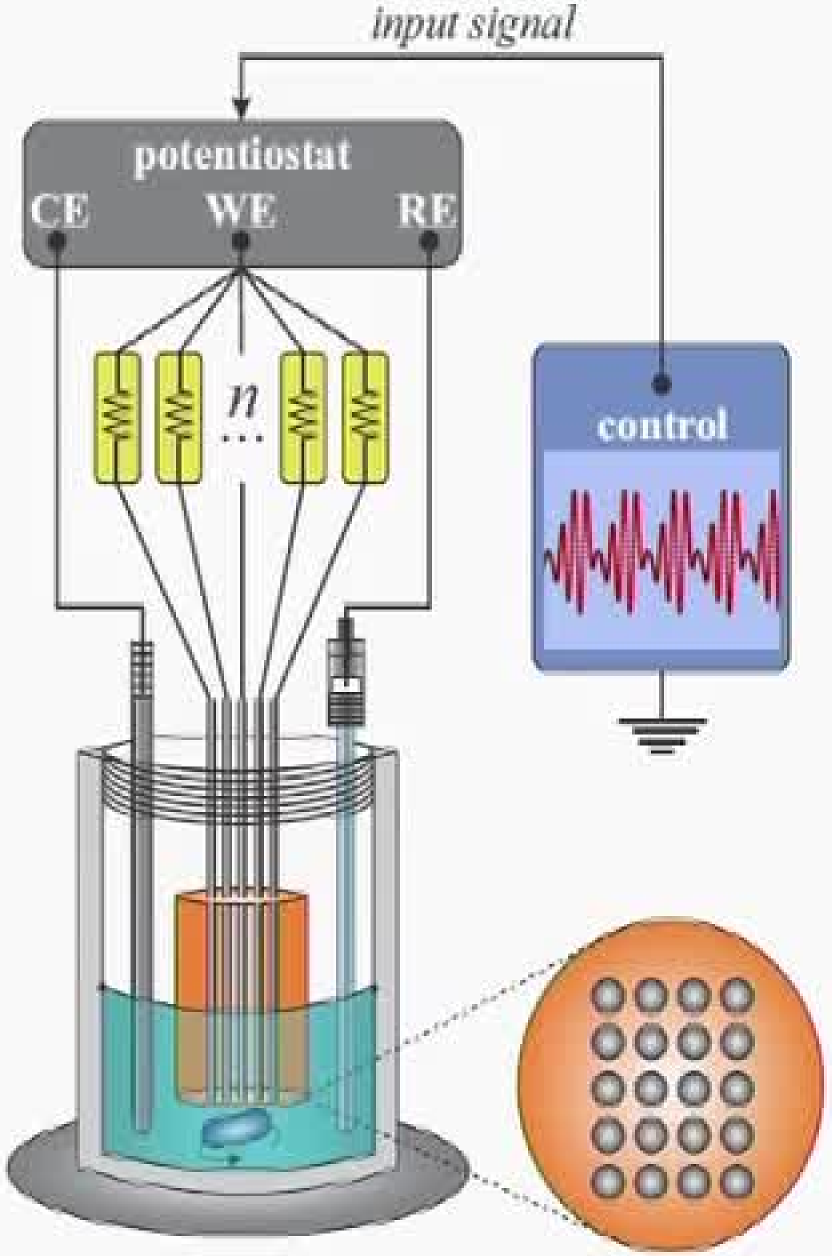

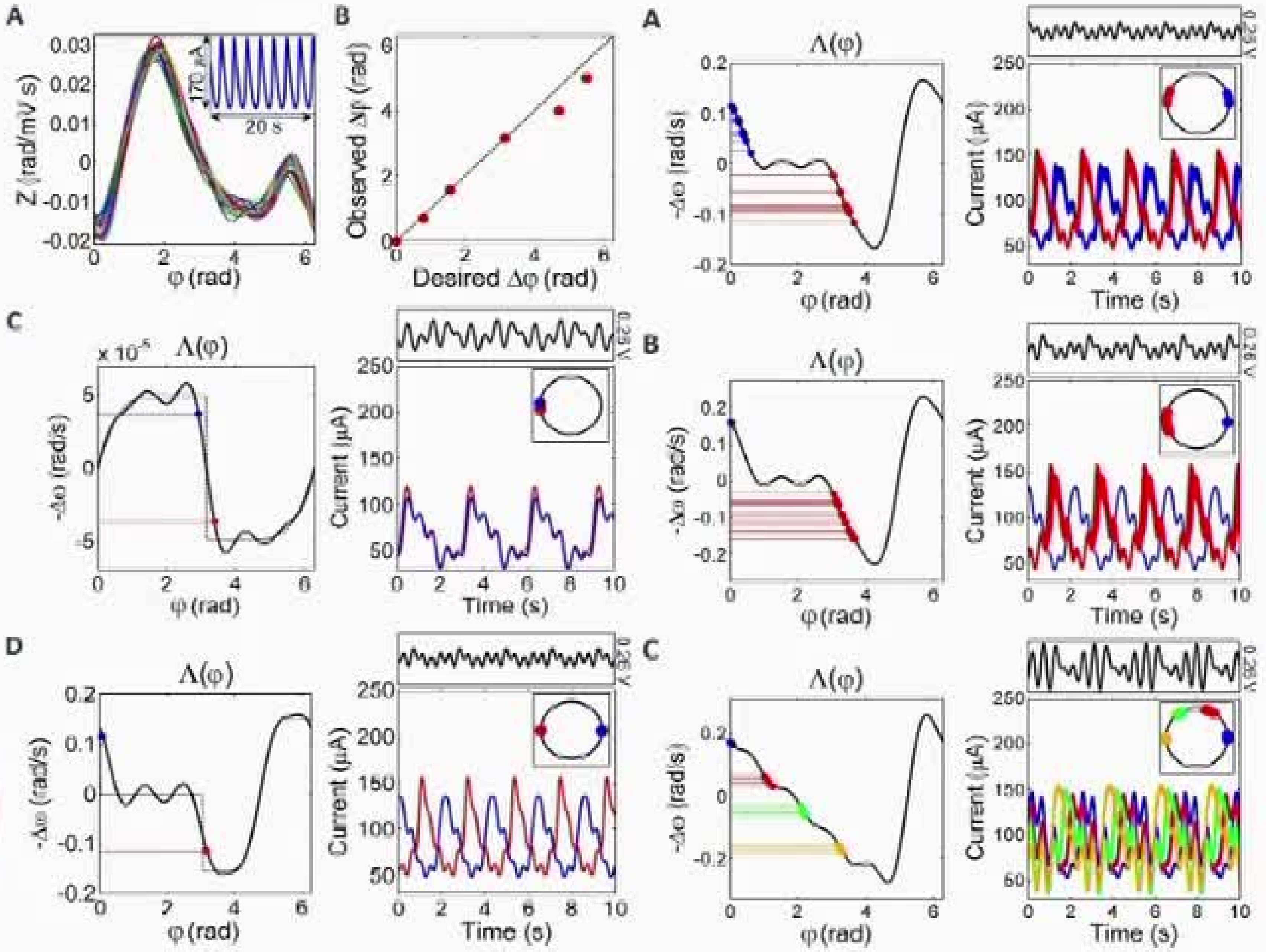

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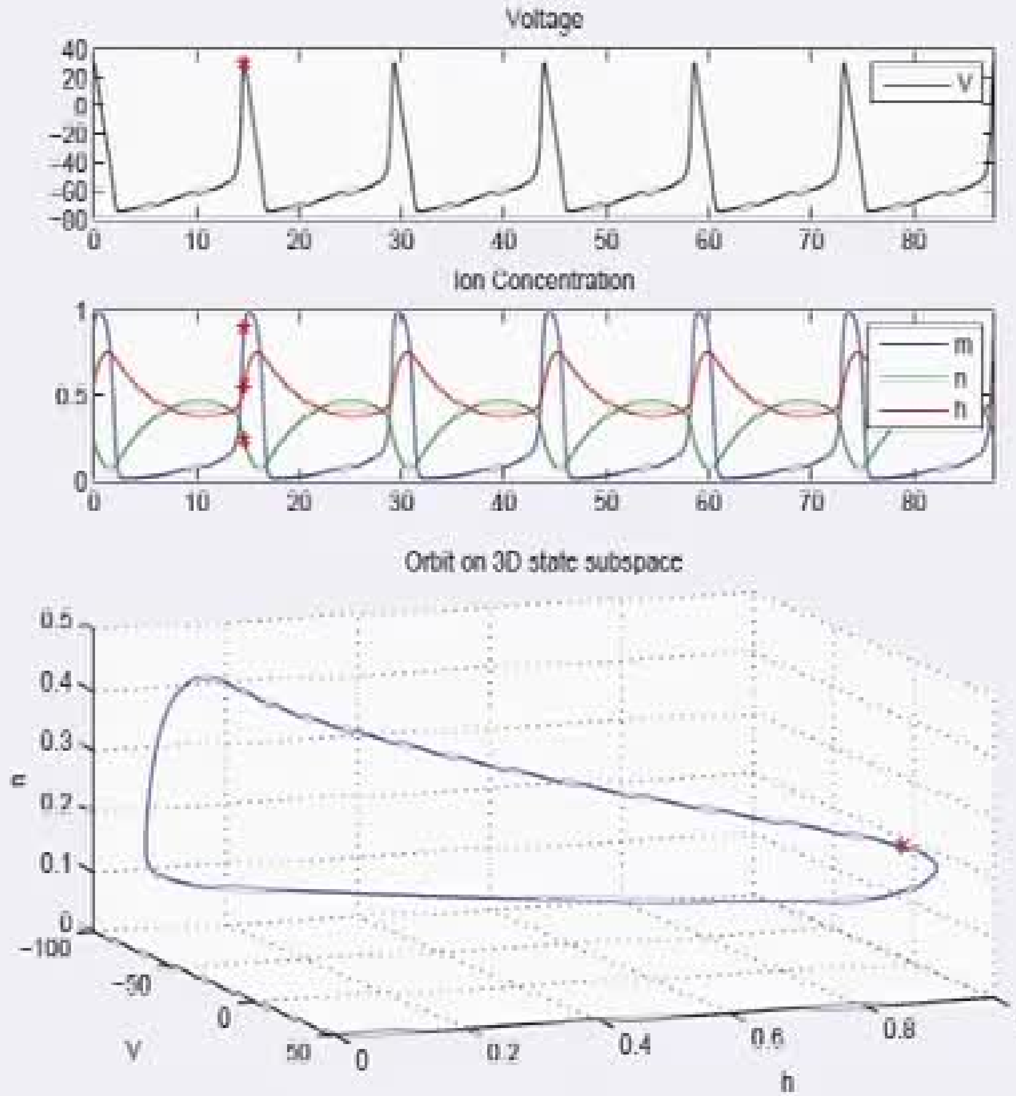

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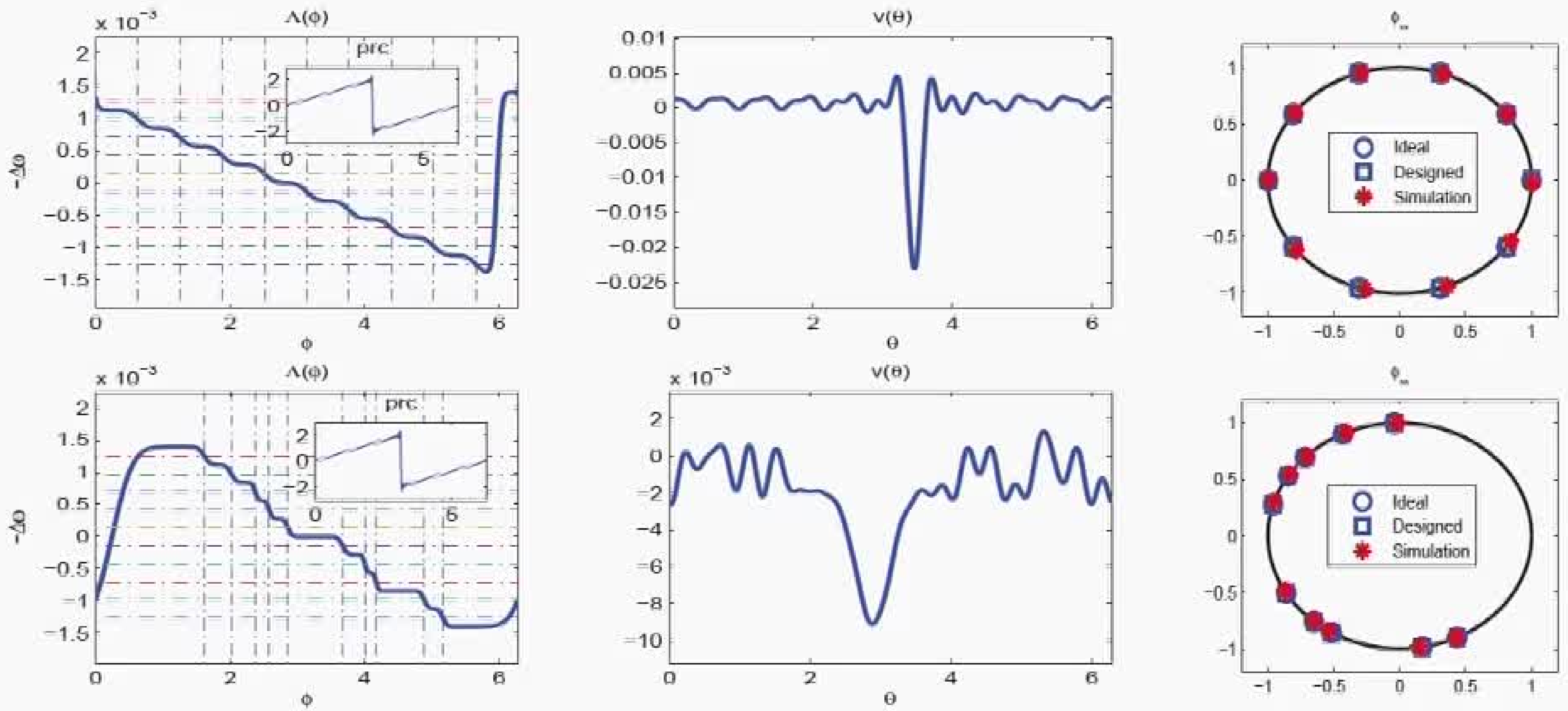
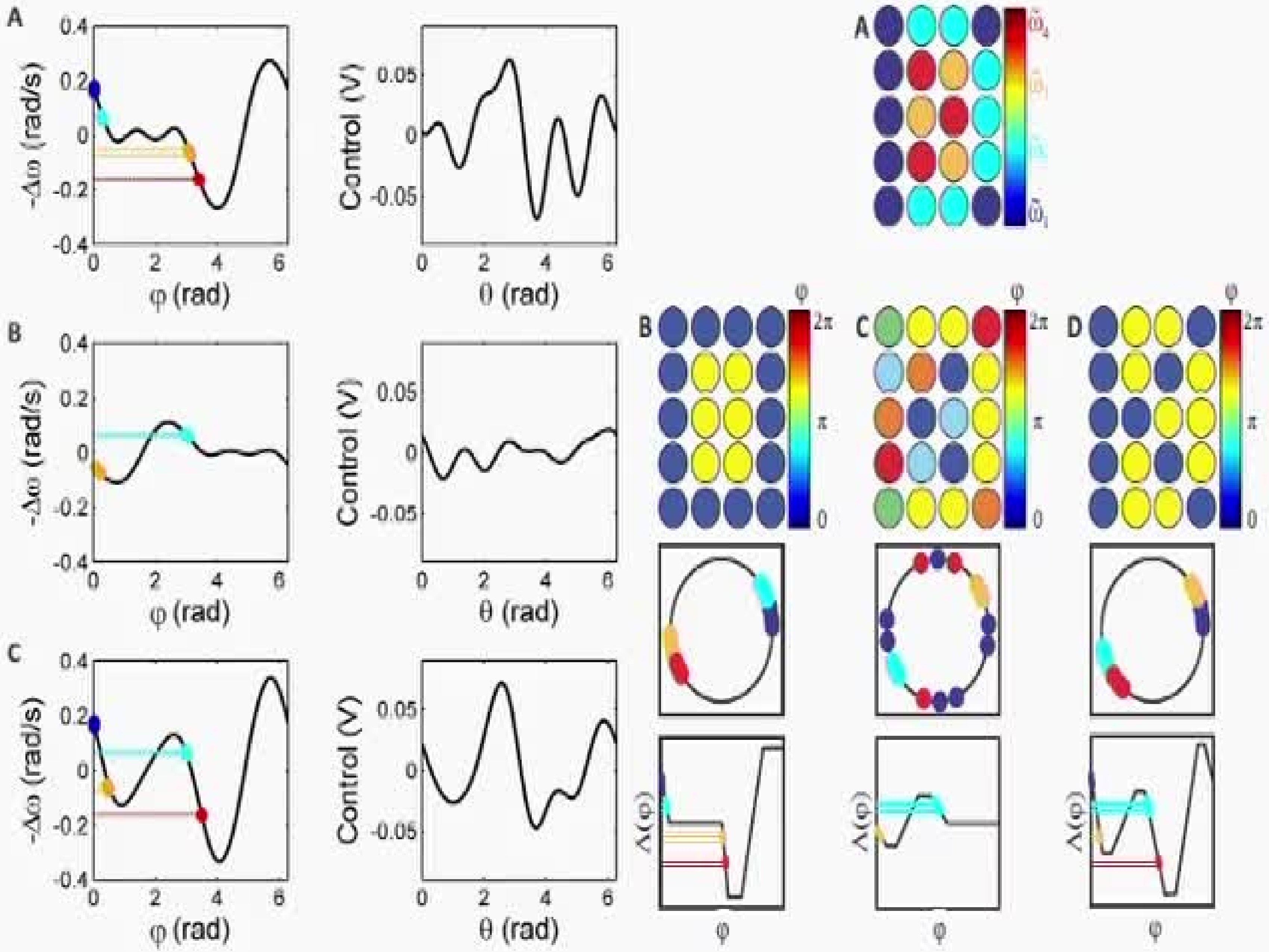


