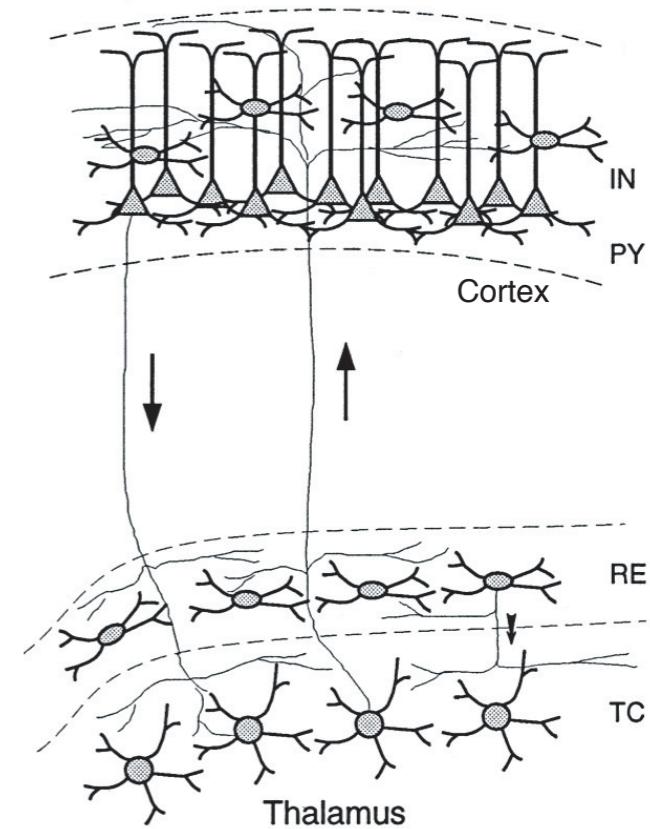


# Mathematical Neuroscience: from neurons to networks



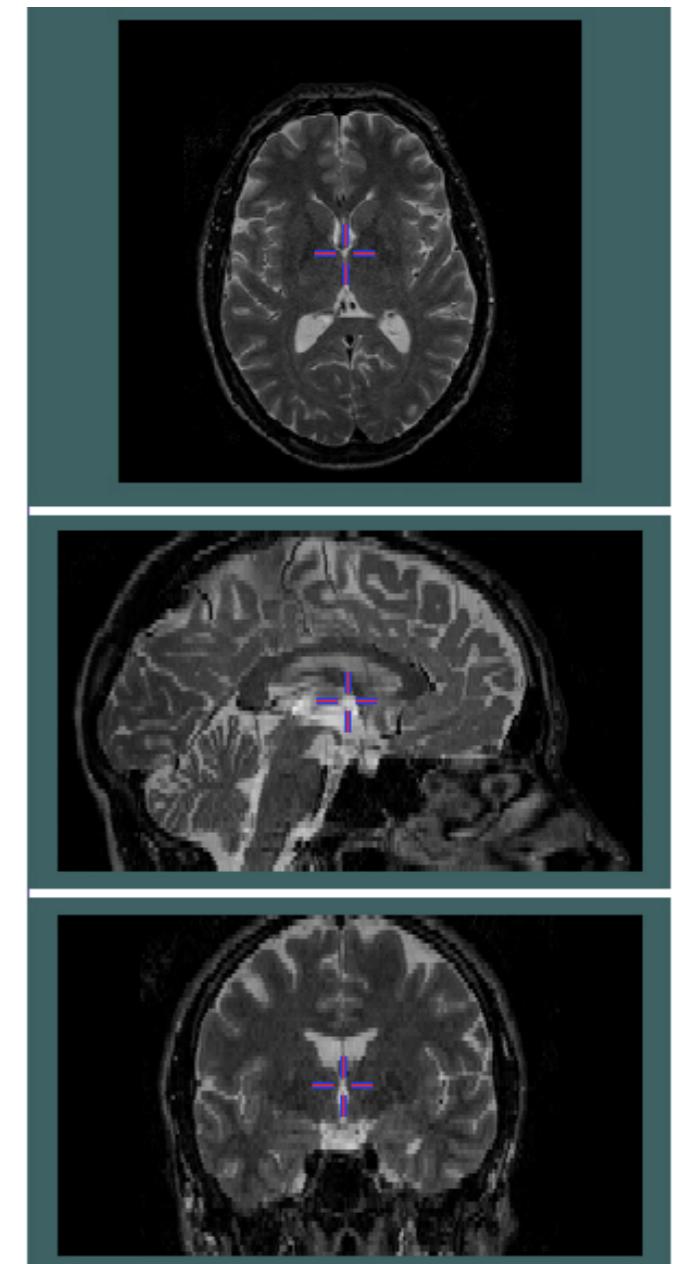
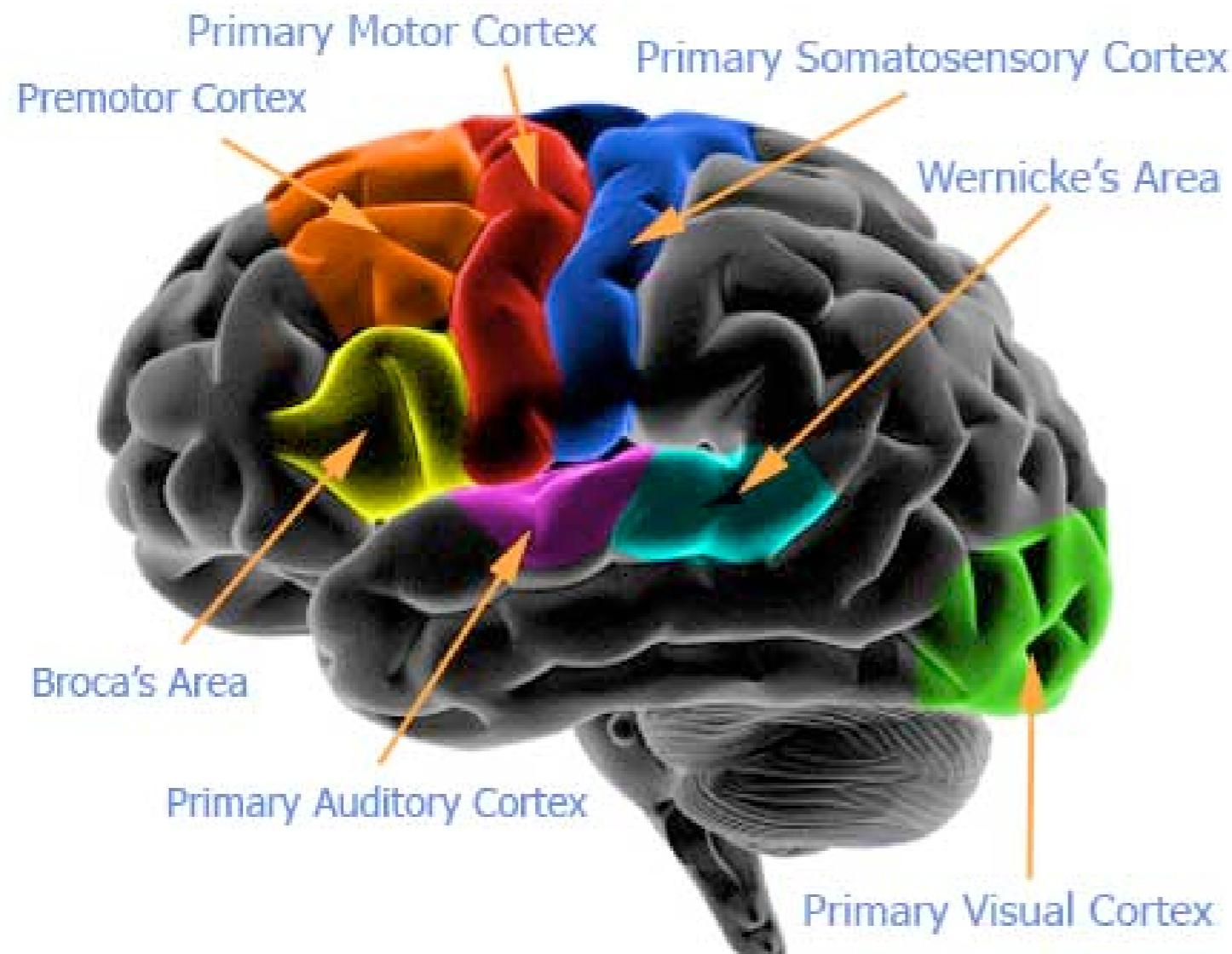
Steve  
Coombes



