Decision Making in Presence of Frustration on Multiagent Antagonistic Networks

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Outline

- Background and motivation
- Signed network: structural balance
- Model for opinion forming
- Application

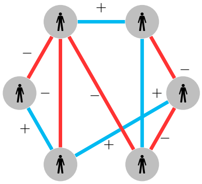
Background

Animal groups*
$\Rightarrow$ decision reached through collaboration

Social Networks
$\Rightarrow$ both cooperative and antagonistic interactions may coexist

*Gray at al., IEEE TCNS, 2018.

Background

 $\Rightarrow$ 

$\Rightarrow$

$$\dot{x} = f(x, \pi)$$

- Signed networks.
  - Cooperative interaction: positive sign.
  - Antagonistic interaction: negative sign.
- Nonlinear model for opinion forming.
  - x: vector of opinions.
  - Equilibrium points: possible decisions.

Signed networks

Signed Laplacian

$\mathcal{G}$ connected signed network, with n nodes and adjacency matrix A.

$$L = \Delta - A : \quad \text{signed Laplacian}$$

$$\mathcal{L} = I - \Delta^{-1} A : \quad \text{normalized signed Laplacian,}$$

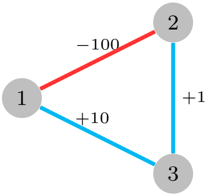
where

$$\Delta = \operatorname{diag}\{\delta_1, \ldots, \delta_n\} : \ \delta_i = \sum_{j=1}^{n} |a_{ij}| > 0 \quad \forall\, i.$$

Example

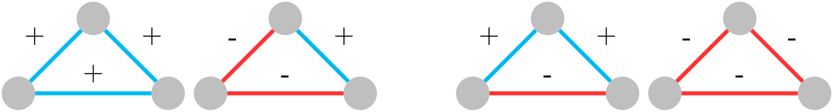
$$A = \begin{bmatrix} 0 & -100 & 10 \\ -100 & 0 & 1 \\ 10 & 1 & 0 \end{bmatrix}, \quad \Delta = \operatorname{diag}\{110, 101, 11\}$$

$$\mathcal{L} = \begin{bmatrix} 1 & 0.909 & -0.091 \\ 0.99 & 1 & -0.01 \\ -0.909 & -0.091 & 1 \end{bmatrix}$$



Signed networks

Structural balance



positive cycles *negative* cycles

Def. A graph $\mathcal{G}$ *structurally balanced* if all its cycles are positive.

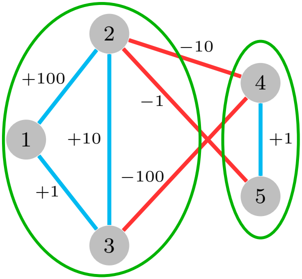
Signed networks

Structural balance: equivalent conditions

$\mathcal{G}$ connected signed graph.

1. $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ s.t. every edge:
   - between $\mathcal{V}_1$ and $\mathcal{V}_2$ is negative;
   - within $\mathcal{V}_1$ or $\mathcal{V}_2$ is positive;

Example



mutual friends

$$\mathcal{L} = \begin{bmatrix} 1 & -0.99 & -0.01 & 0 & 0 \\ -0.83 & 1 & -0.08 & 0.08 & 0.01 \\ -0.01 & -0.09 & 1 & 0.9 & 0 \\ 0 & 0.09 & 0.9 & 1 & -0.01 \\ 0 & 0.5 & 0 & -0.5 & 1 \end{bmatrix}$$

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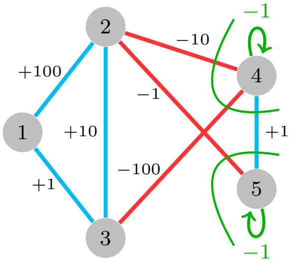
2. $\exists$ signature matrix $S = \text{diag}\{s_1, \dots, s_n\}$ with $s_i = \pm 1$, s.t. $S\mathcal{L}S$ has all nonpositive off-diagonal entries;