Phase Selection with Electrochemical Reactions

- Pattern switching control



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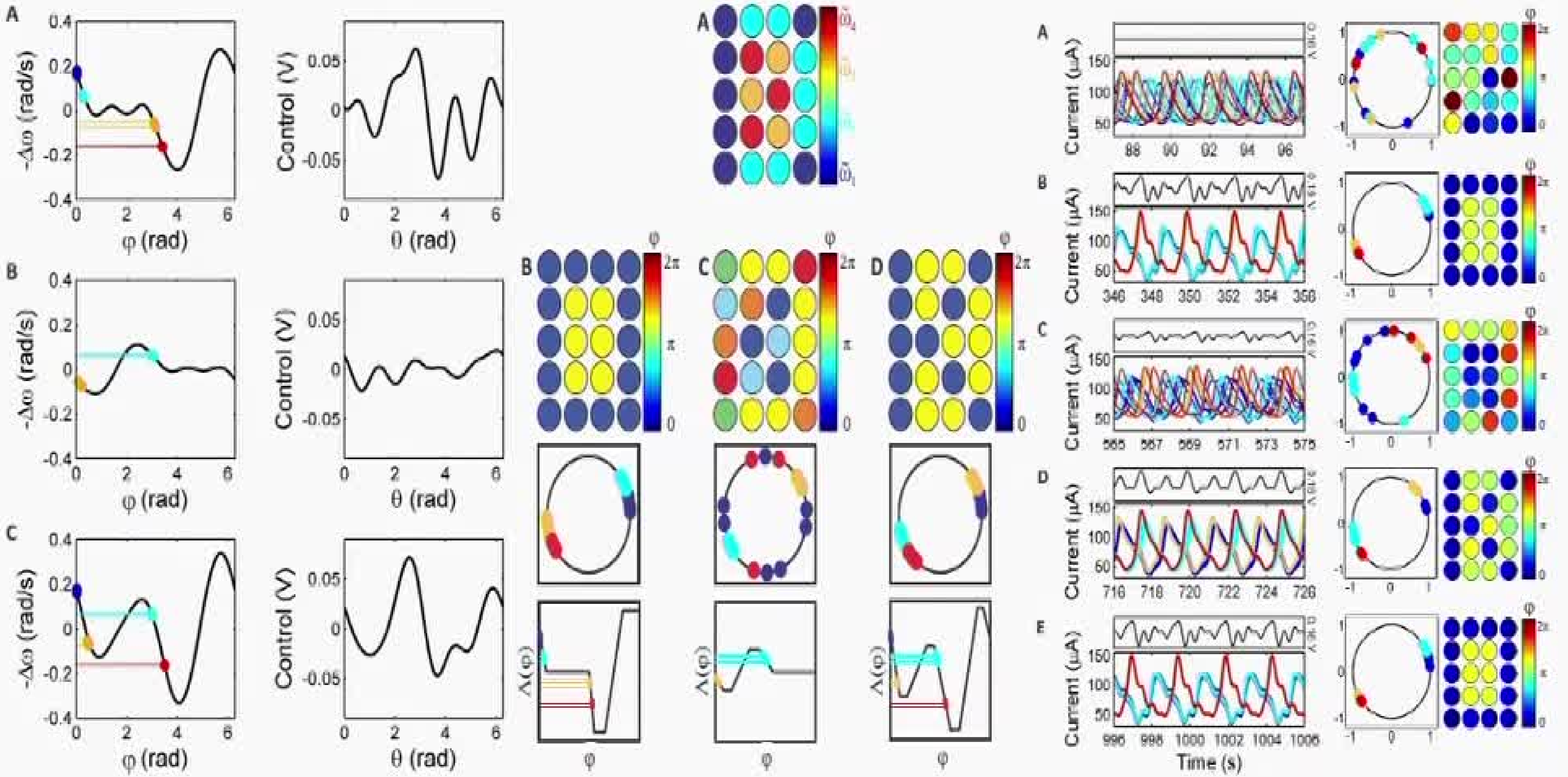

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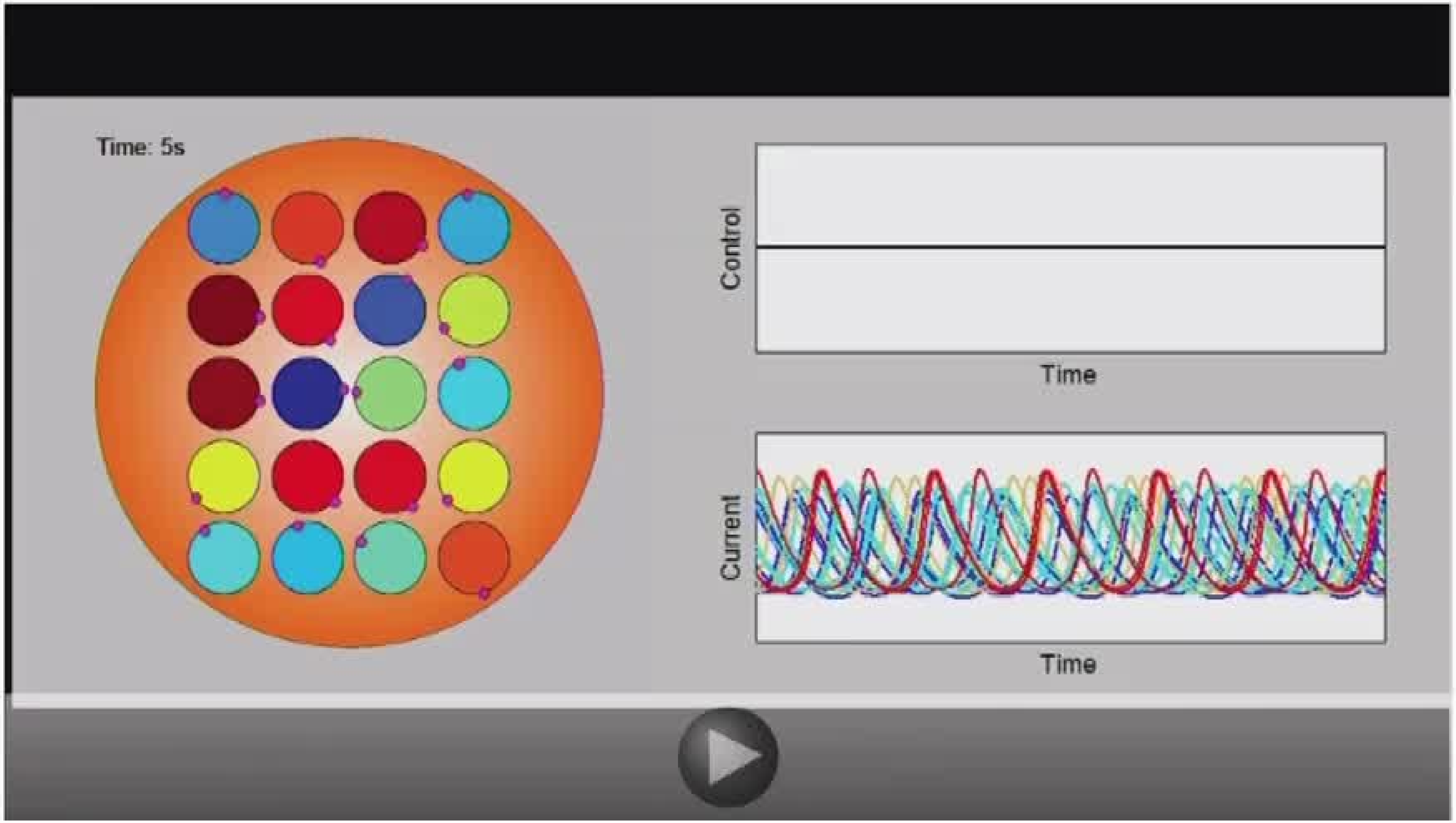

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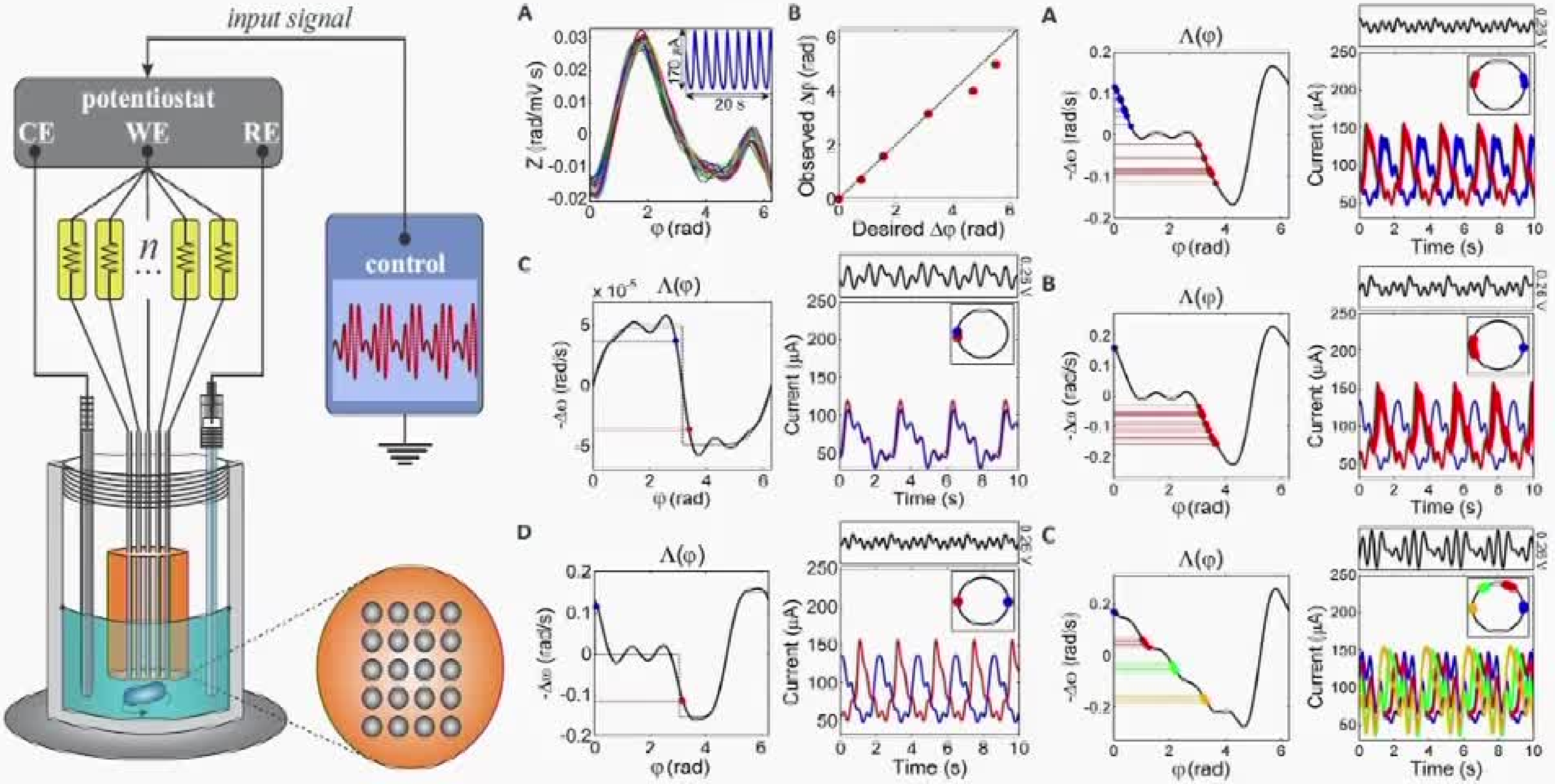

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2 Phase-Selective Entrainment in Oscillator Assemblies