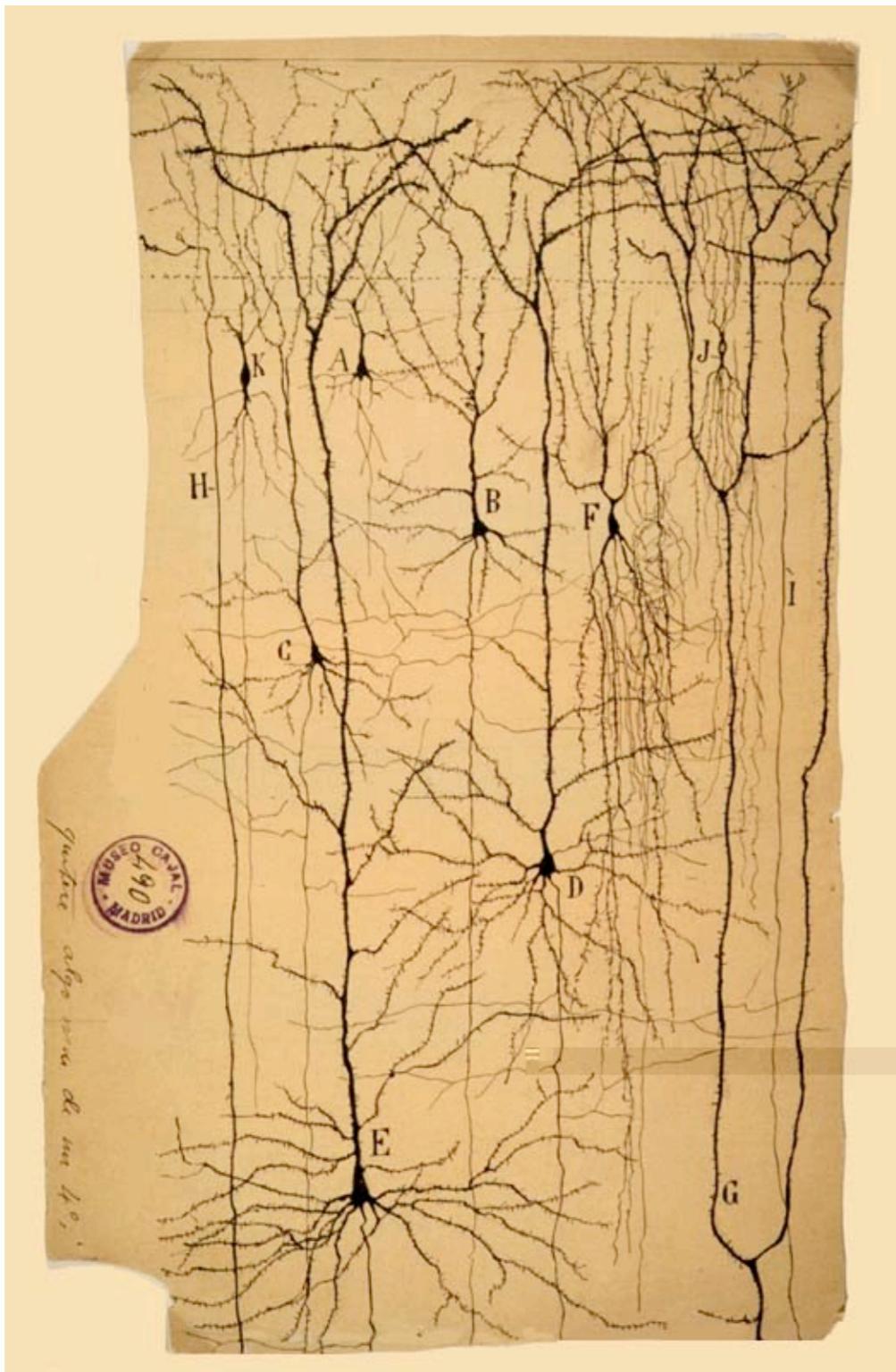
The University of  
Nottingham

School of Mathematical  
Sciences