Example



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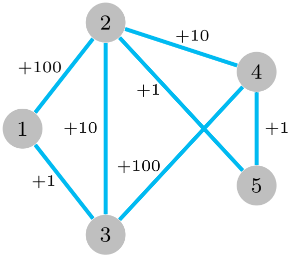

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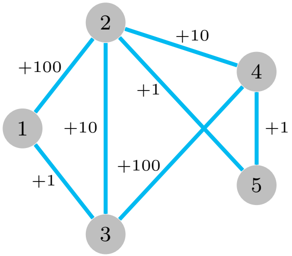
$\Rightarrow \mathcal{G}$ is structurally unbalanced iff
$$\lambda_1(\mathcal{L}) > 0$$

Example



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and $\lambda_1(\mathcal{L}) = 0.004 > 0.$

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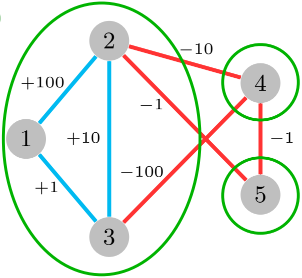

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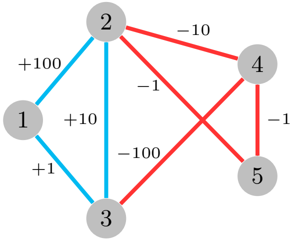

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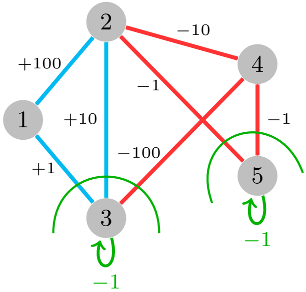

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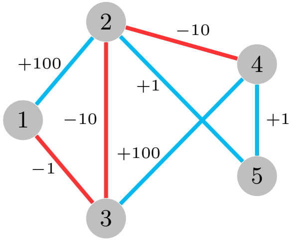

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we obtain

$$\begin{aligned} 0.36 &= \textstyle\sum \text{positive (off-diagonal) elements of } S\mathcal{L}S \\ &= \text{minimum possible sum!} \end{aligned}$$

Signed networks

Frustration index, algebraic conflict

Task

Characterize the graph distance from structurally balanced state

- Frustration Index

$$\epsilon(\mathcal{G}) = \min_{\substack{S=\mathrm{diag}\{s_1,\dots,s_n\},\\ s_i=\pm 1}} \frac{1}{2}\sum_{i\neq j}\left[\,|\mathcal{L}| + S\mathcal{L}S\,\right]_{ij}$$

Computation: NP-hard problem

- Algebraic Conflict

$$\xi(\mathcal{G}) = \lambda_1(\mathcal{L})$$

$\lambda_1(\mathcal{L})$ good approximation of $\epsilon(\mathcal{G})$

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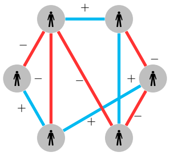
Model for opinion forming

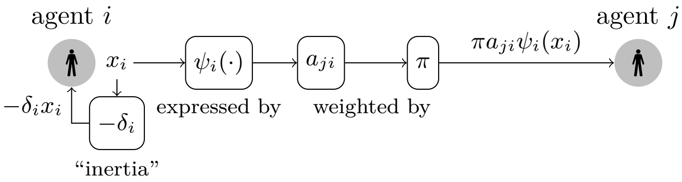
- Signed network $\mathcal{G}$ with n agents;
- $x \in \mathbb{R}^n$ vector of opinions.



$$\dot{x} = -\Delta x + \pi A \psi(x), \quad x \in \mathbb{R}^n$$

where:

- A adjacency matrix, $\Delta = \mathrm{diag}\{\delta_1, \dots, \delta_n\}$
- $\psi(x) = [\psi_1(x_1) \ \dots \ \psi_n(x_n)]^T$
- $\pi > 0$ scalar



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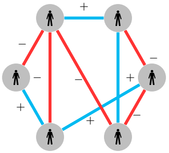