3 Demand Response for Power Distribution

Phase Selection with Electrochemical Reactions

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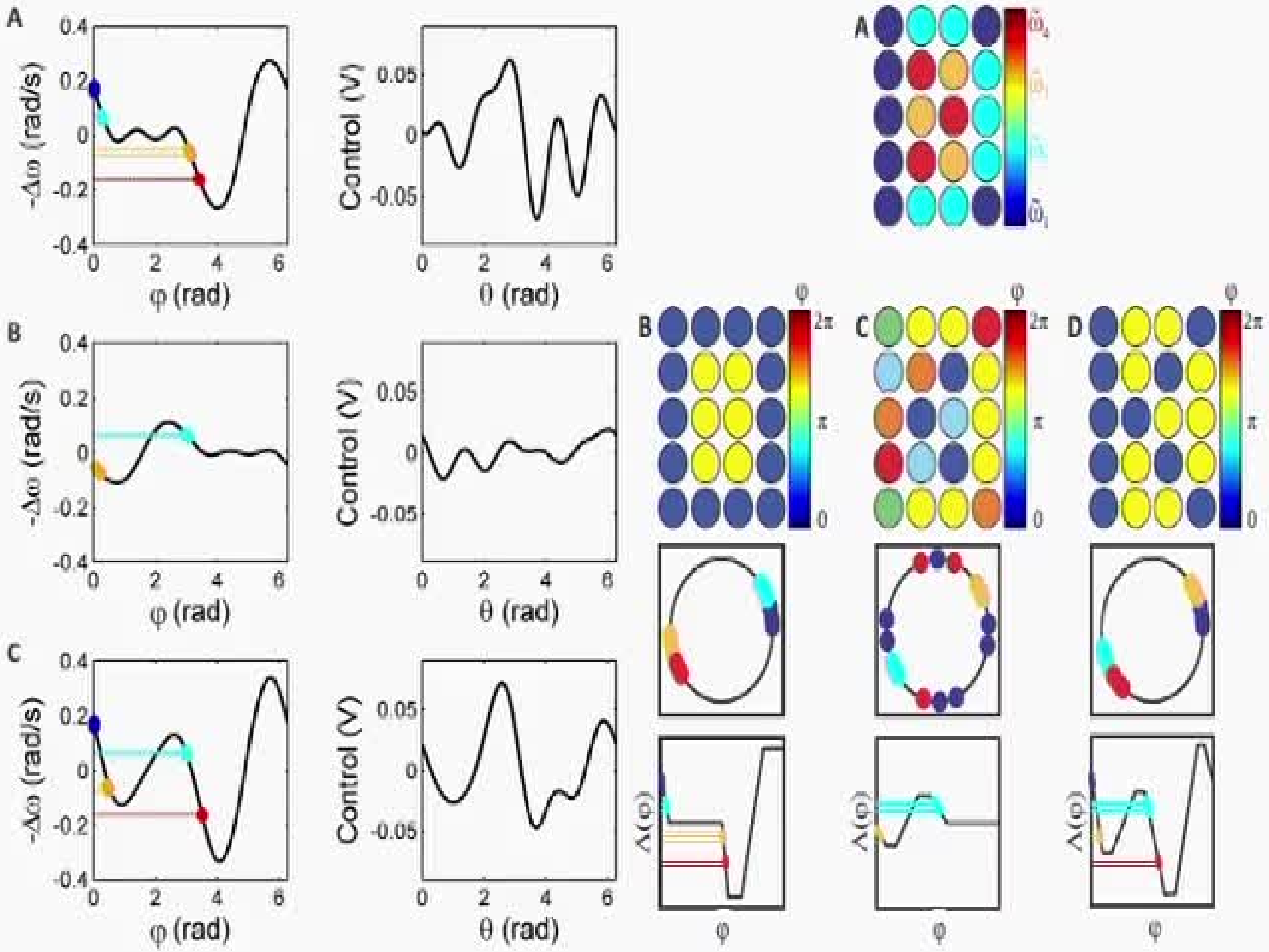

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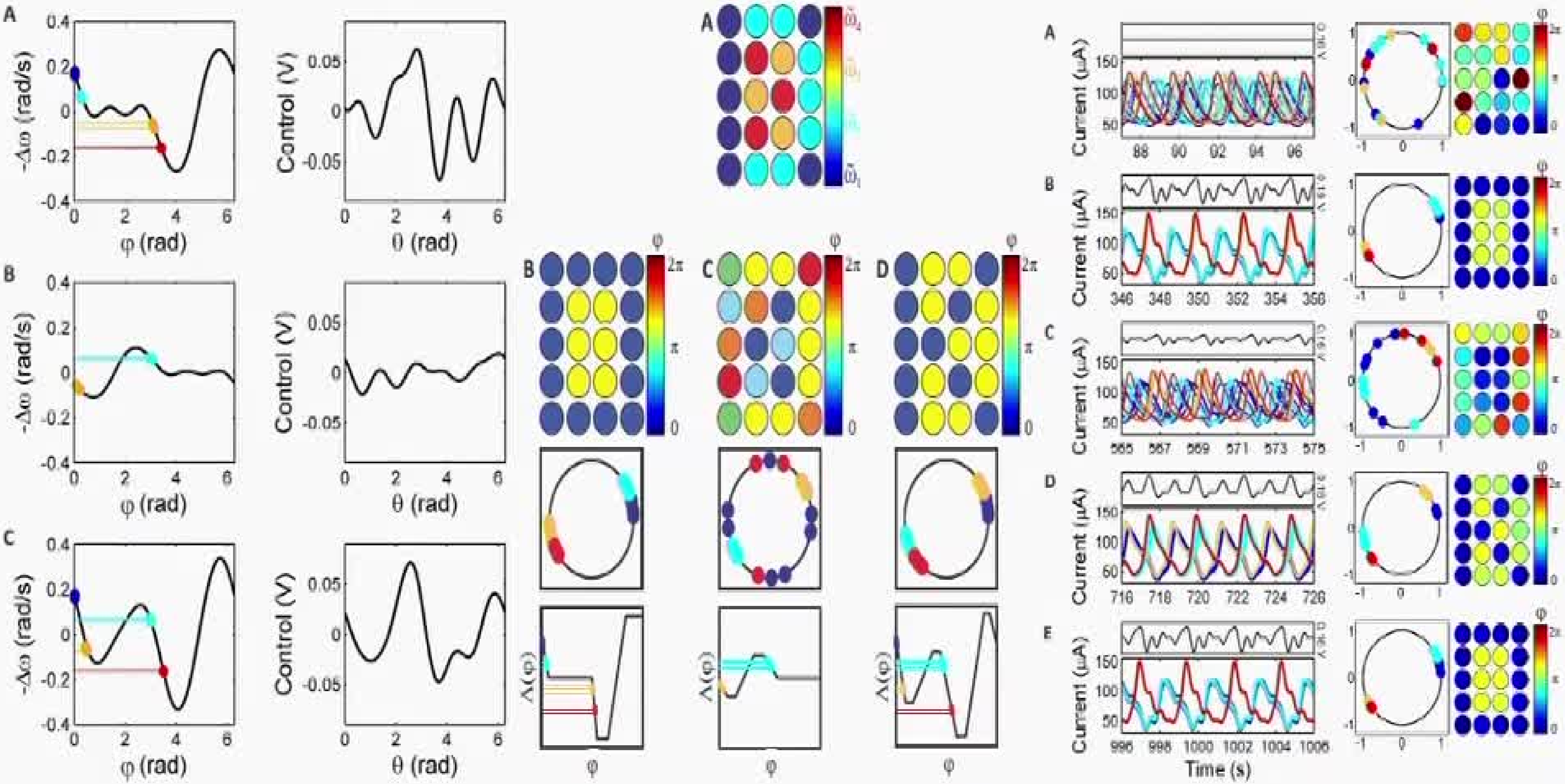

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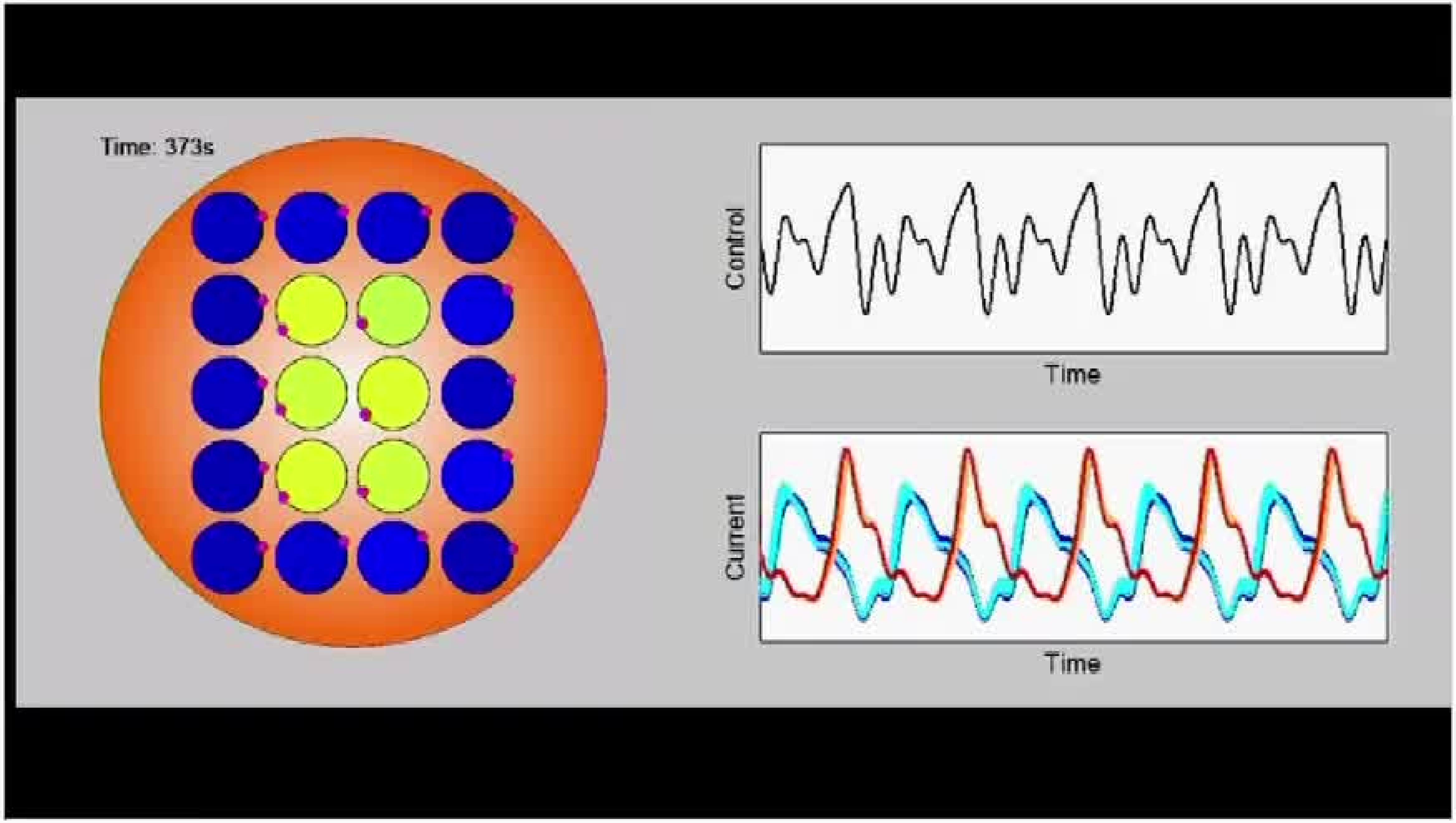

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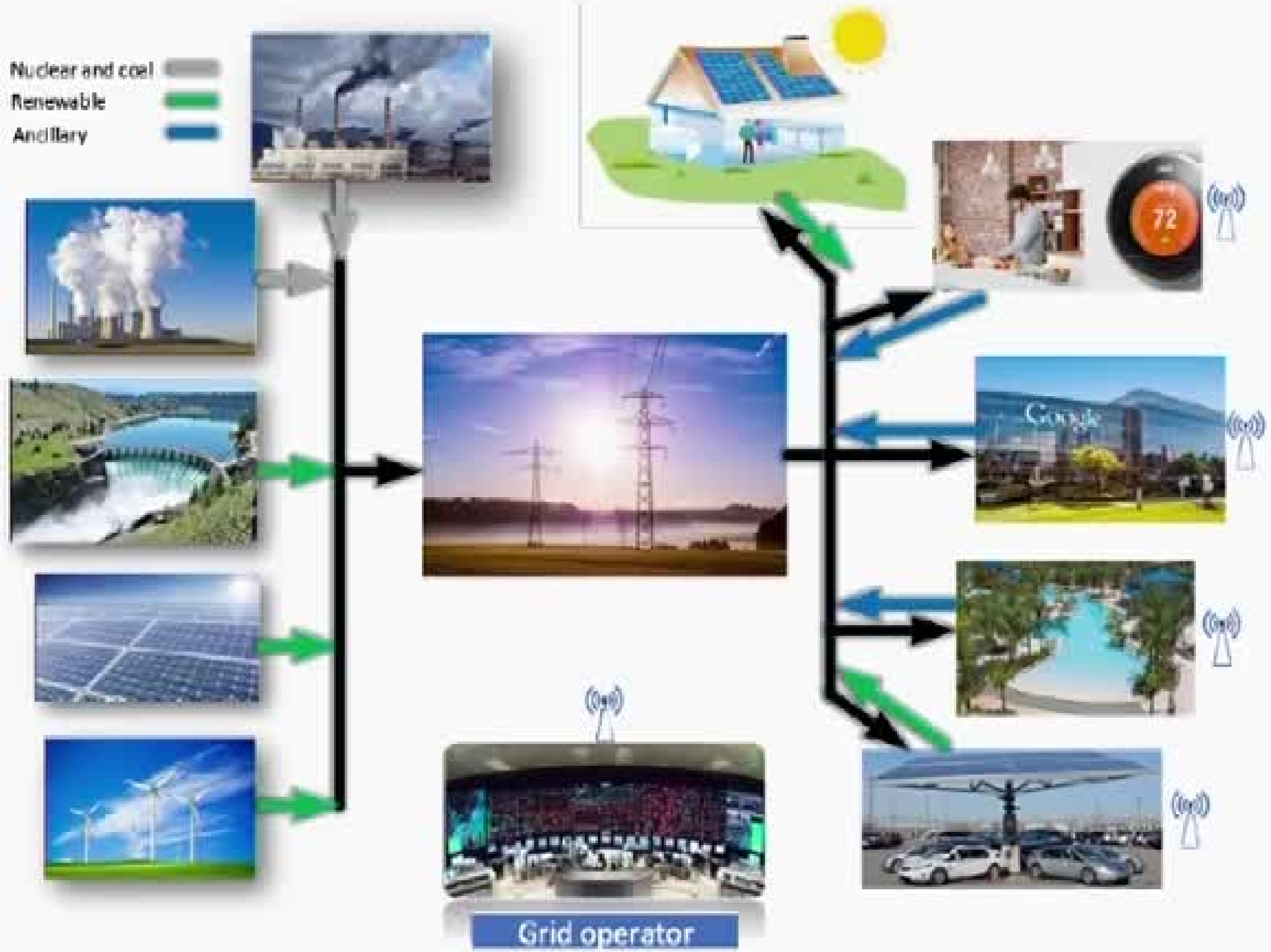
Demand Response

- Increasing penetration of clean, renewable energy (20% renewables by 2030)