# Brain and Cortex

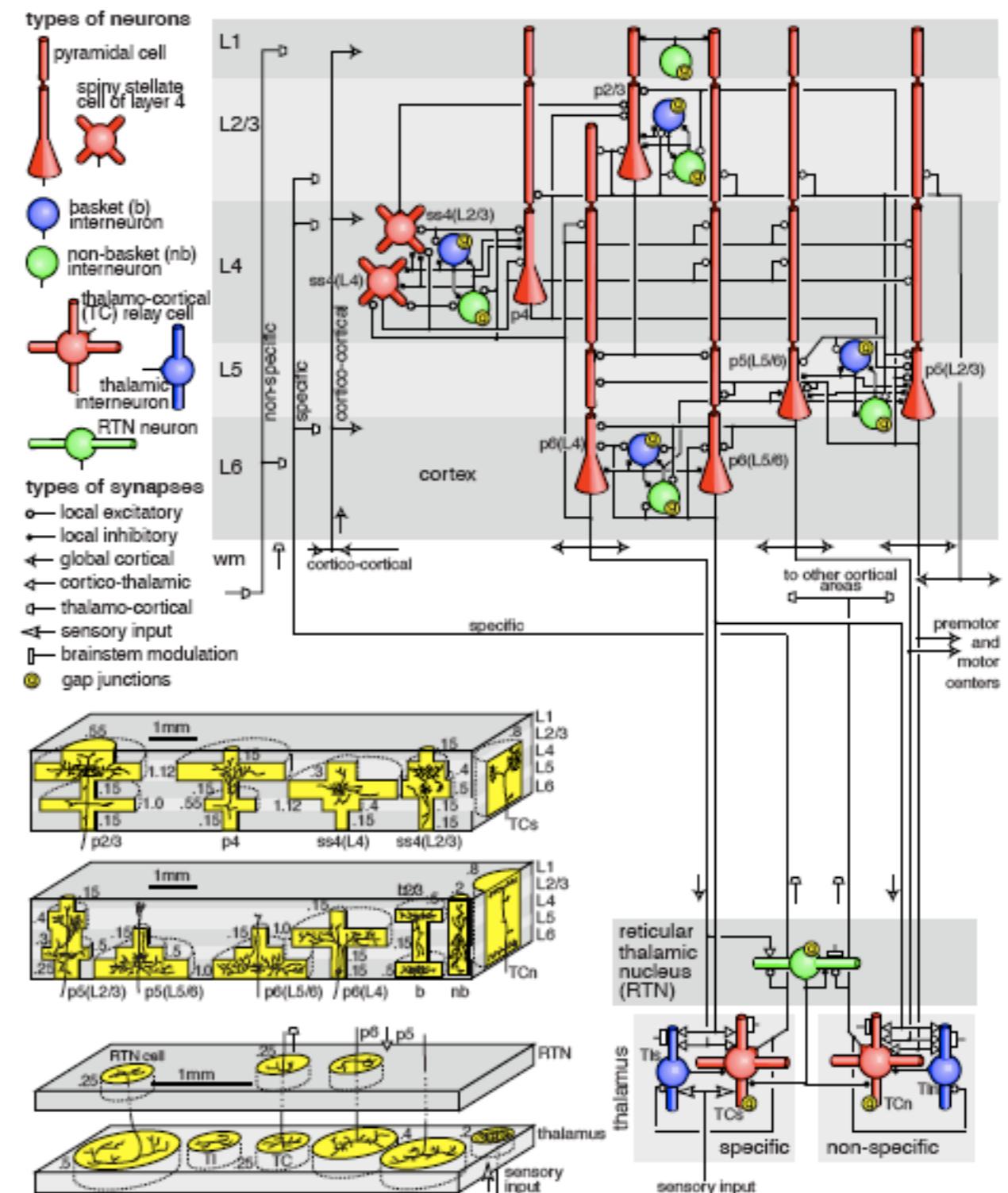
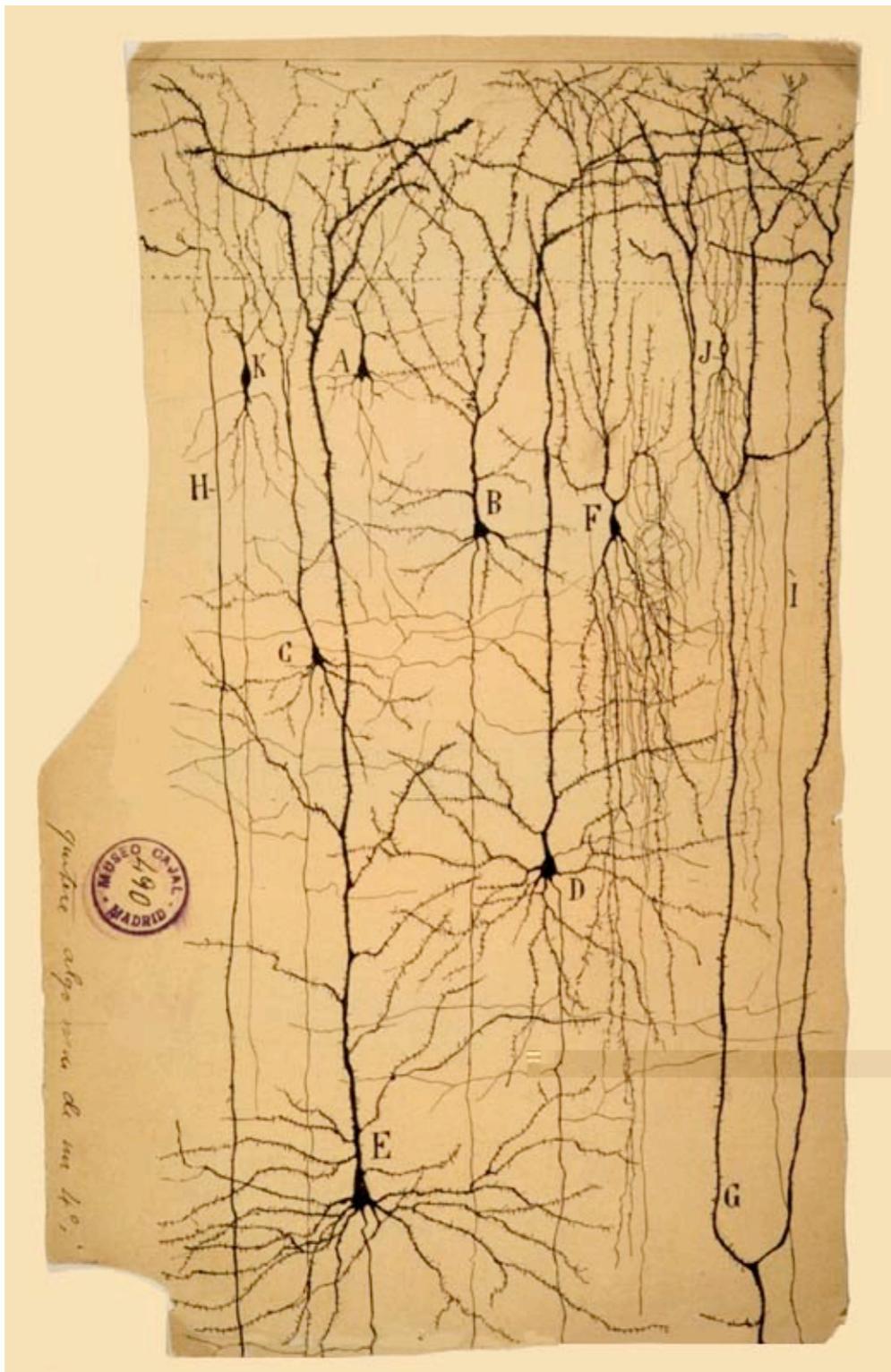