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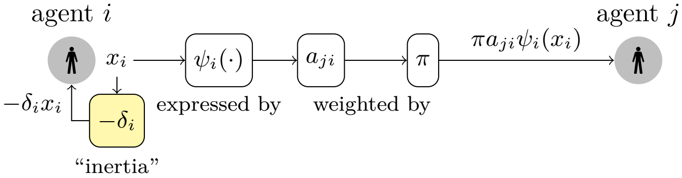

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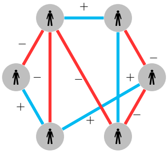

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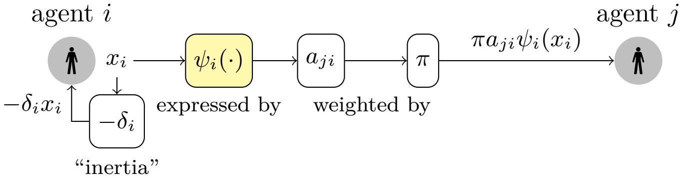

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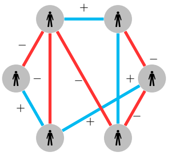

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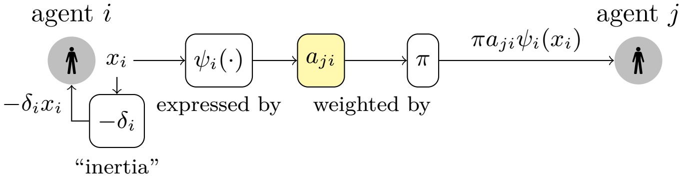

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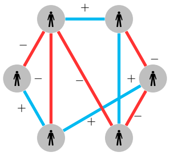

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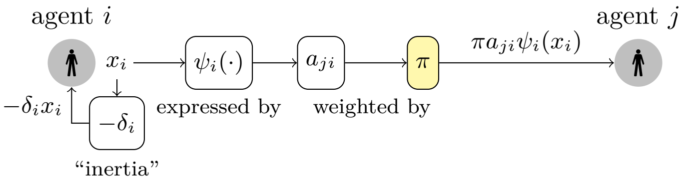
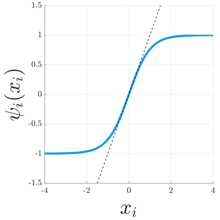


Model for opinion forming

Assumptions:

- $\mathcal{G}$ undirected, connected, without self-loops (A is *symmetric*, *irreducible*, with *null diagonal*).
- signed Laplacian-like assumption: $\delta_i = \sum_j |a_{ij}| > 0$.
- "S-shape" for each $\psi_i(x_i) : \mathbb{R} \to \mathbb{R}$ (odd, monotonically increasing with $\frac{\partial \psi_i}{\partial x_i}(0) = 1$, saturated, sigmoidal)



$$(\star) \quad \dot{x} = -\Delta x + \pi A \psi(x) = \Delta \left[-x + \pi H \psi(x) \right], \quad x \in \mathbb{R}^n,$$

$$\text{with } H := \Delta^{-1} A \quad \Rightarrow \quad \mathcal{L} = I - H.$$

Then

$$(\star) \text{ is monotone} \quad \Leftrightarrow \quad \mathcal{G} \text{ is structurally balanced} \quad \Leftrightarrow \quad \lambda_1(\mathcal{L}) = 0.$$

Task

Investigate how the social effort parameter π affects the existence and stability of the equilibrium points of the system

$$\dot{x} = \Delta \left[-x + \pi H \psi(x) \right], \quad x \in \mathbb{R}^n.$$

In particular:

- Find π_1 s.t. for $\pi \in (0, \pi_1)$ nontrivial equilibria cannot appear.
- Investigate what happens for $\pi > \pi_1$.
  Find π_2 s.t. for $\pi \in (\pi_1, \pi_2)$ there exist only three equilibria.