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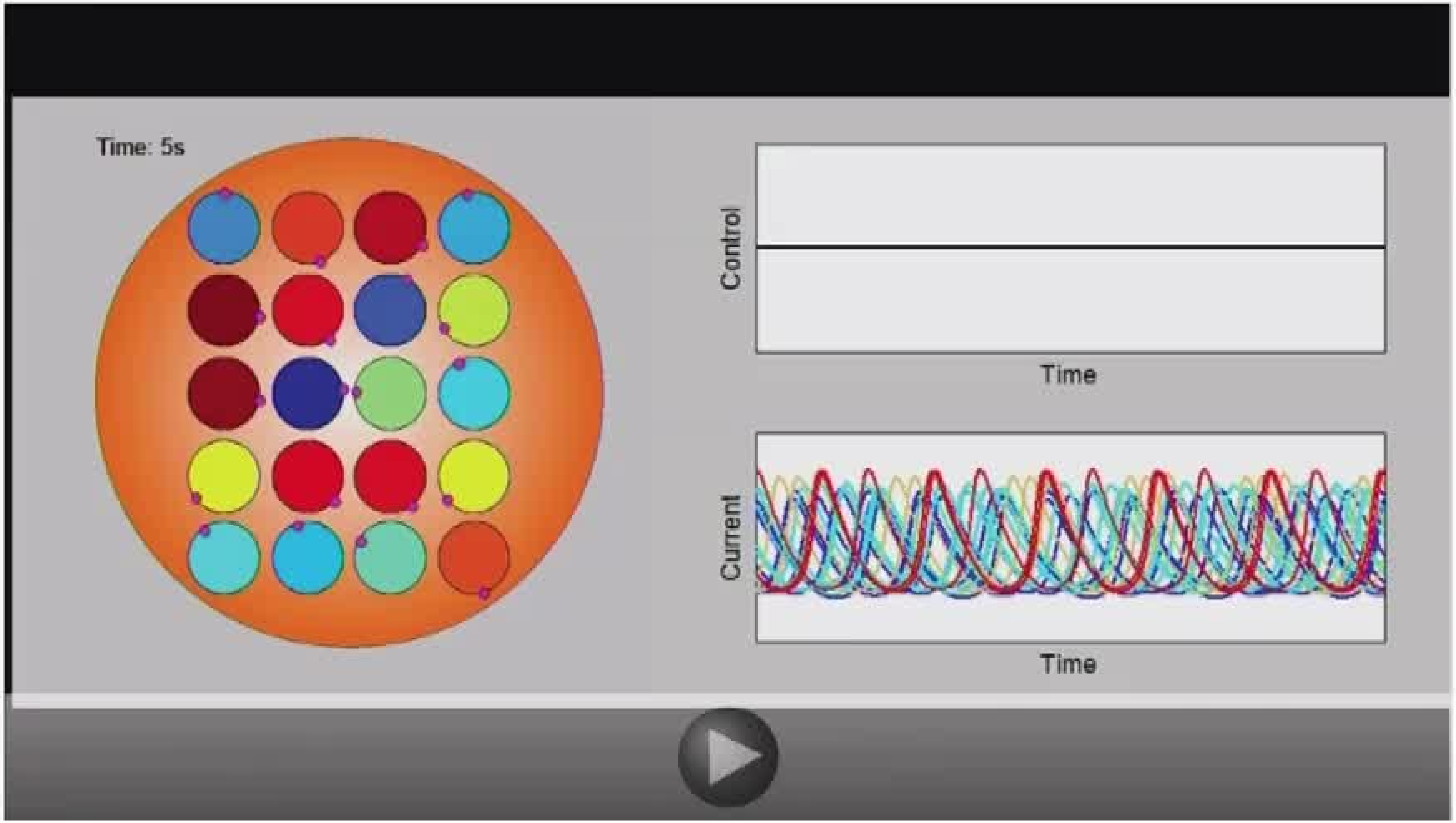

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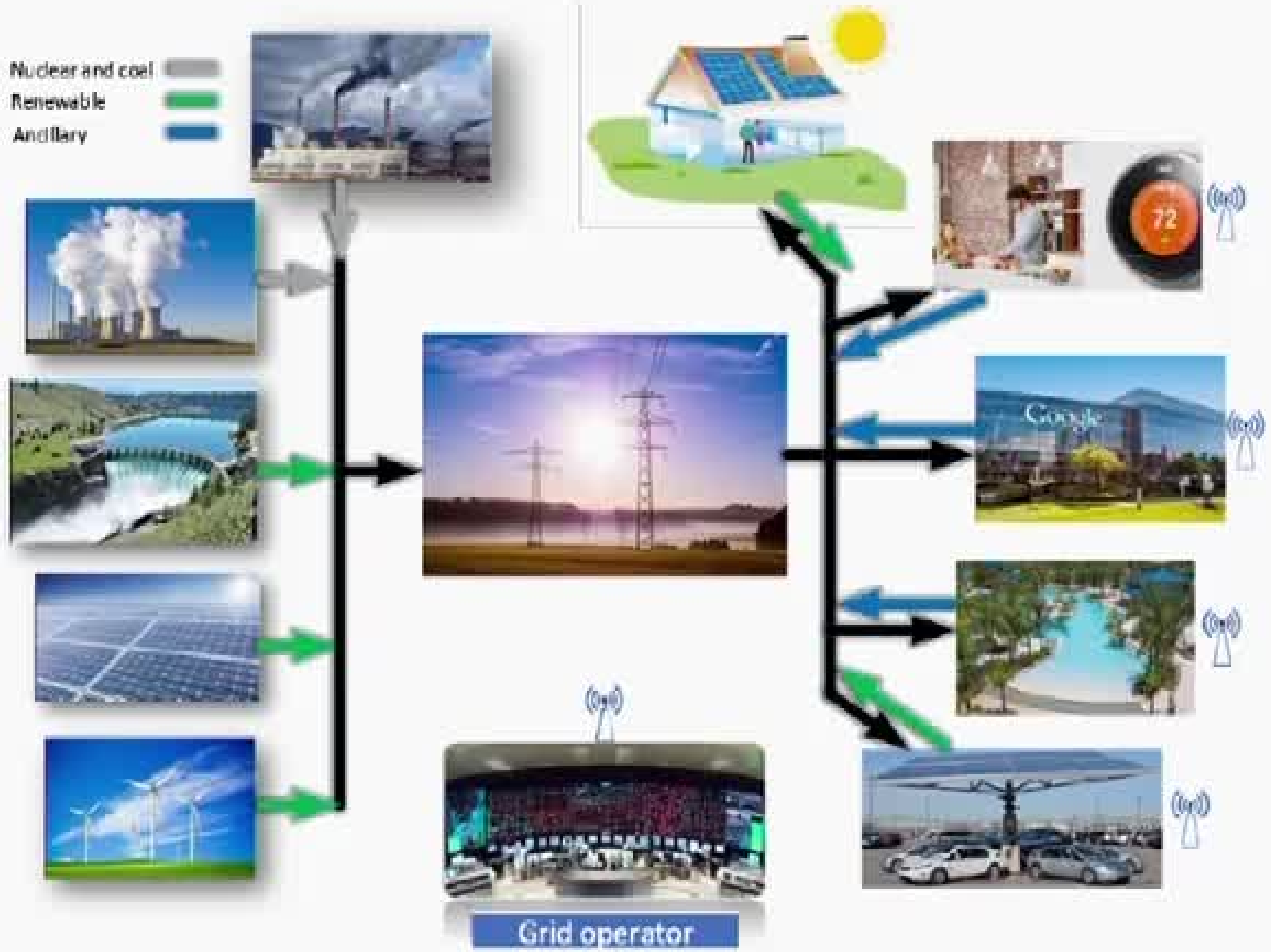

- Distributed generation, microgrids, new technology - "smart grid"
- **Demand Response** provides "ancillary services" to balance loads on the grid

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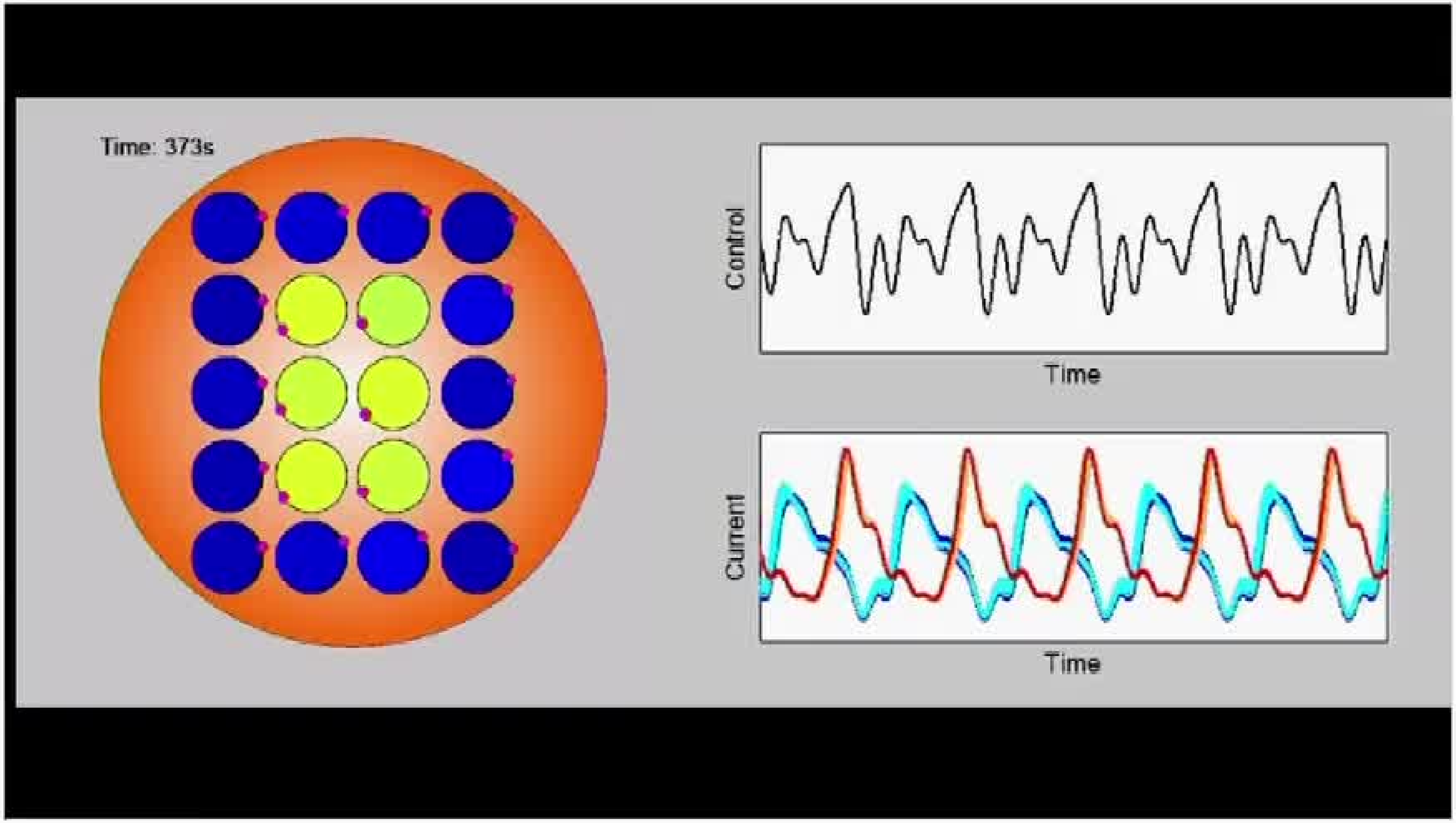

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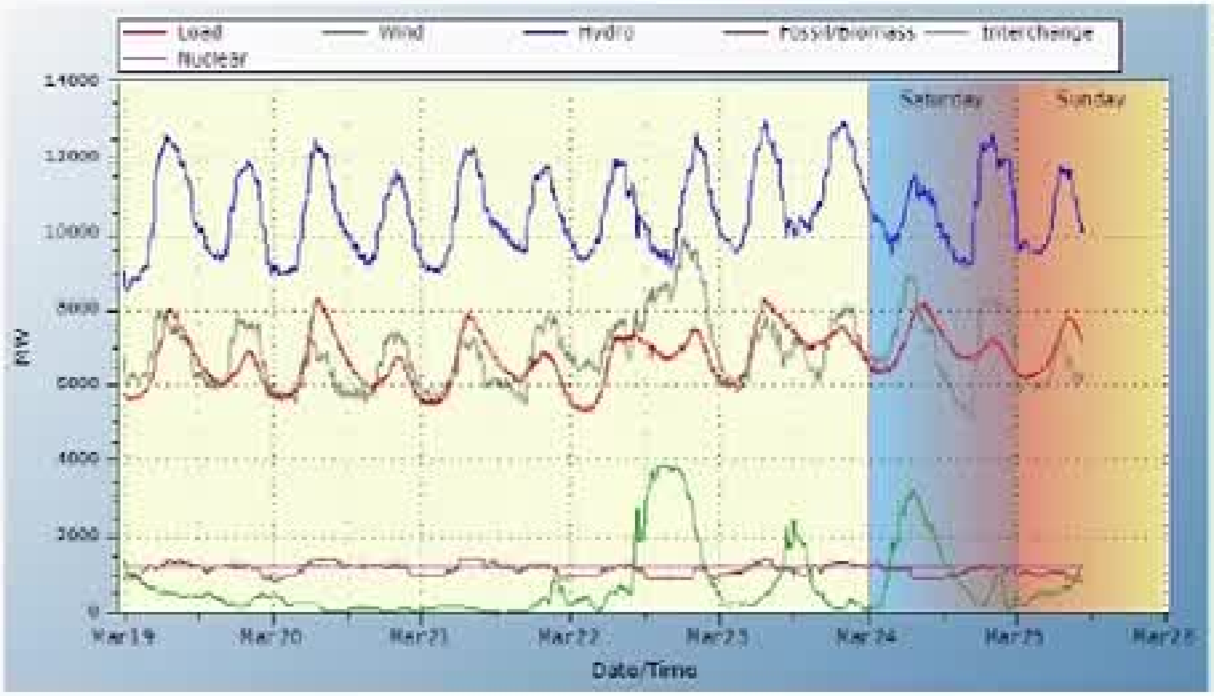
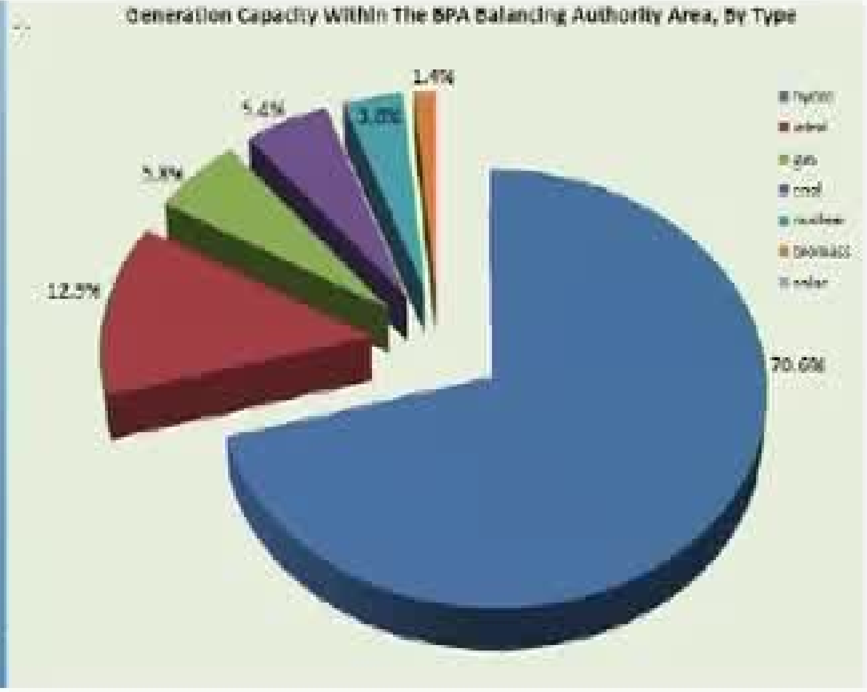
Demand Response

