Symmetry and Synthesis of Agreement and Disagreement Dynamics

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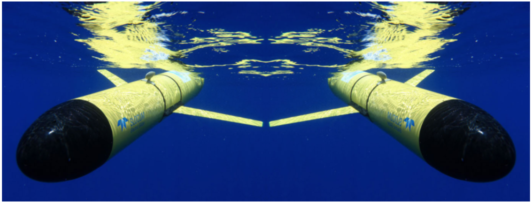
Motivation: Robotic and Natural Systems



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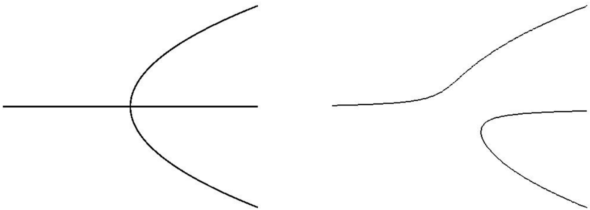
Symmetry, Sensitivity, and Robustness



- Singularity theory[1] classifies unfolding of bifurcation diagrams near singular points
- Near singularity: sensitivity to input; far from singularity: robustness to perturbation and uncertainty

[1] M. Golubitsky and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory (Applied Mathematical Sciences 51). New York, NY, *USA: Springer-Verlag*, 1985.

2 Agents and 2 Options (Tasks)

Agent i at time t has opinion $x_i(t)$:

$$\begin{cases} x_i(t) > 0 & \text{preference for option A} \\ x_i(t) = 0 & \text{uncommitted} \\ x_i(t) < 0 & \text{preference for option B} \end{cases}$$

Opinion dynamics described by

$$\dot{x}_i = f_i(x_i, x_j) \quad i, j \in \{1, 2\} \quad i \neq j \tag{1}$$

S_2 symmetry in options: $-f_i(x_i, x_j) = f_i(-x_i, -x_j)$

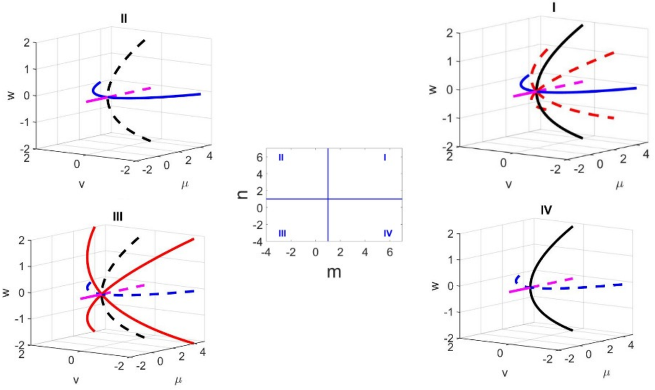
S_2 symmetry in agents: $\big(f_1(x_2, x_1), f_2(x_2, x_1)\big) = \big(f_2(x_1, x_2), f_1(x_1, x_2)\big)$

$S_2 \times S_2$ Symmetry: 2 Agents, 2 Options

$$\boxed{\begin{aligned} \frac{dv}{dt} &= f_1(v, w, \mu) = v^3 + mvw^2 - \mu v \quad (2) \\ \frac{dw}{dt} &= f_2(v, w, \mu) = w^3 + nwv^2 - \mu w \quad (3) \end{aligned}}$$

$$\frac{dv}{dt} = f_1(v, w, \mu) = v^3 + mvw^2 - \mu v \tag{2}$$

$$\frac{dw}{dt} = f_2(v, w, \mu) = w^3 + nwv^2 - \mu w \tag{3}$$



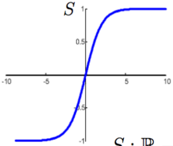
$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities[3]

Sigmoids are everywhere! Extension of model with S_2 symmetry[2]:

$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \tag{4}$$

$$i, j \in \{1, 2\}, \quad i \neq j, \quad u, c \in \mathbb{R}$$

u bifurcation parameter, c selects topology of the bifurcation diagram



$S : \mathbb{R} \to (-1, 1)$

[2] Franci, Srivastava, Leonard (2015). A realization theory for bio-inspired collective decision-making, arXiv:1503.08526v1

[3] Bizyaeva, Franci, Leonard (2019). Flexible allocation dynamics for two agents and two options. In preparation.