# Principal cells and interneurons



Santiago Ramón y Cajal  
1900

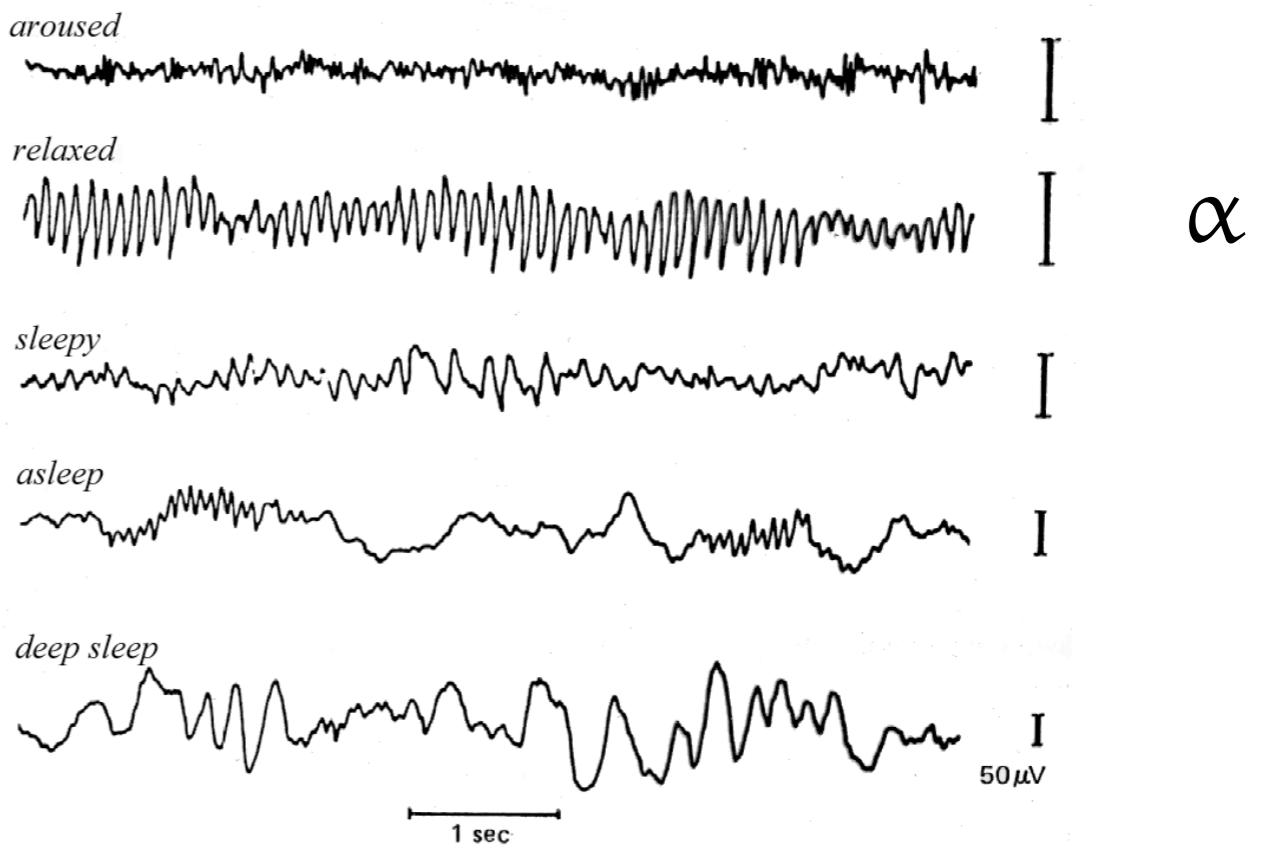
# Principal cells and interneurons



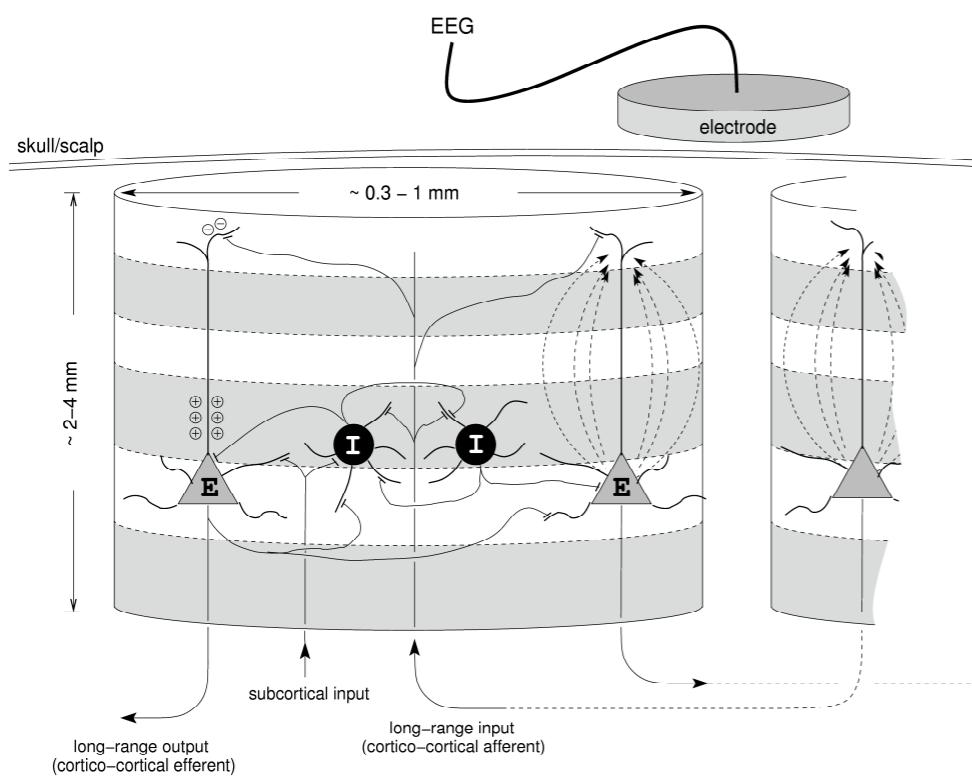