- Bonneville Power Administration Balancing Authority Load and Total Wind, Hydro, Fossil/Biomass, Nuclear Generation, and Net Interchange





- We want responsive regulation using deferable loads
- Thermostatically Controlled Loads (TCLs), e.g. air conditioners, refrigerators
- The control policy should be **open loop** (no state feedback)
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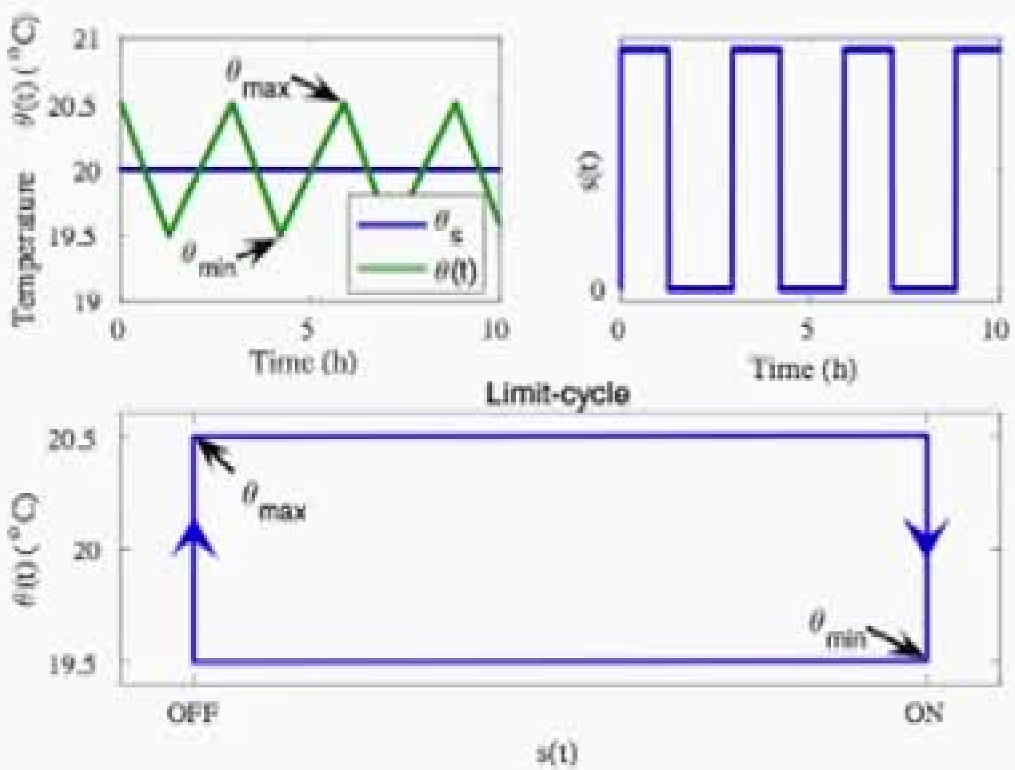
Modeling TCL as an Oscillator

- Consider air conditioner units
- Hybrid-states dynamical model

$$\dot{\theta}(t) = -\frac{1}{RC}[\theta(t) - \theta_a + s(t)PR]$$

$$s(t) = \begin{cases} 0 & \text{if } \theta(t) < \theta_{\min} \\ 1 & \text{if } \theta(t) > \theta_{\max} \\ s(t) & \text{otherwise} \end{cases}$$



Parameter	Meaning	Value
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R	thermal resistance	2°C/kW
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P	energy transfer rate	14 kW
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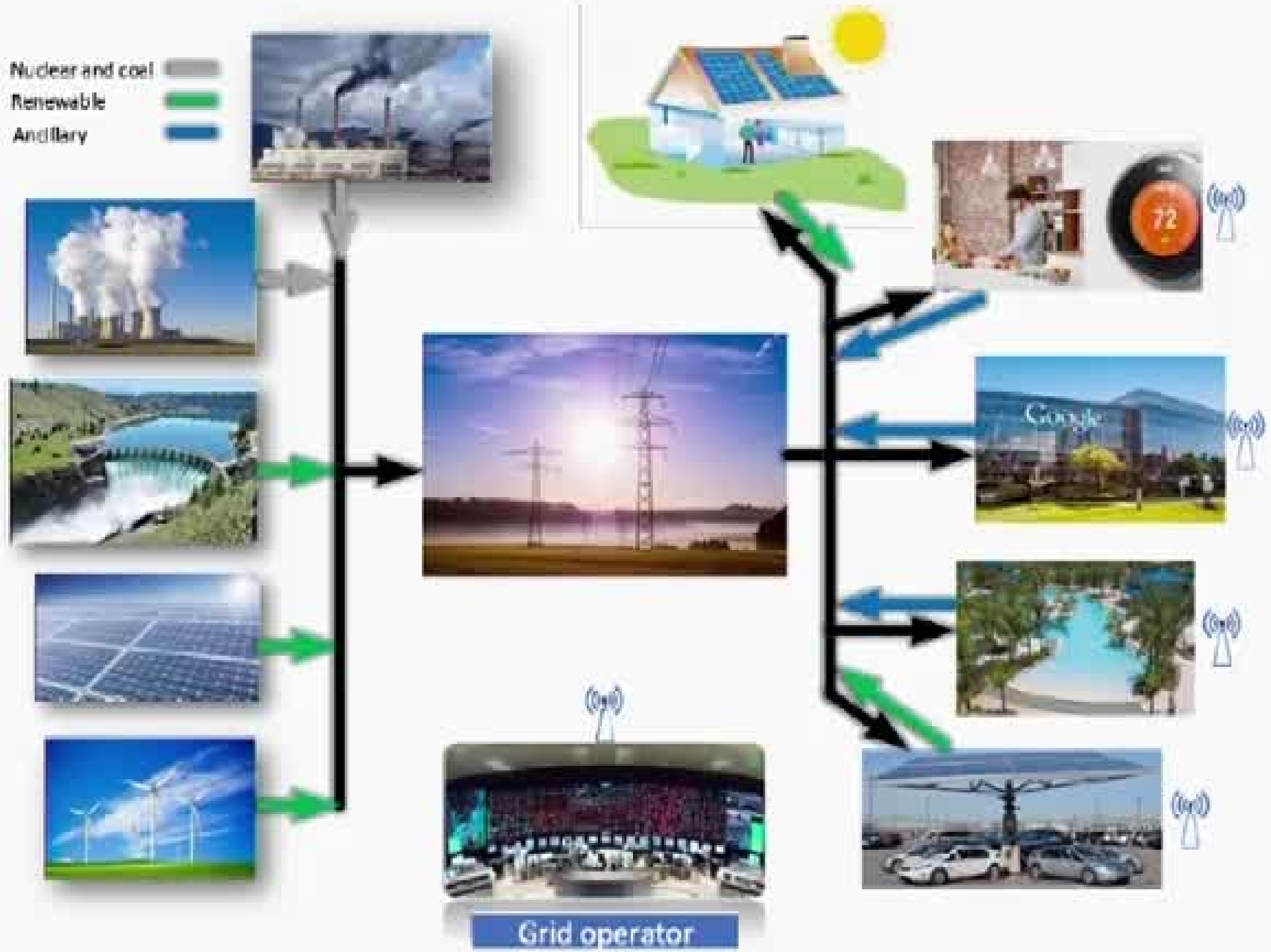

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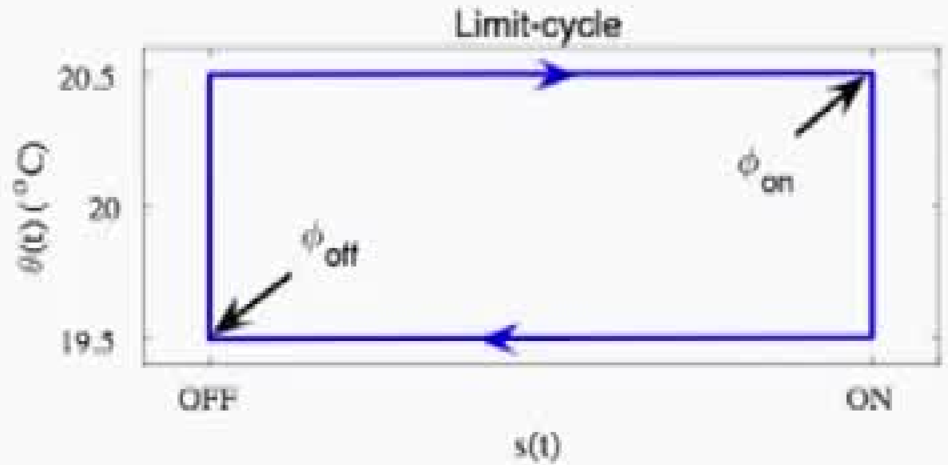
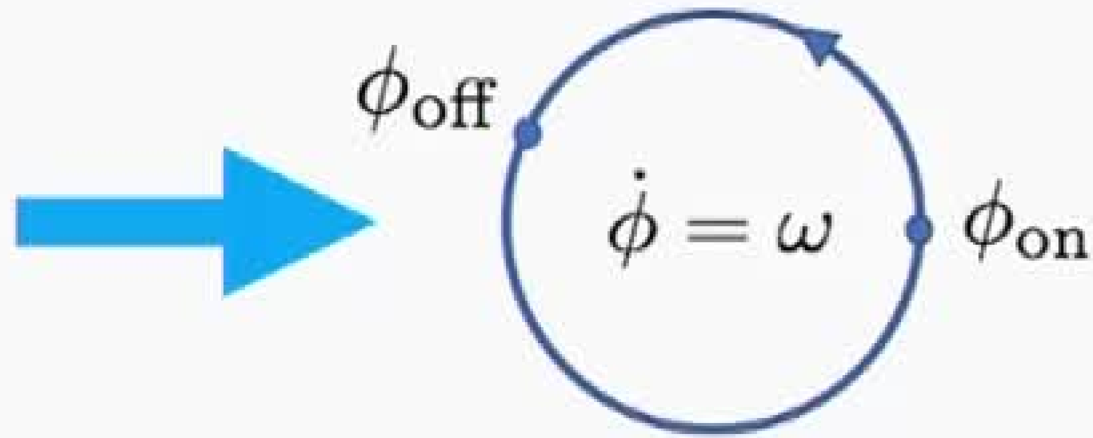
- Use phase coordinate transformation





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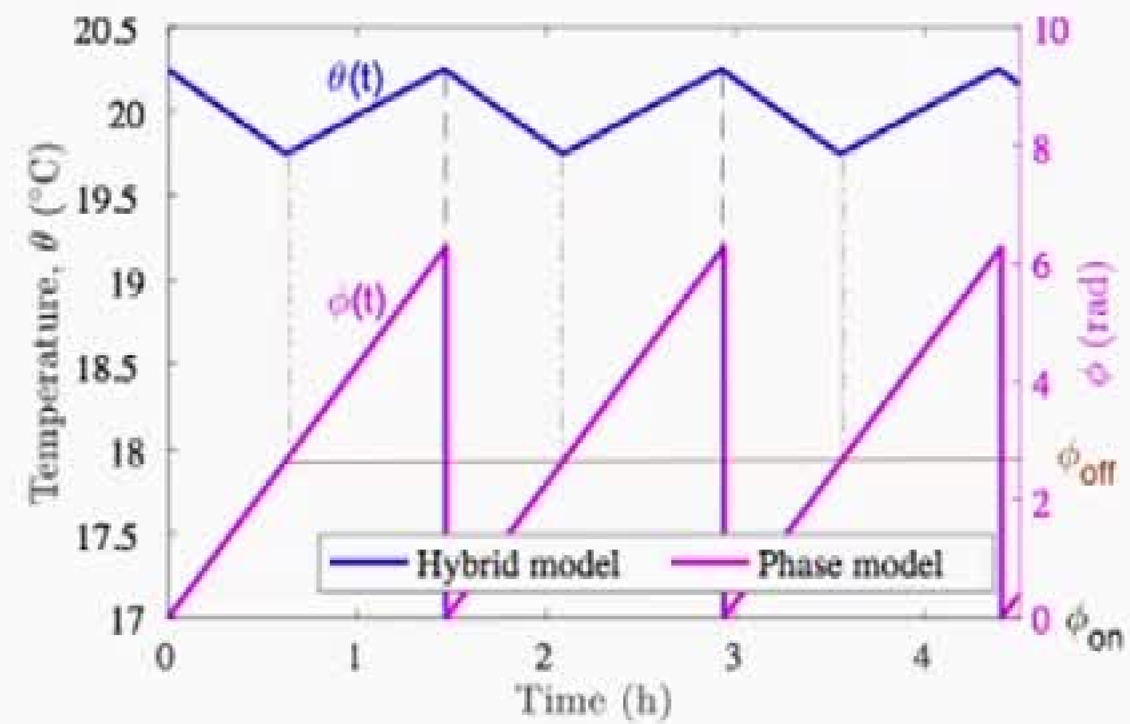
$$\frac{d\phi}{dt} = \omega + Z(\phi)u(t)$$

$\omega = \frac{2\pi}{T}$: free running frequency

$Z(\phi)$: PRC computed numerically



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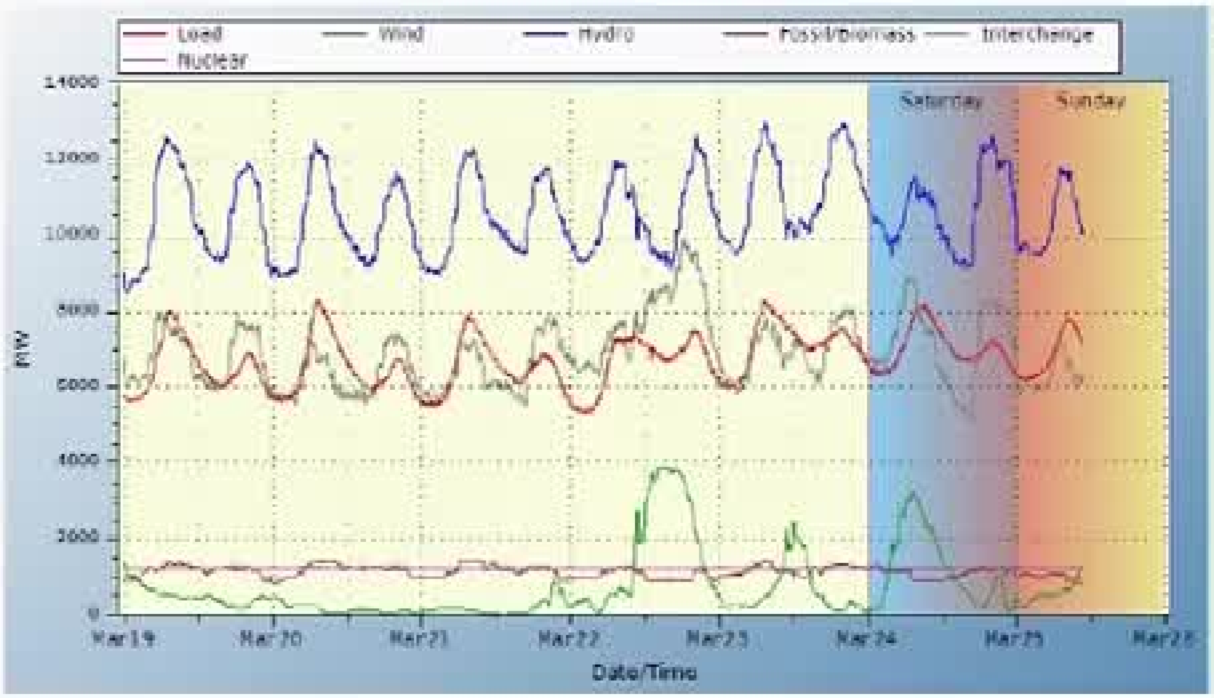