R. Gray, A. Franci, V. Srivastava, N.E. Leonard, *Multiagent Decision-Making Dynamics Inspired by Honeybees*, IEEE TCNS, v. 5 (2), pp. 793-806, 2018.

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Tools:

- matrix theory: symmetrizable matrices;
- bifurcation theory ($\mathcal{L}$ has simple eigenvalues).

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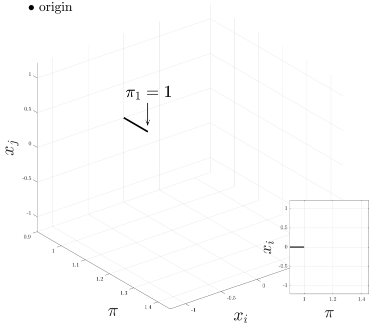
Bifurcation analysis

Structurally balanced networks

$$\dot{x} = \Delta\left[-x + \pi H \psi(x)\right], \quad x \in \mathbb{R}^n.$$

- $\pi < 1$: $x = 0$ only eq. point (GAS).
- $\pi = 1$: pitchfork bifurcation
  - $x = 0$ saddle point;
  - two more equilibria: x^* and $-x^*$ s.t. $|x^*| = \alpha \mathbb{1}_n$ (loc. AS $\forall \pi > 1$).
- $\pi = \pi_2 = \frac{1}{1-\lambda_2(\mathcal{L})}$: (second) pitchfork bifurcation
  - new equilibria (stable/unstable).



Bifurcation diagram (x_i, π, x_j)

A. Fontan and C. Altafini, "Multiequilibria analysis for a class of collective decision-making networked systems", *IEEE TCNS*, vol. 5 (4), pp. 1931-1940, 2018.

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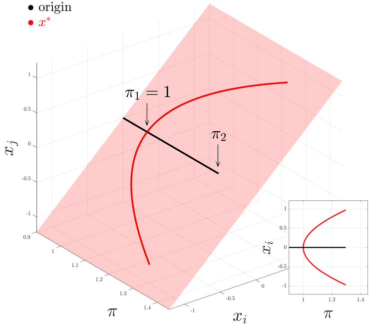

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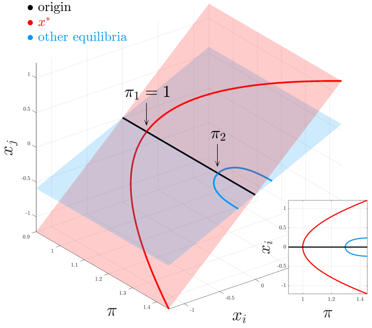

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Bifurcation analysis

Structurally unbalanced networks

$$\dot{x} = \Delta\left[-x + \pi H \psi(x)\right], \quad x \in \mathbb{R}^n.$$

$$\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}, \qquad \pi_2 = \begin{cases} \frac{1}{1-\lambda_2(\mathcal{L})}, & \lambda_2(\mathcal{L}) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

- $\pi < \pi_1$: $x = 0$ only eq. point (GAS).
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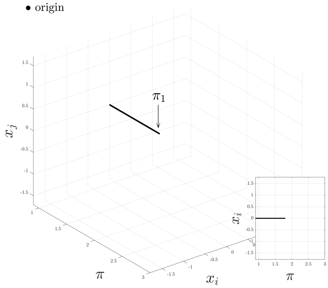

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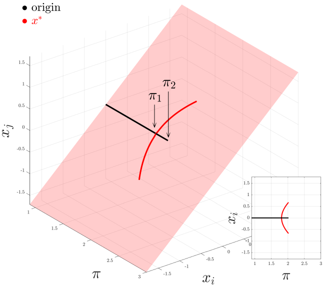
$$\pi_1 = \frac{1}{1 - \lambda_1(\mathcal{L})}, \qquad \pi_2 = \begin{cases} \frac{1}{1-\lambda_2(\mathcal{L})}, & \lambda_2(\mathcal{L}) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

- $\pi < \pi_1$: $x = 0$ only eq. point (GAS).
- $\pi = \pi_1$: pitchfork bifurcation
  - $x = 0$ saddle point;
  - two new equilibria x^* and $-x^*$.
- $\pi = \pi_2$: (second) pitchfork bifurcation
  - new equilibria (stable/unstable).