Santiago Ramón y Cajal  
1900

Eugene Izhikevich  
2008

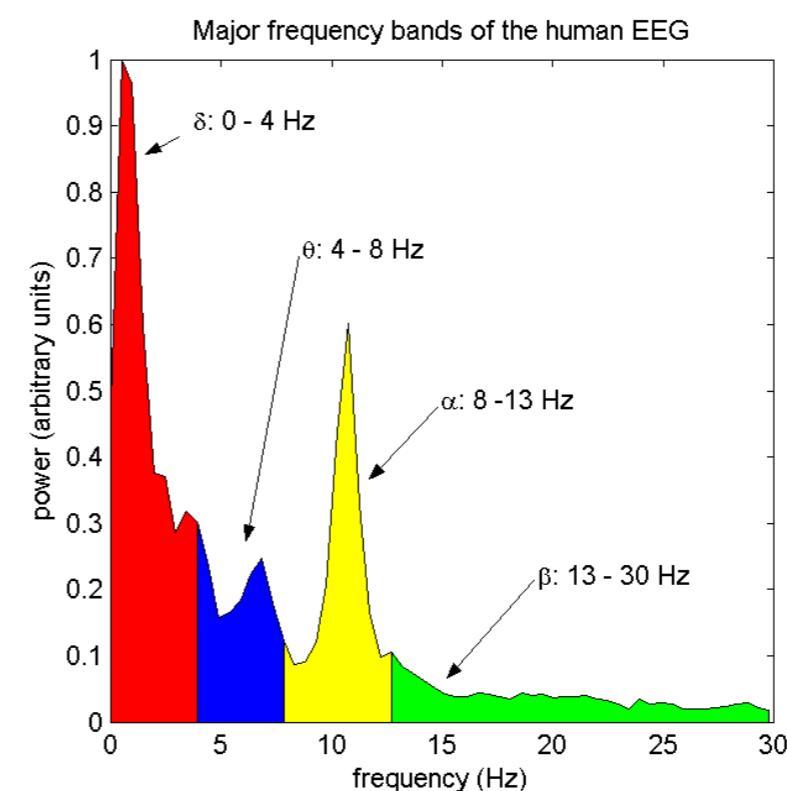
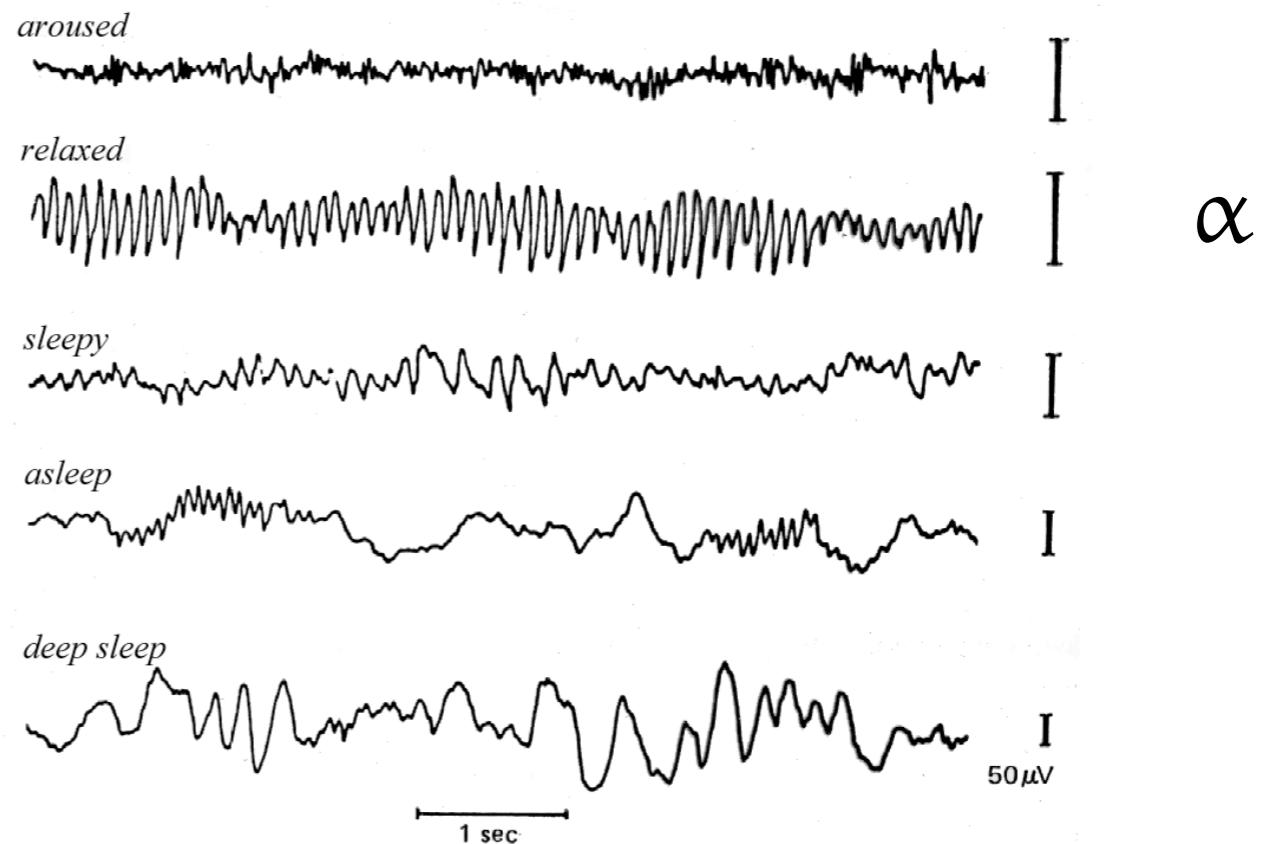
# Electroencephalogram (EEG) power spectrum