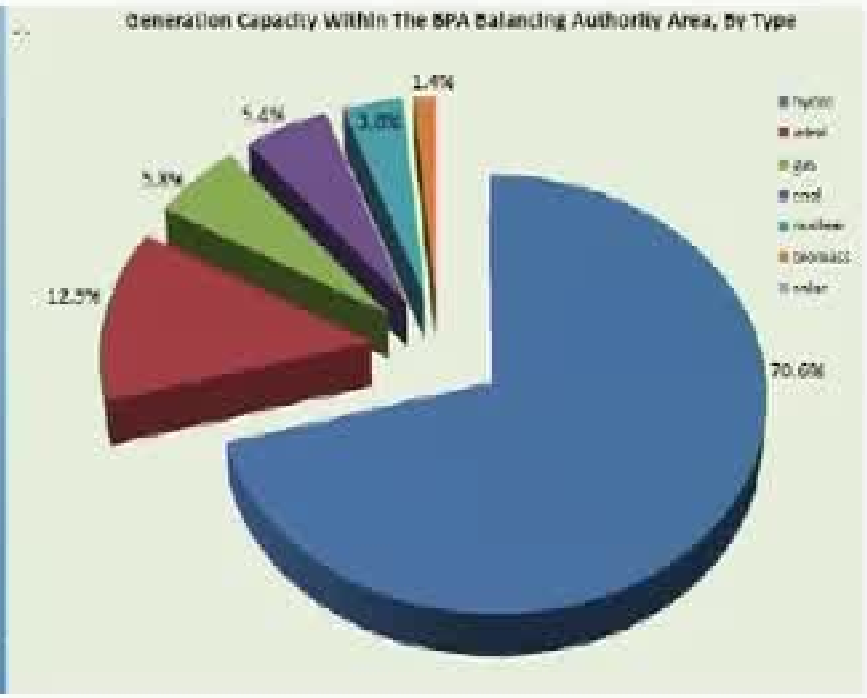

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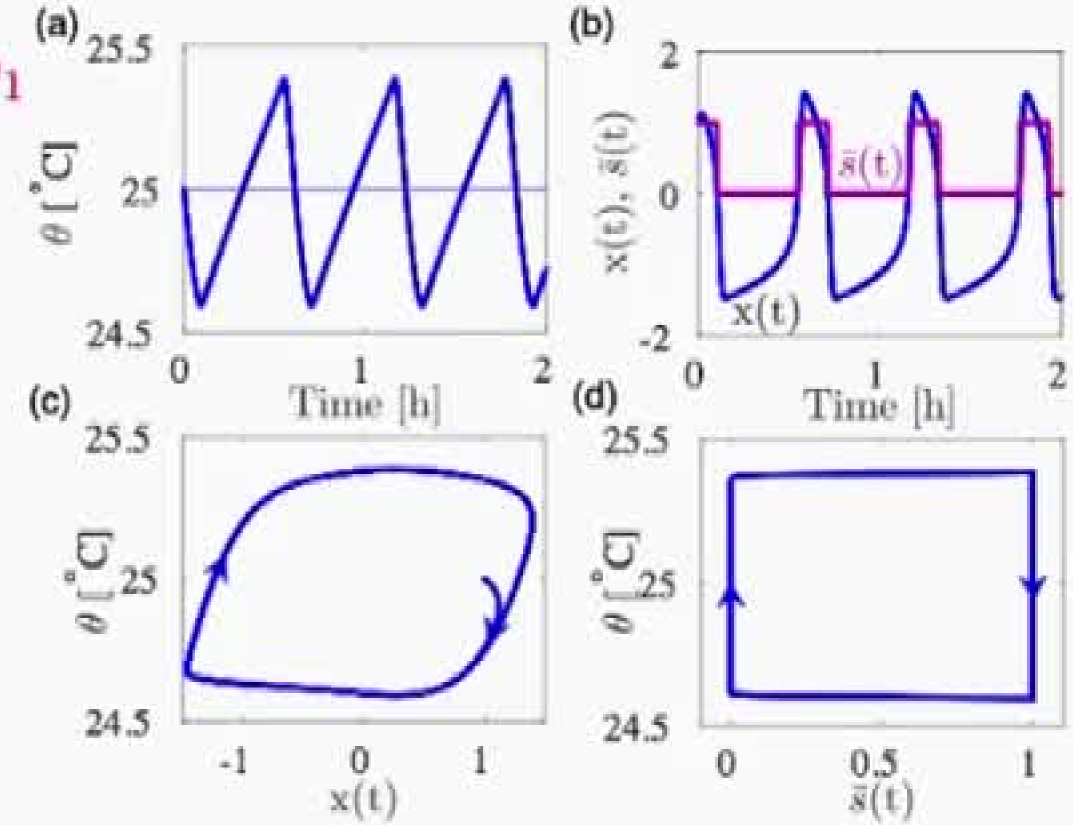
Modeling TCL as an Oscillator

- 2-D TCL continuous model (FitzHugh-Nagumo neuron model)*

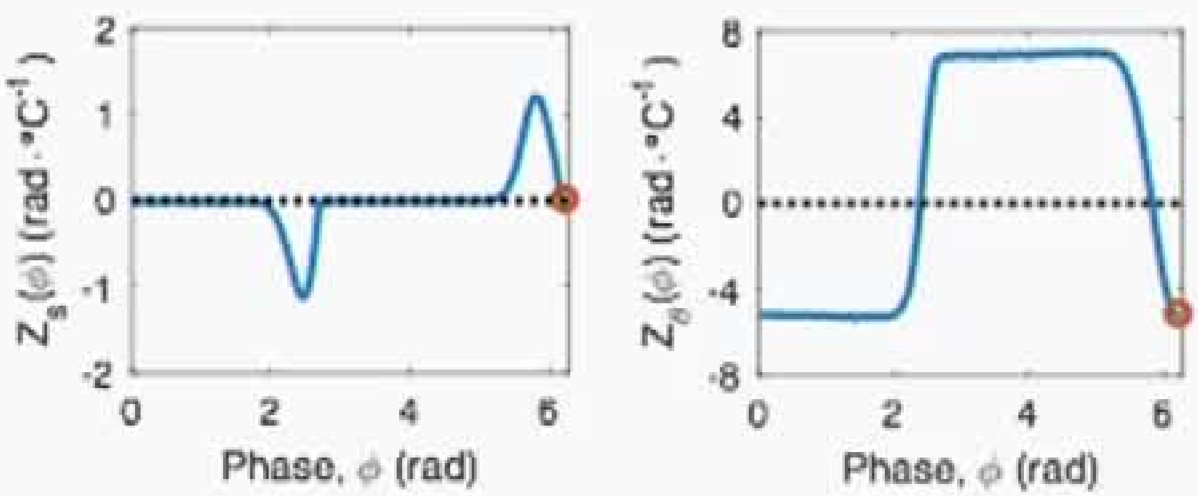
$$\dot{x}(t) = \mu\left((\frac{\delta}{2}+\sigma)x - \frac{x^3}{3} + \theta - \theta_s\right) + v_1$$

$$\dot{\theta}(t) = -\frac{1}{RC}(\theta - \theta_a + \bar{s}(t)PR) + v_2$$



- Heaviside step function smooth approximation

$$\bar{s}(t) = \frac{1}{2}(1 + \tanh(kx))$$



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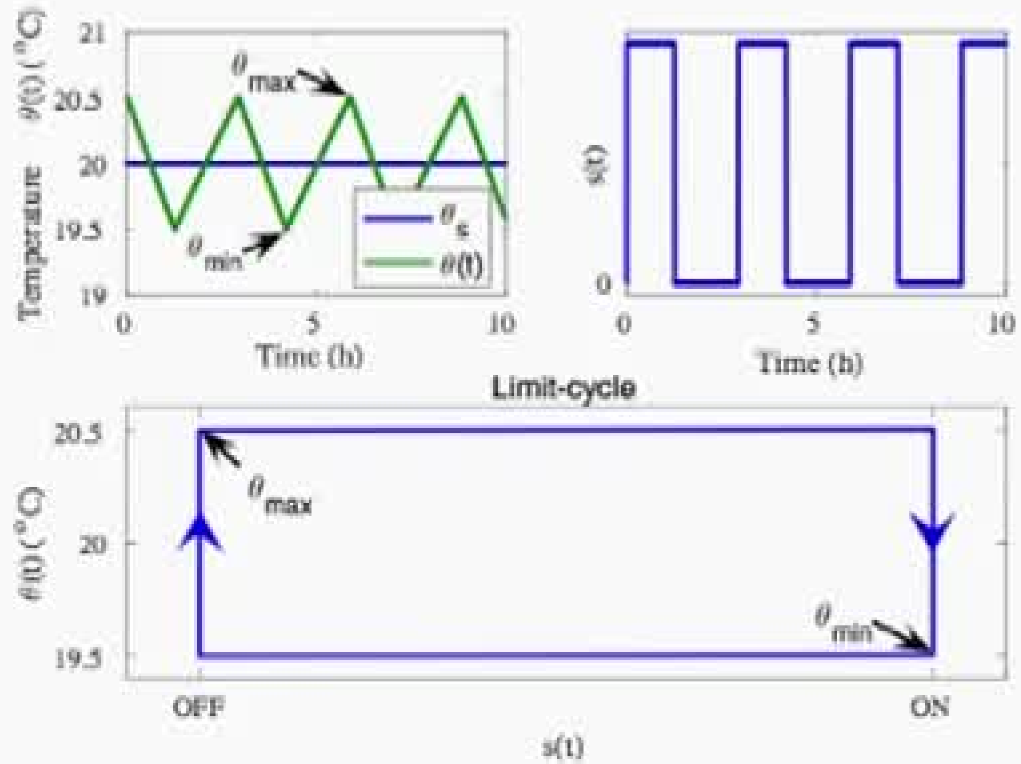

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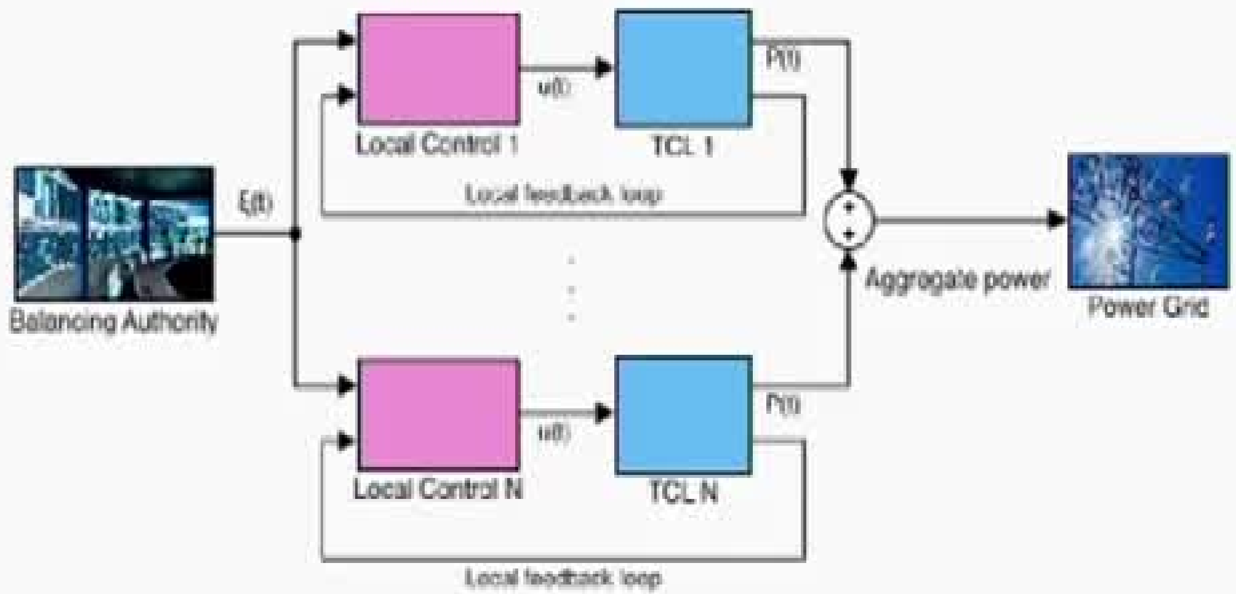
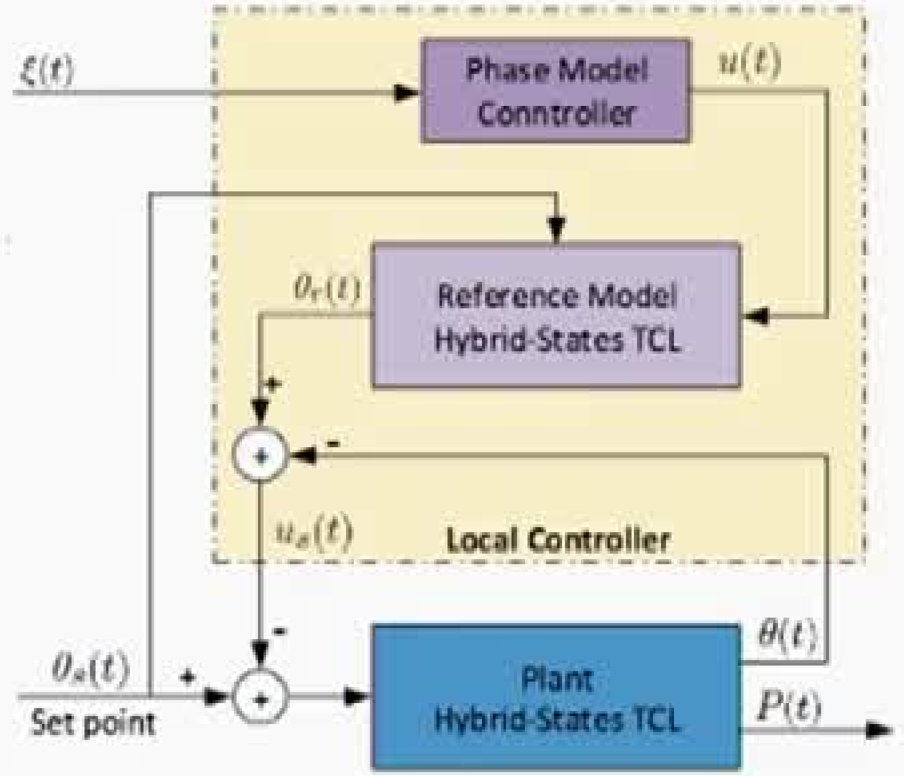
Control design for TCL ensemble regulation