Bifurcation diagram (x_i, π, x_j)

Bifurcation analysis

Structurally unbalanced networks

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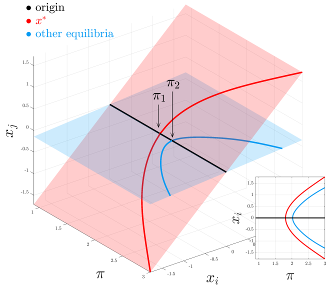
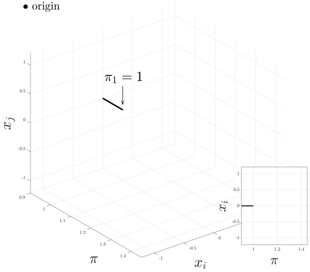


Bifurcation diagram (x_i, π, x_j)

Interpretation of the results

$\mathcal{G}$ structurally balanced



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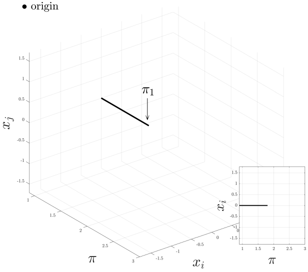

- $\pi < \pi_1$: no decision;

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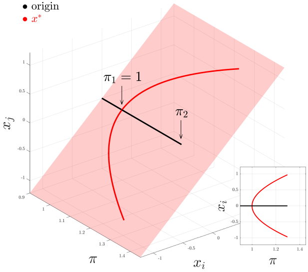

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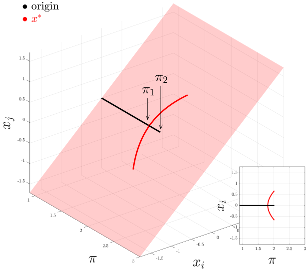

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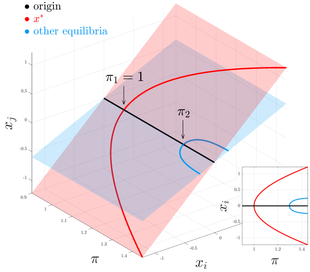

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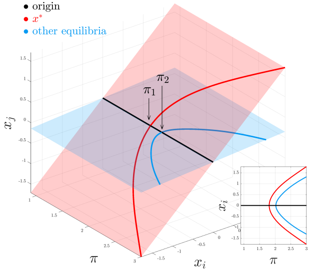

- $\pi < \pi_1$: no decision;
- $\pi \in (\pi_1, \pi_2)$: two (alternative) decisions;
- $\pi > \pi_2$: several decisions.

Interpretation of the results

Model for opinion forming: $\dot{x} = \Delta \left[-x + \pi H \psi(x)\right]$.

$\pi \in (\pi_1, \pi_2)$: two alternative decisions (eq. points x^* and $-x^*$)

$$\pi_1 = \frac{1}{1 - \lambda_1(\mathcal{L})}, \qquad \pi_2 = \begin{cases} \frac{1}{1-\lambda_2(\mathcal{L})}, & \lambda_2(\mathcal{L}) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

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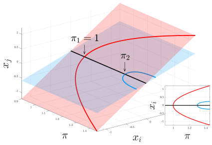
- Structurally balanced $\mathcal{G}$: $\lambda_1(\mathcal{L}) = 0$.
  - $\pi_1 = 1$ fixed
  - π_2 depends on $\lambda_2(\mathcal{L})$: algebraic connectivity of $\mathcal{G}$



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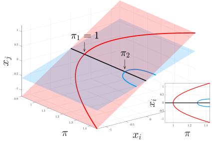
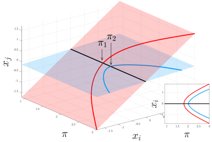