$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities

$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \qquad (4)$$

(4) is a bifurcation problem in two state variables with bifurcation parameter u and bifurcation point $u_0 = 1$. For $-2.7024 < c < 2.7024$ it is locally equivalent to the $(S_2 \times S_2)$-symmetric nondegenerate normal form with a subcritically stable trivial solution and supercritical bifurcations of pure mode solutions.

$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities

$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \tag{4}$$

$(S_2 \times S_2)$-symmetry in model $\rightarrow$ expect steady-state agreement ($x_1 = x_2$) and/or disagreement ($x_1 = -x_2$) solutions.[4]

Introduce new rotated coordinates:
$x_{avg} = \frac{1}{2}(x_1 + x_2)$ (average opinion of the agents)
$x_{dif} = \frac{1}{2}(x_1 - x_2)$ (average disagreement of the agents).

[4] A. Franci, M. Golubitsky, N. Leonard. Flexibility and stability of multi-agent, multi-option decision making: a nonlinear dynamics perspective. In Preparation.

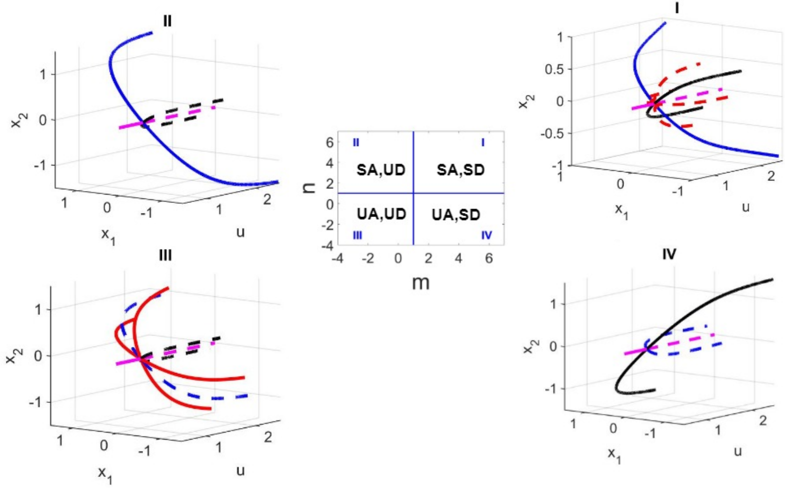
$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities

In x_{avg}, x_{dif} variables the model is locally equivalent to the $S_2 \times S_2$-symmetric nondegenerate normal form[5]

$$\begin{aligned} \dot{x}_{avg} &= x_{avg}^3 + m(c) x_{avg} x_{dif}^2 - (u-1) x_{avg} \\ \dot{x}_{dif} &= x_{dif}^3 + n(c) x_{avg}^2 x_{dif} - (u-1) x_{dif} \end{aligned} \tag{5}$$

[5] M. Golubitsky and D. G. Schaeffer, Singularities and Groups in Bifurcation Theory (Applied Mathematical Sciences 51). New York, NY, *USA: Springer-Verlag*, 1985.

$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities



$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities

$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \tag{4}$$

$$\begin{aligned} \dot{x}_{avg} &= x_{avg}^3 + m(c)x_{avg}x_{dif}^2 - (u-1)x_{avg} \\ \dot{x}_{dif} &= x_{dif}^3 + n(c)x_{avg}^2 x_{dif} - (u-1)x_{dif} \end{aligned} \tag{5}$$

A single parameter, c, of (4) determines the local bifurcation diagram of the system from four possible topologically distinct configurations.