- We want to track the regulation signal for modulating the aggregate power, sent from the balancing authority

  - Open-loop control





  - Phase model based local controllers

$\xi(t)$: Power regulation signal
$u(t)$: Control signal to ref. rodel
$u_s(t)$: Control signal to the plant
$\theta_r(t)$: Ref. temperature
$\theta(t)$: TCL temperature
$\theta_s(t)$: ste point temperature

$$\dot{u}(t) = I_1[I_2 \mathrm{sgn}(\tfrac{1}{2} - s(t))Z(\phi(t))\xi(t) - u(t)]$$

$$u_{k+1} = u_k + I_1 h[I_2 \mathrm{sgn}(\tfrac{1}{2} - s_k)Z(\phi_k)\xi_k - u_k]$$

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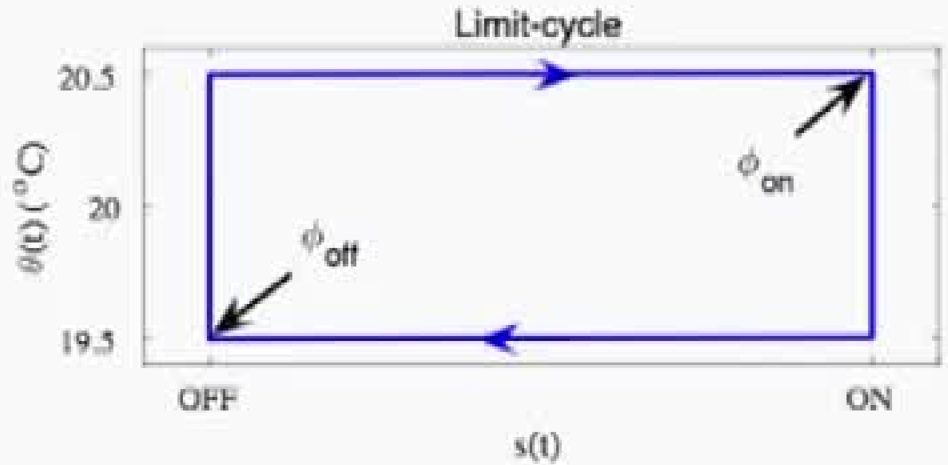

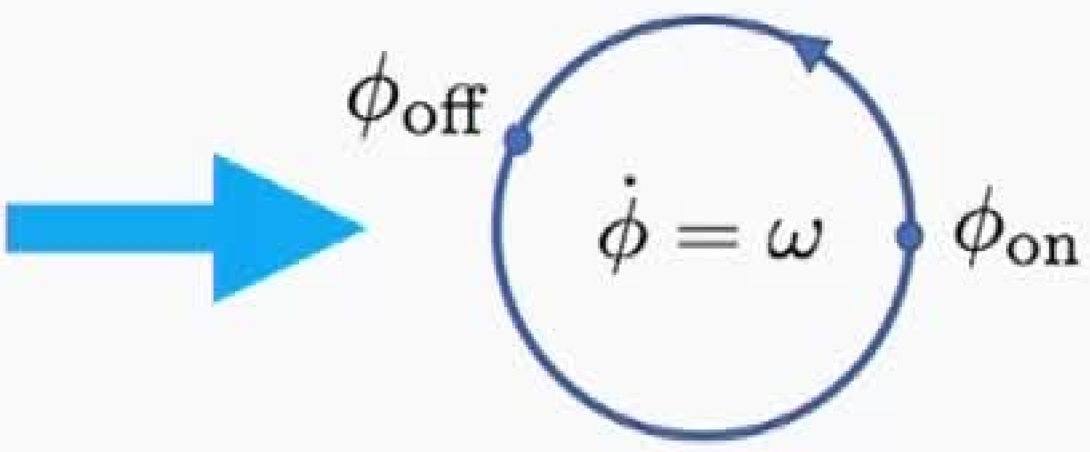

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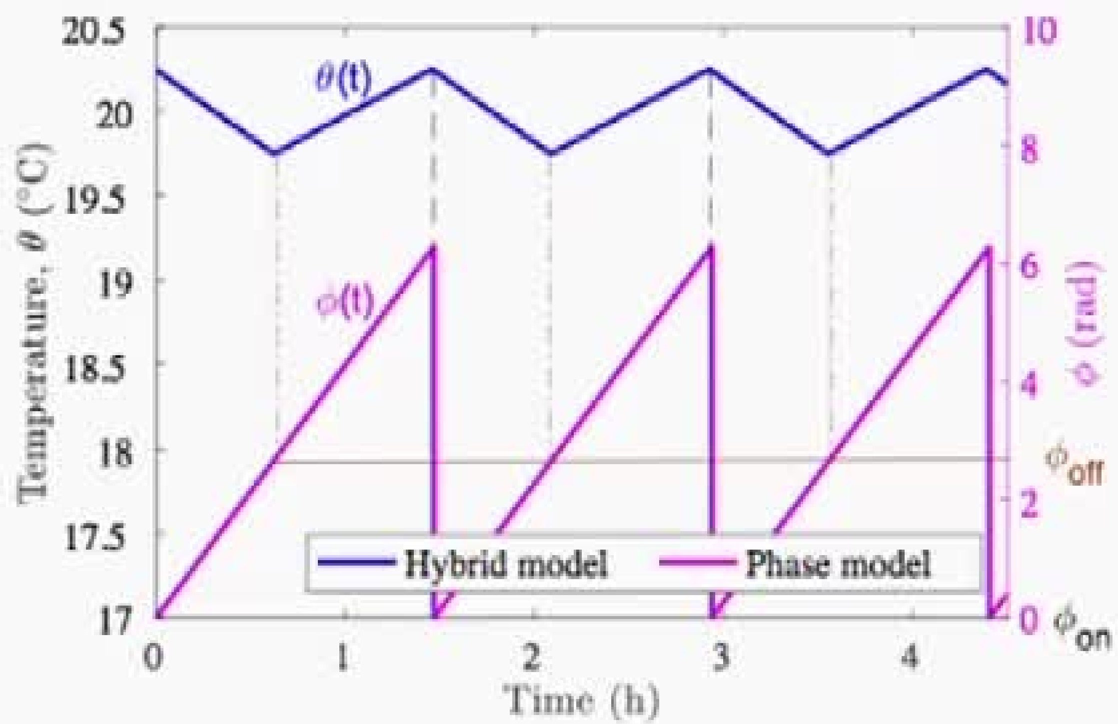
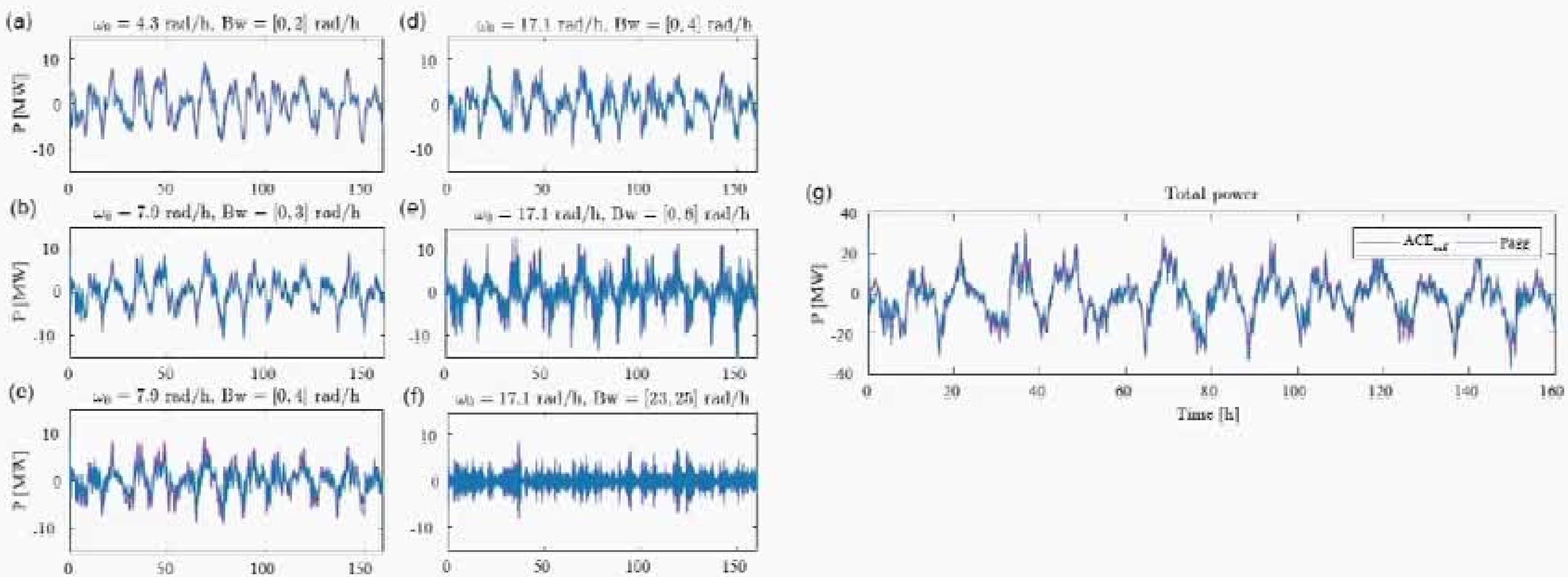
$\omega = \frac{2\pi}{T}$: free running frequency

$Z(\phi)$: PRC computed numerically



Modeling TCL as an Oscillator

- Filter the regulation signal by frequency bands, and use loads with different natural cycling frequencies to track each band



Modeling TCL as an Oscillator

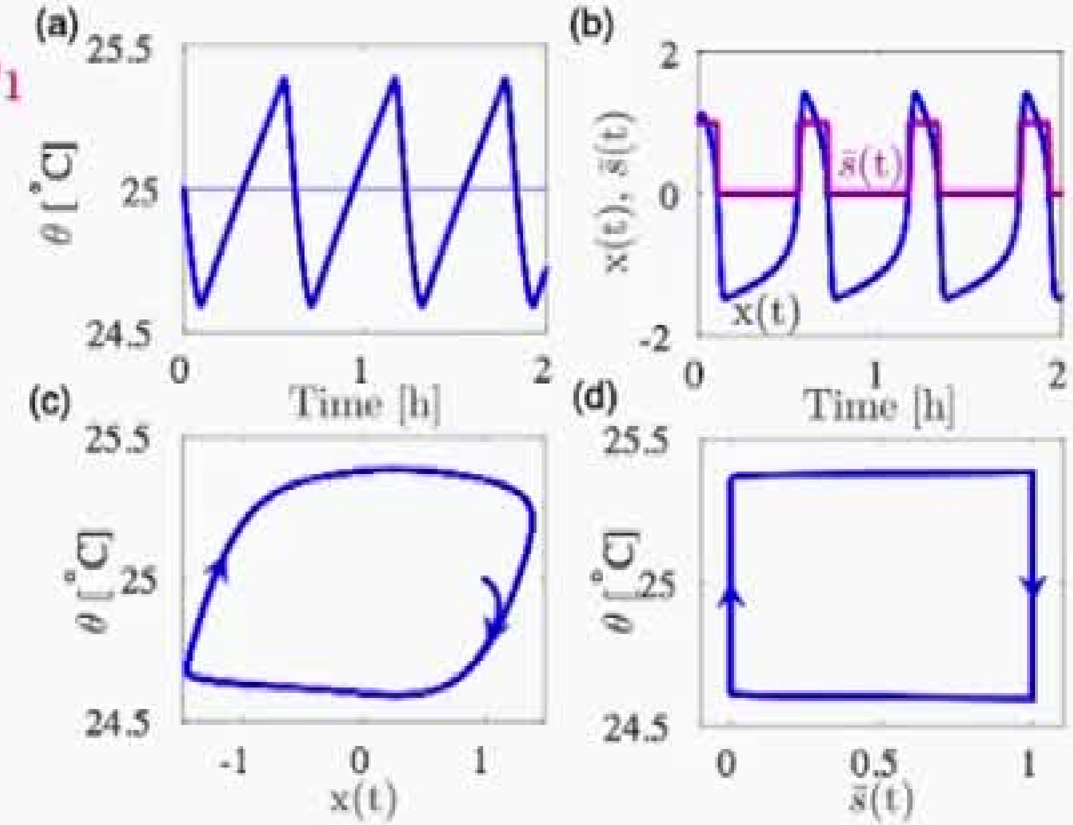
- 2-D TCL continuous model (FitzHugh-Nagumo neuron model)*