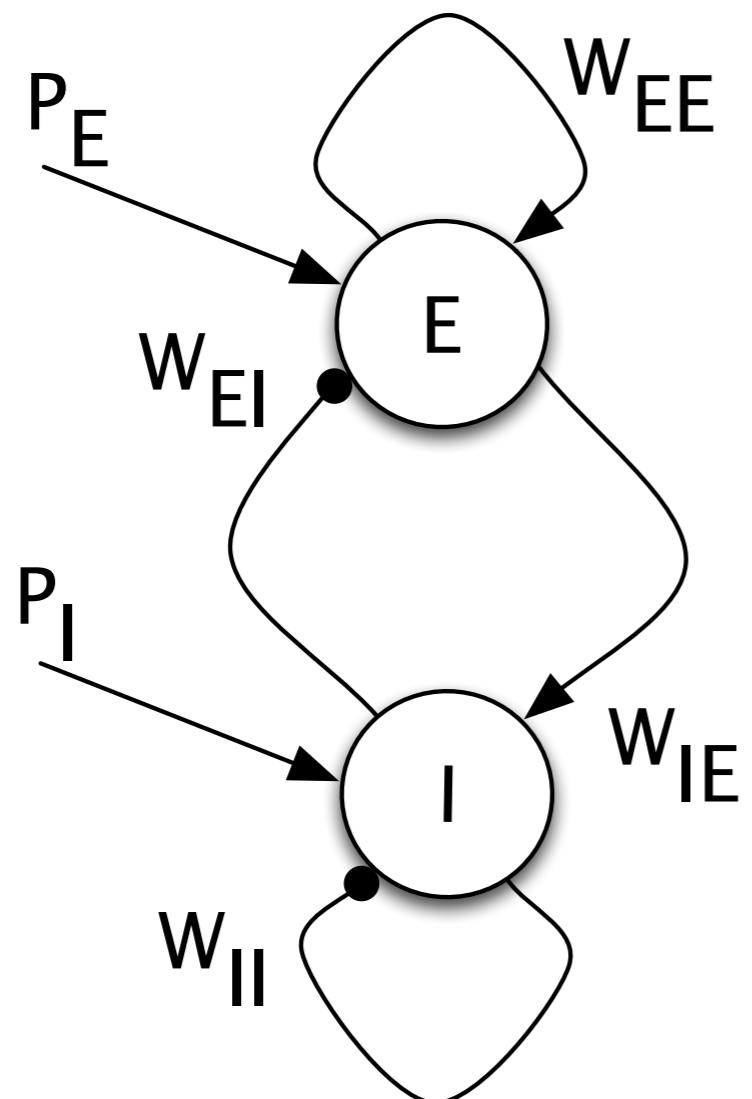
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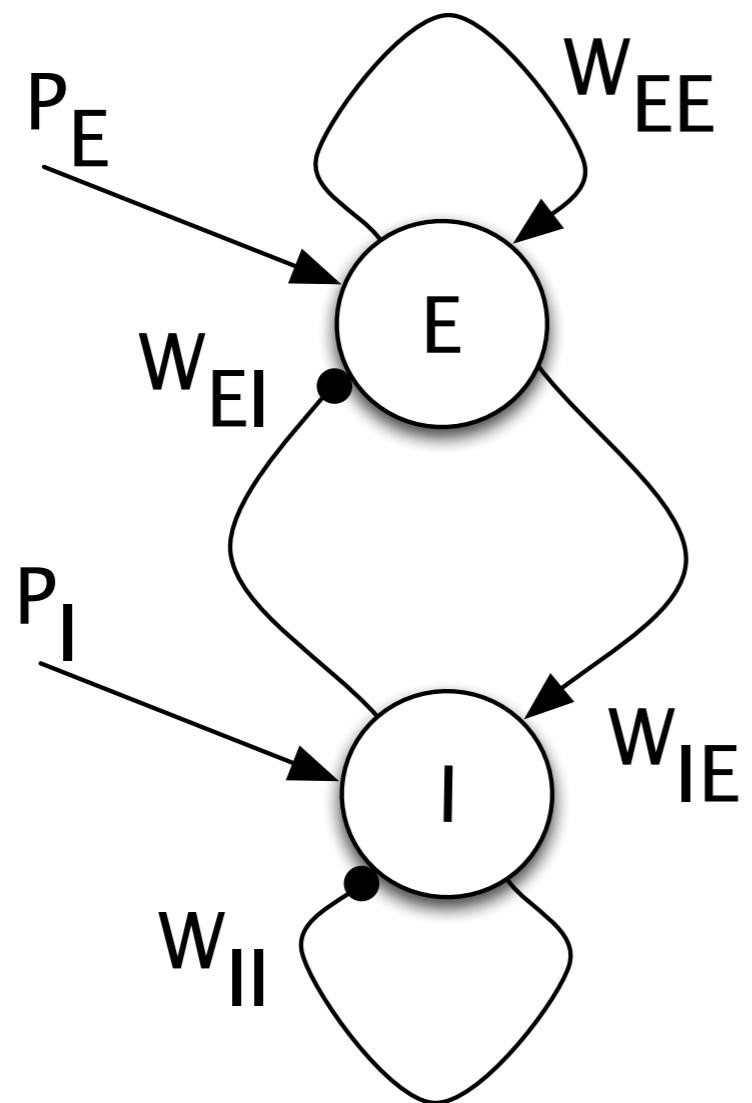
EEG records the activity of  $\sim 10^6$  pyramidal neurons.