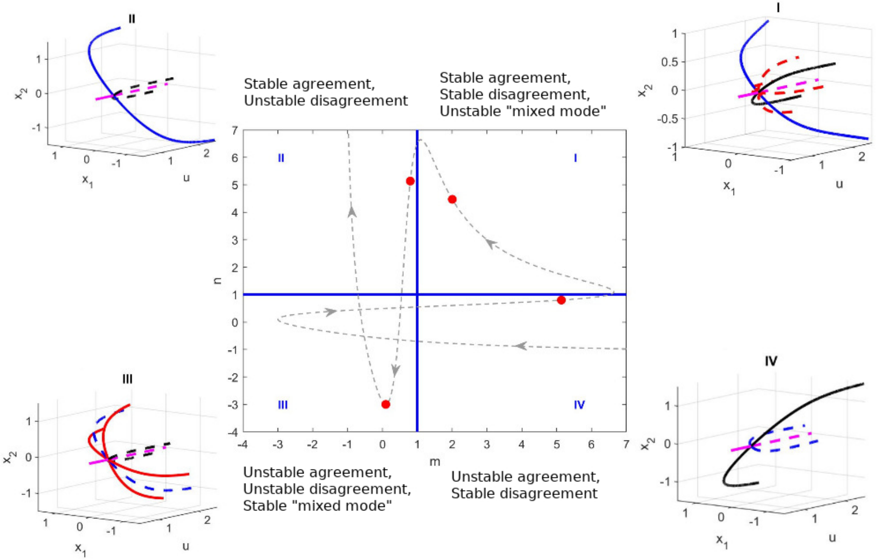
$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities

$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \tag{4}$$

$$\begin{aligned} m(c) &= \frac{-3(3c^3 + 4c^2 - 4c - 4)}{3c^3 + 12c^2 + 12c + 4} \\ n(c) &= \frac{3(3c^3 - 4c^2 - 4c + 4)}{-(3c^3 - 12c^2 + 12c - 4)} \end{aligned} \tag{6}$$

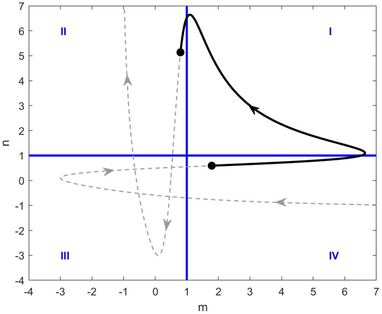
$$-2.7024 < c < 2.7024 \tag{7}$$

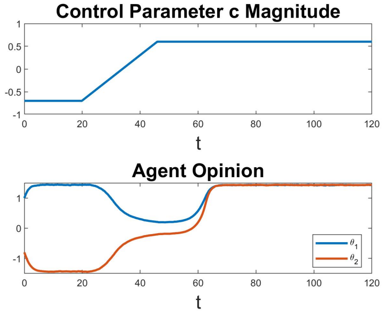
$S_2 \times S_2$ Realization with Sigmoidal Nonlinearities



Model Parameters as Control Parameters

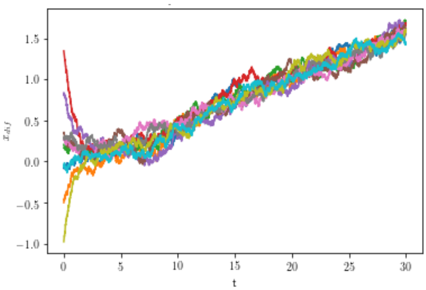
$$\boxed{\dot{x}_i = -x_i + u\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big)} \quad (4)$$

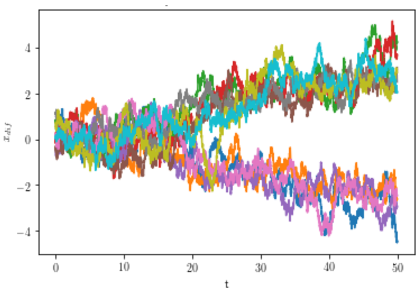




Tunable Sensitivity and Robustness

$$\boxed{\dot{x}_i = -x_i + u(t)\Big(S(cx_j) + \frac{1}{2}S(2(x_i - cx_j))\Big) + \beta_i(t)} \qquad (8)$$





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