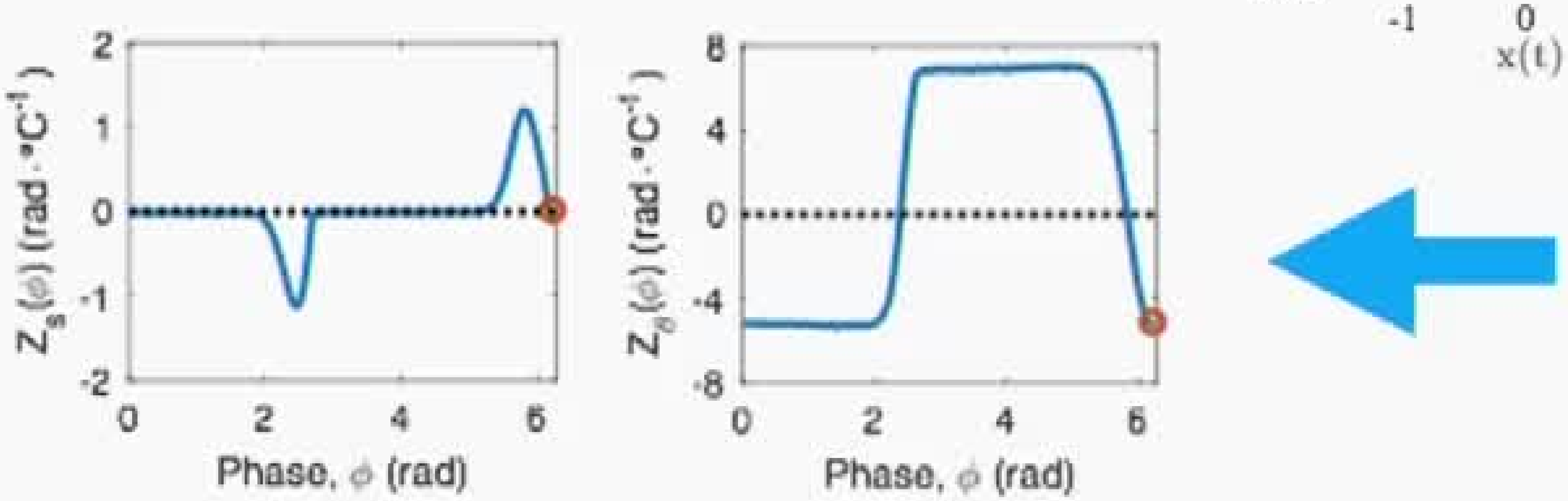
$$\dot{x}(t) = \mu\left((\frac{\delta}{2}+\sigma)x - \frac{x^3}{3} + \theta - \theta_s\right) + v_1$$

$$\dot{\theta}(t) = -\frac{1}{RC}(\theta - \theta_a + \bar{s}(t)PR) + v_2$$

- Heaviside step function smooth approximation

$$\bar{s}(t) = \frac{1}{2}(1 + \tanh(kx))$$

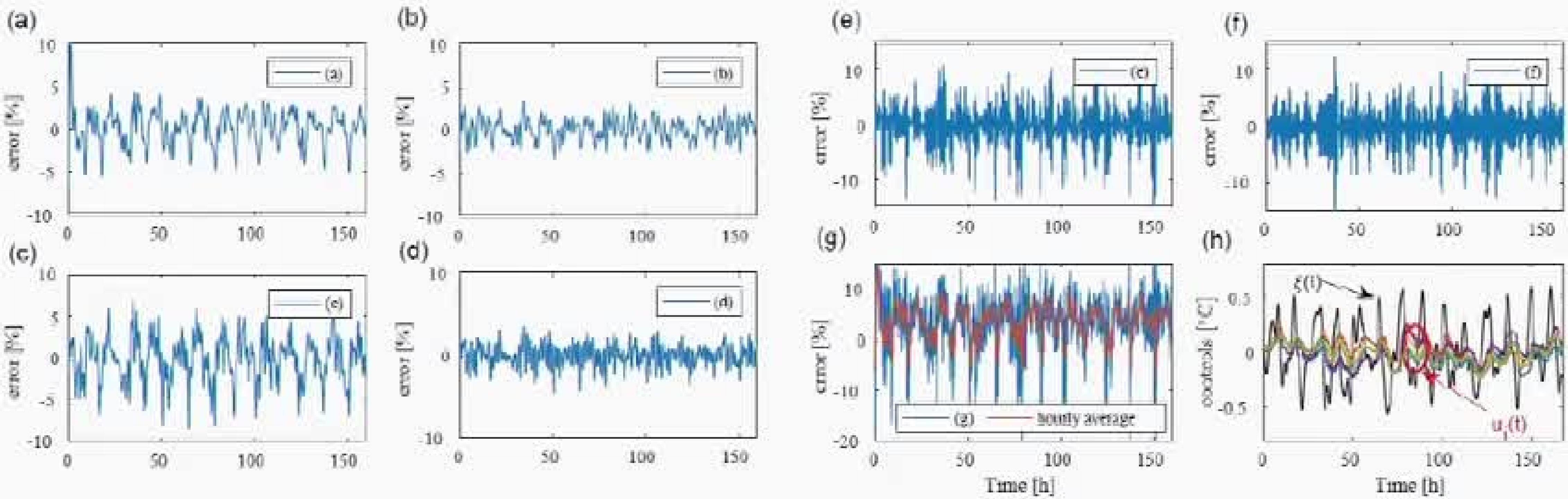




PRCs

Modeling TCL as an Oscillator

- Good tracking performance for each frequency band, and overall $< 6\%$ RMSE



$$\text{RMSE \%} = \sqrt{\frac{1}{T}\frac{\int_0^T (P_{ref}(t) - P_{agg}(t))^2 dt}{P_{base}^2}} \times 100$$

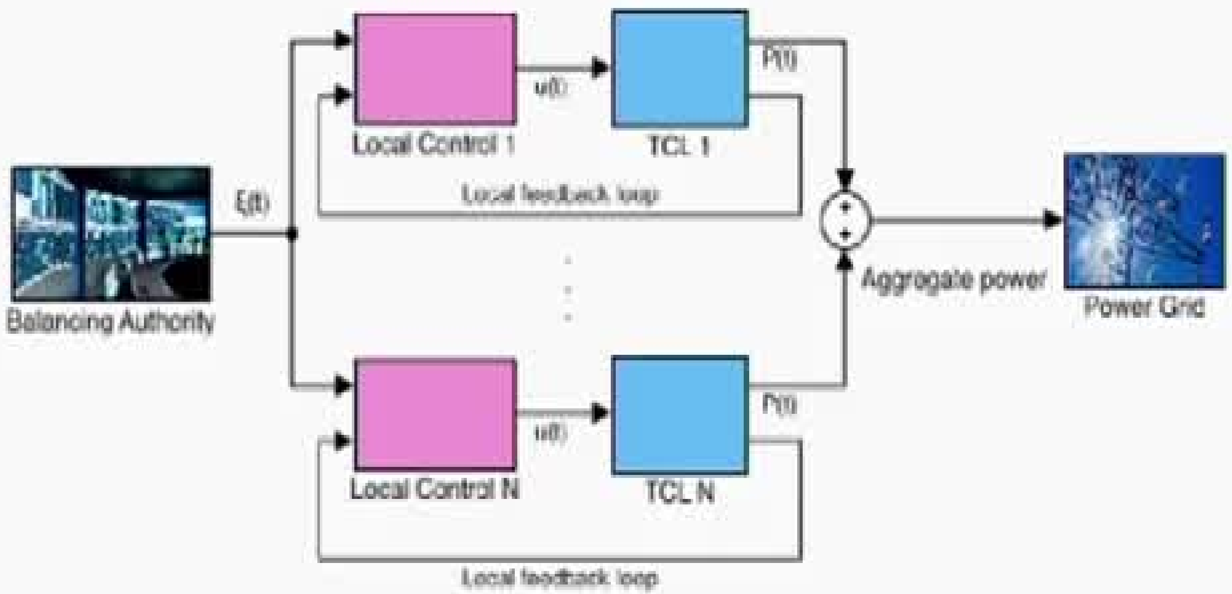
TCL groups	(a)	(b)	(c)	(d)	(e)	(f)	(g)
P_{max} [MW]	10.3	11.2	10.5	9.85	16.5	8.72	39.5
RMSE %	2.31	1.92	3.09	1.38	2.92	3.39	5.89

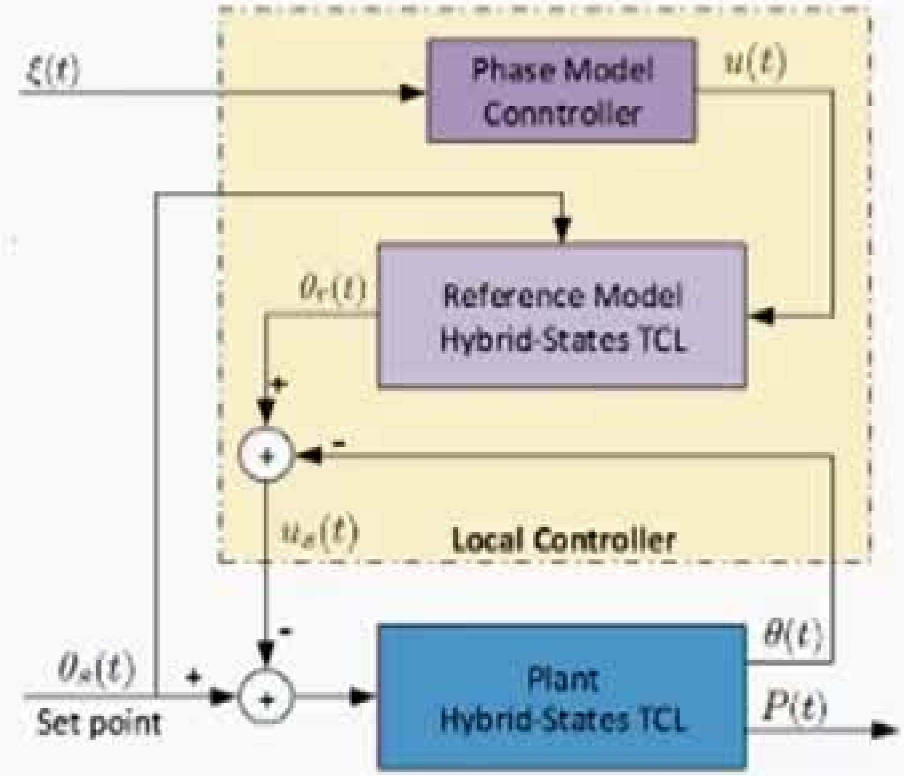
Bomela, Walter, Anatoly Zlotnik, and Jr-Shin Li. "A phase model approach for thermostatically controlled load demand response." Applied Energy 228(15), 667-680 (2018).

Control design for TCL ensemble regulation

- We want to track the regulation signal for modulating the aggregate power, sent from the balancing authority

  - Open-loop control





  - Phase model based local controllers

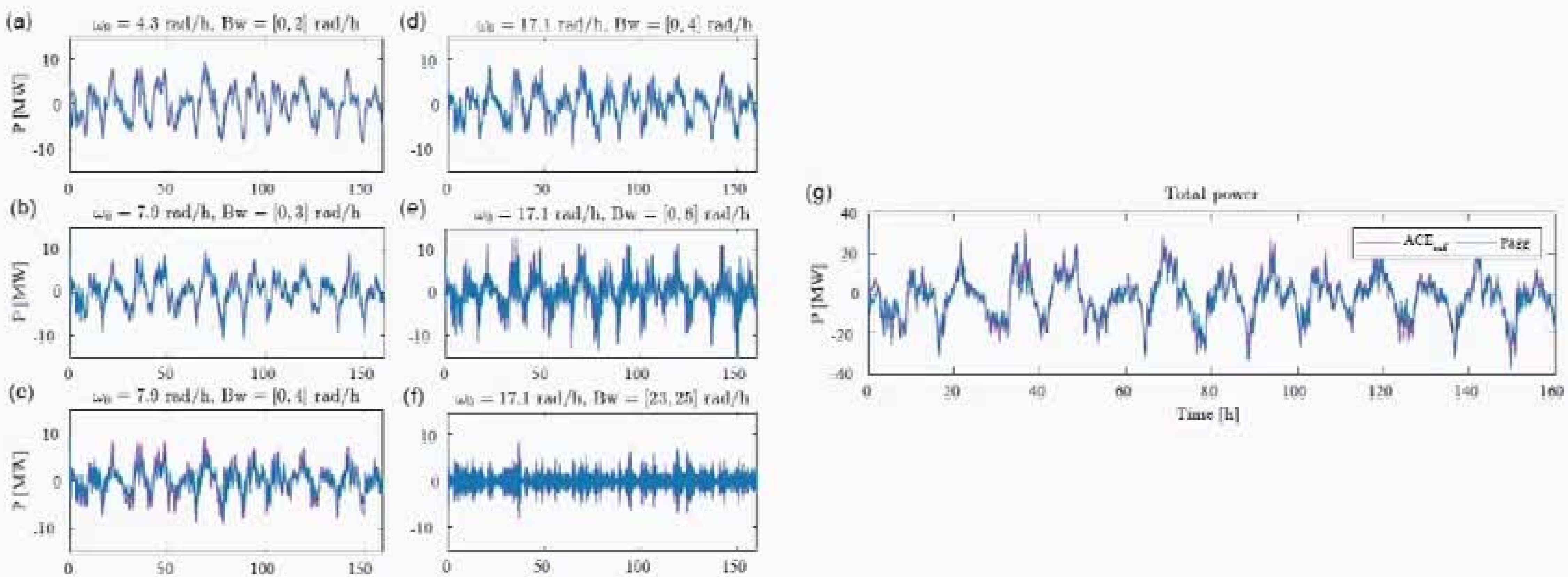
$\xi(t)$: Power regulation signal
$u(t)$: Control signal to ref. rodel
$u_s(t)$: Control signal to the plant
$\theta_r(t)$: Ref. temperature
$\theta(t)$: TCL temperature
$\theta_s(t)$: ste point temperature

$$\dot{u}(t) = I_1[I_2 \mathrm{sgn}(\tfrac{1}{2} - s(t))Z(\phi(t))\xi(t) - u(t)]$$

$$u_{k+1} = u_k + I_1 h[I_2 \mathrm{sgn}(\tfrac{1}{2} - s_k)Z(\phi_k)\xi_k - u_k]$$

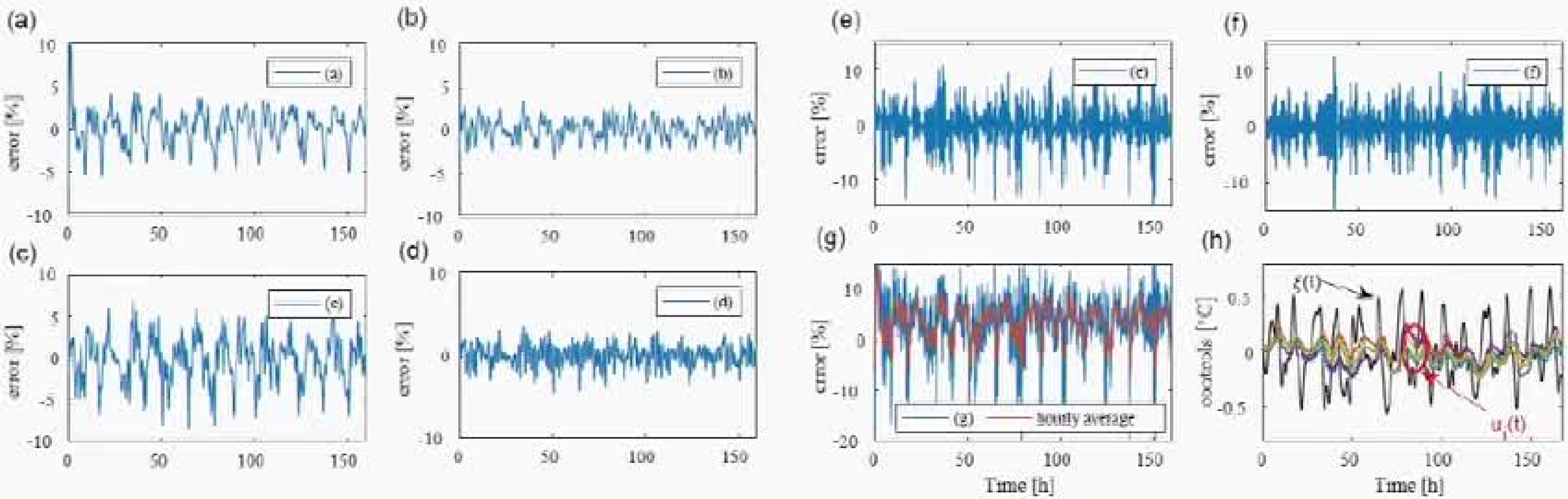
Modeling TCL as an Oscillator

- Filter the regulation signal by frequency bands, and use loads with different natural cycling frequencies to track each band



Modeling TCL as an Oscillator

- Good tracking performance for each frequency band, and overall $< 6\%$ RMSE



$$\text{RMSE}\ \% = \sqrt{\frac{1}{T}\frac{\int_0^T (P_{ref}(t) - P_{agg}(t))^2 dt}{P_{base}^2}} \times 100$$

TCL groups	(a)	(b)	(c)	(d)	(e)	(f)	(g)
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