# Population model

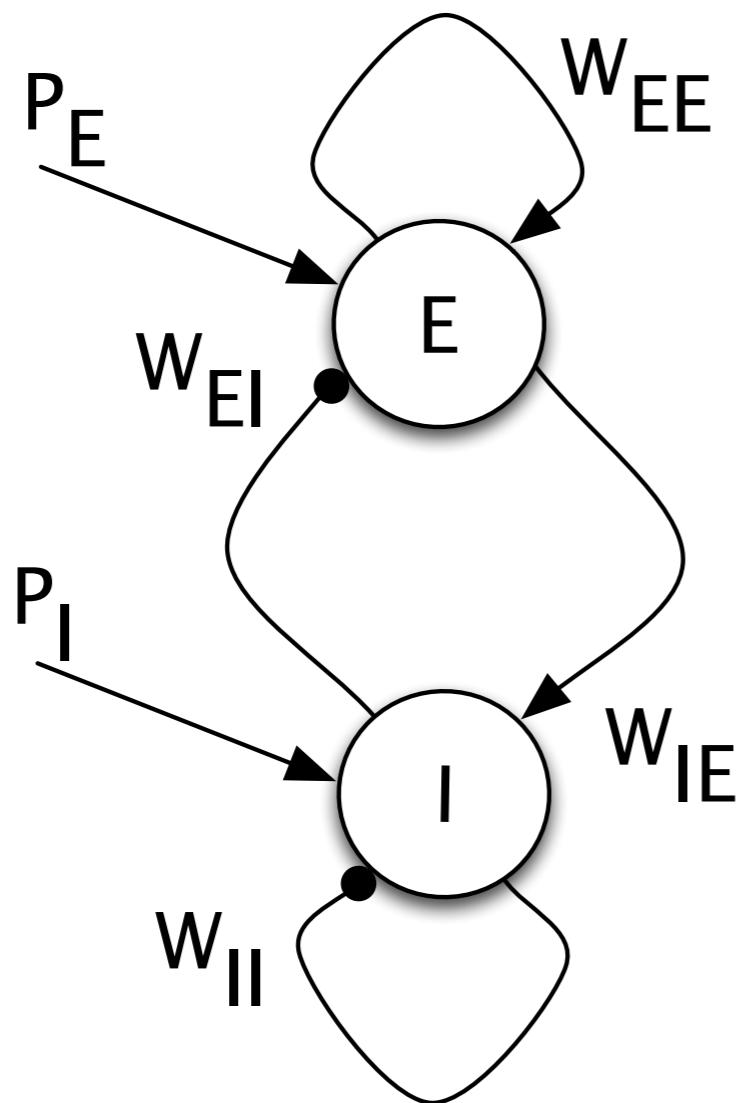


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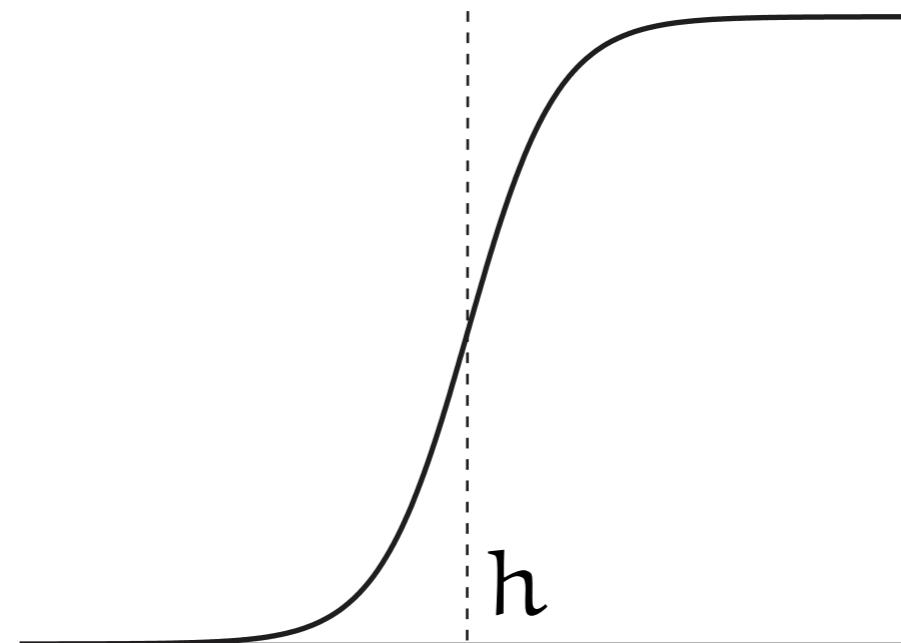


$$\dot{E} = -\frac{E}{\tau_E} + W_{EE}g_{EE}(A^+ - E) + W_{EI}g_{EI}(A^- - E) + P_E$$