- Structurally unbalanced $\mathcal{G}$: $\lambda_1(\mathcal{L}) > 0$.
  - $\lambda_1(\mathcal{L}) \approx \epsilon(\mathcal{G})$: measure of the structural imbalance of $\mathcal{G}$
  - $\lambda_2(\mathcal{L})$: independent from $\epsilon(\mathcal{G})$

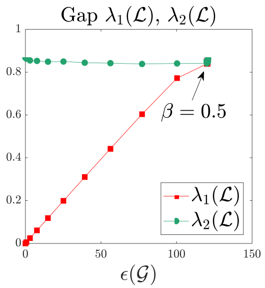


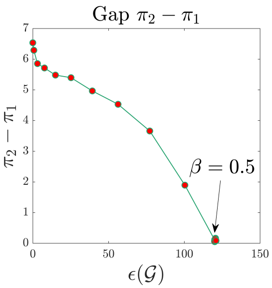
Example

Sequence of signed Erdős-Rényi graphs $\mathcal{G}$ with $n = 500$ nodes.

$$\beta = \text{percentage of edges with negative sign}$$
$$\epsilon(\mathcal{G}) = \text{frustration of the network}$$

$$\pi_1 = \frac{1}{1-\lambda_1(\mathcal{L})}, \quad \pi_2 = \begin{cases} \frac{1}{1-\lambda_2(\mathcal{L})}, & \lambda_2(\mathcal{L}) < 1 \\ +\infty, & \text{otherwise.} \end{cases}$$

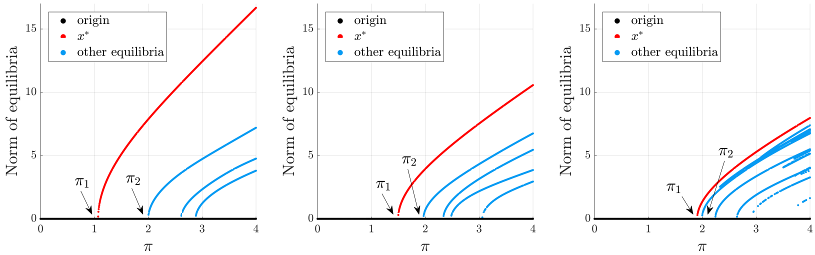




Example

Consider three signed networks $\mathcal{G}$ with $n = 20$ nodes and different percentages of edges with negative sign given by $\beta = 0.2$, $\beta = 0.4$, $\beta = 0.7$.

	β	frustration $\epsilon(\mathcal{G})$	$\lambda_1(\mathcal{L})$	$\lambda_2(\mathcal{L})$	π_1	π_2	$\pi_2 - \pi_1$
(a)	0.2	0.666	0.065	0.500	1.069	2.000	0.930
(b)	0.4	4.285	0.332	0.491	1.496	1.966	0.470
(c)	0.7	5.536	0.475	0.499	1.905	1.995	0.090



(a) $\epsilon(\mathcal{G}) = 0.666$ (b) $\epsilon(\mathcal{G}) = 4.285$ (c) $\epsilon(\mathcal{G}) = 5.536$

Summary

Model for opinion forming:

- signed network
- saturated sigmoidal nonlinearities
- social effort parameter π

Results

- Nontrivial decision: $\pi > \pi_1$, π_1 grows with the frustration.
- Two alternative decisions: $\pi \in (\pi_1, \pi_2)$. The interval (π_1, π_2) becomes smaller as the frustration grows.

Application

From parliamentary networks to government formation

Parliamentary elections
in 29 European countries
(1978-2019)

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From parliamentary networks to government formation

Parliamentary elections
in 29 European countries
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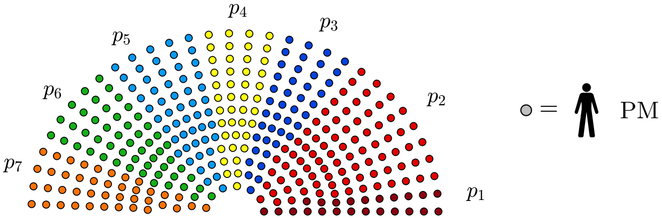
Characterized by:

- negotiation periods;
- coalition governments (enjoying the confidence of the Parliament).

Parliamentary networks

p_i: political parties winning seats in the Parliament (different sizes)



Parliamentary networks

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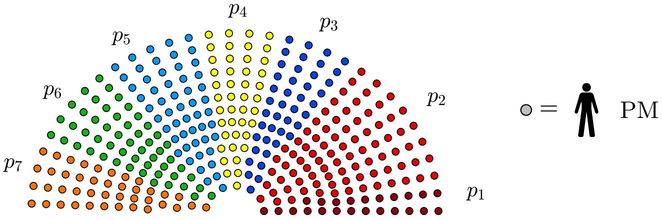
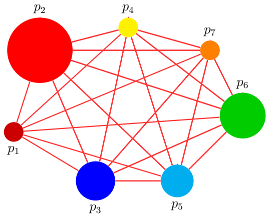
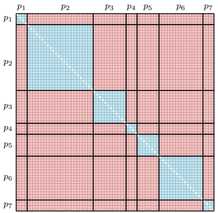


$\Downarrow$



parliamentary network $\mathcal{G}$



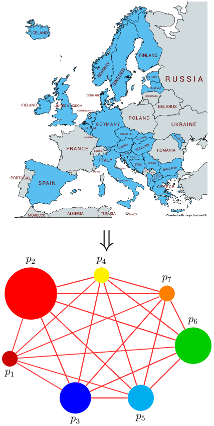
adjacency matrix A

scenario: all-against-all

Process of government formation

Model for opinion forming: $\dot{x} = -\Delta x + \pi A \psi(x)$

- π: duration of negotiation
- decision: vote of confidence to candidate cabinet



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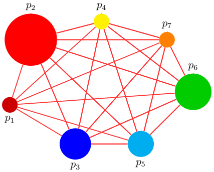
- $\pi_1 \propto$ frustration

$\Downarrow$



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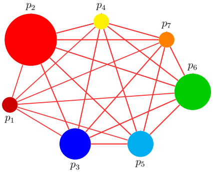
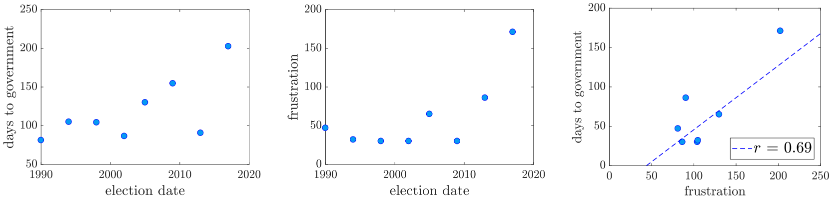


Aim

To predict the duration of "negotiation" period before the government formation

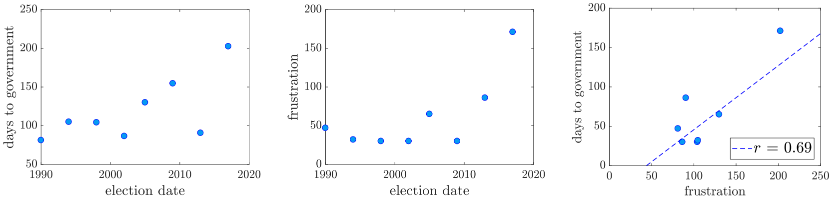
Results: duration of "negotiation" period

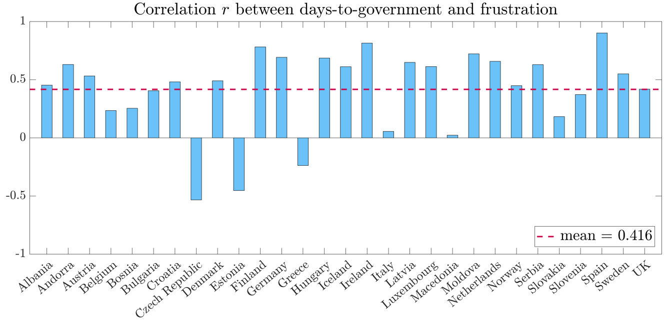
Example: Germany



Results: duration of "negotiation" period

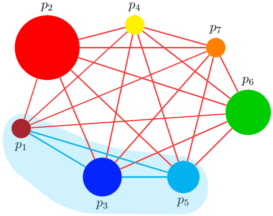
Example: Germany



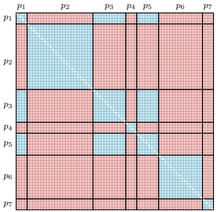


Results: duration of "negotiation" period

scenario: pre-electoral coalitions

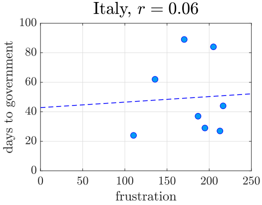


parliamentary network $\mathcal{G}$

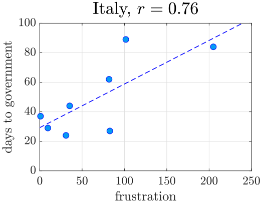


adjacency matrix A

Example: Italy



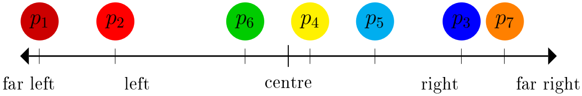
all-against-all

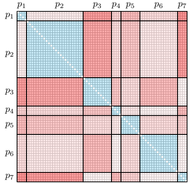


pre-electoral coalitions

Results: duration of "negotiation" period

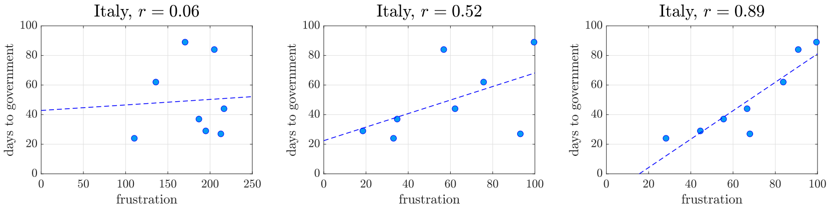
scenario: all-against-all, weighted





left-right positions: rile (electoral manifestos*), random

Example: Italy

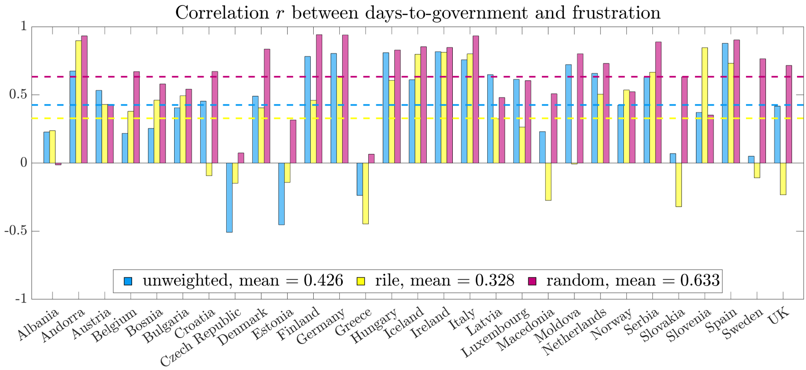


unweighted rile random

*Volkens et al. (2018): Manifesto Project, doi: 10.25522/manifesto.mpds.2018b

Results: duration of "negotiation" period

scenario: pre-electoral coalitions



Conclusions

Model for opinion forming:

- signed network
- saturated sigmoidal nonlinearities
- social effort parameter

Results

- The social effort required to reach a decision grows with the frustration of the network.

  The interval for the social effort parameter for which only two alternative decisions are possible becomes smaller as the frustration grows.

- Application: process of government formation in 29 parliamentary democracies

Thank you!

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