# Population model

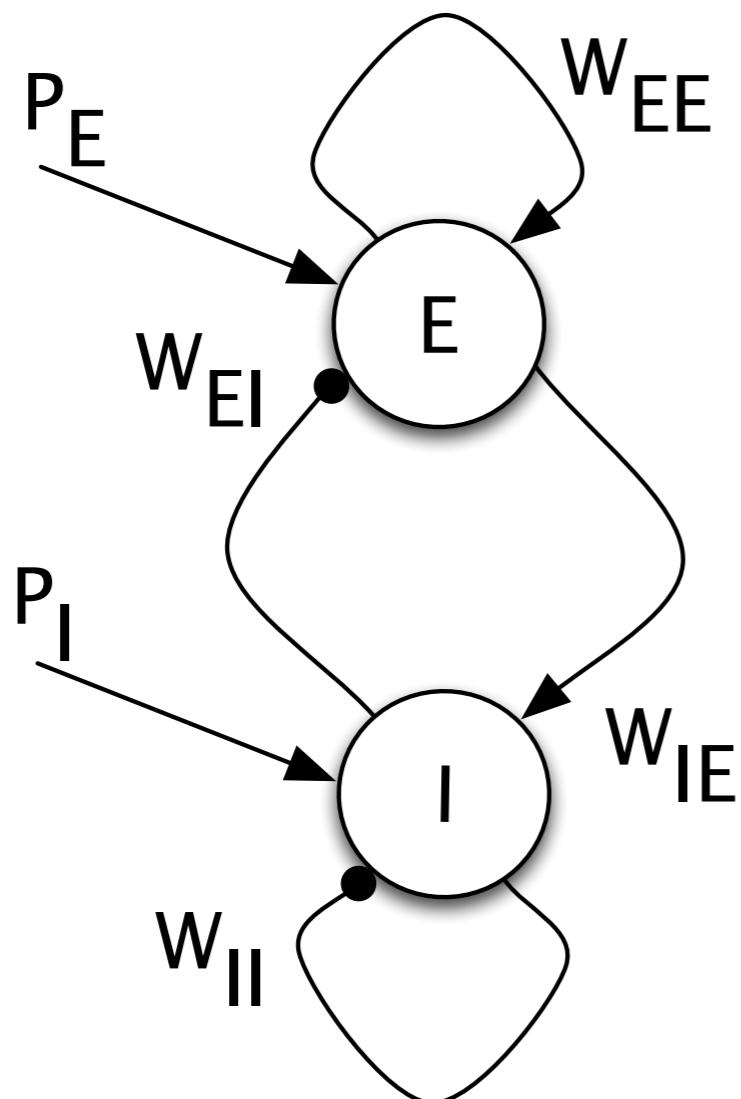


Firing rate activity  $f(E)$

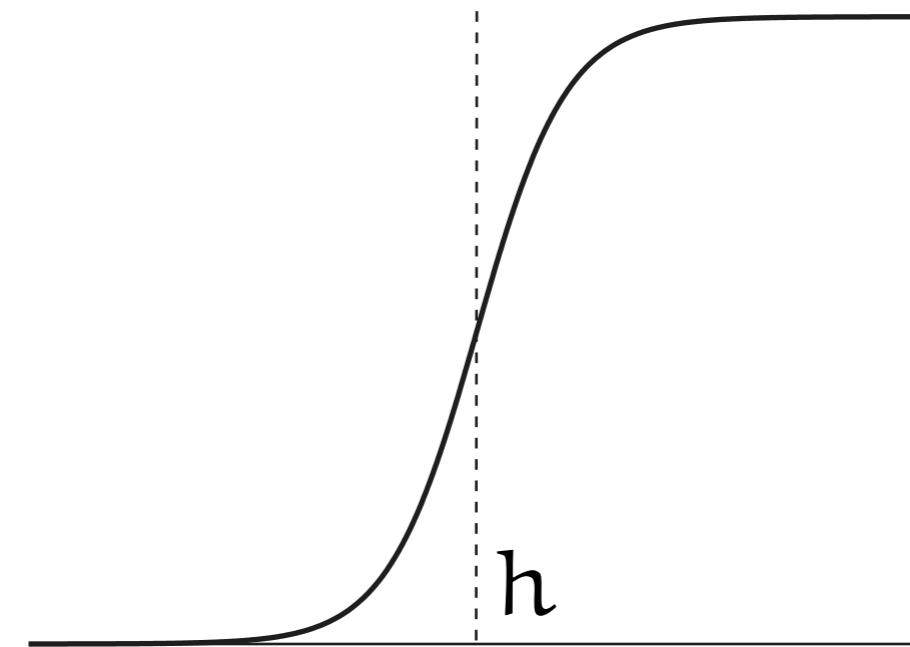


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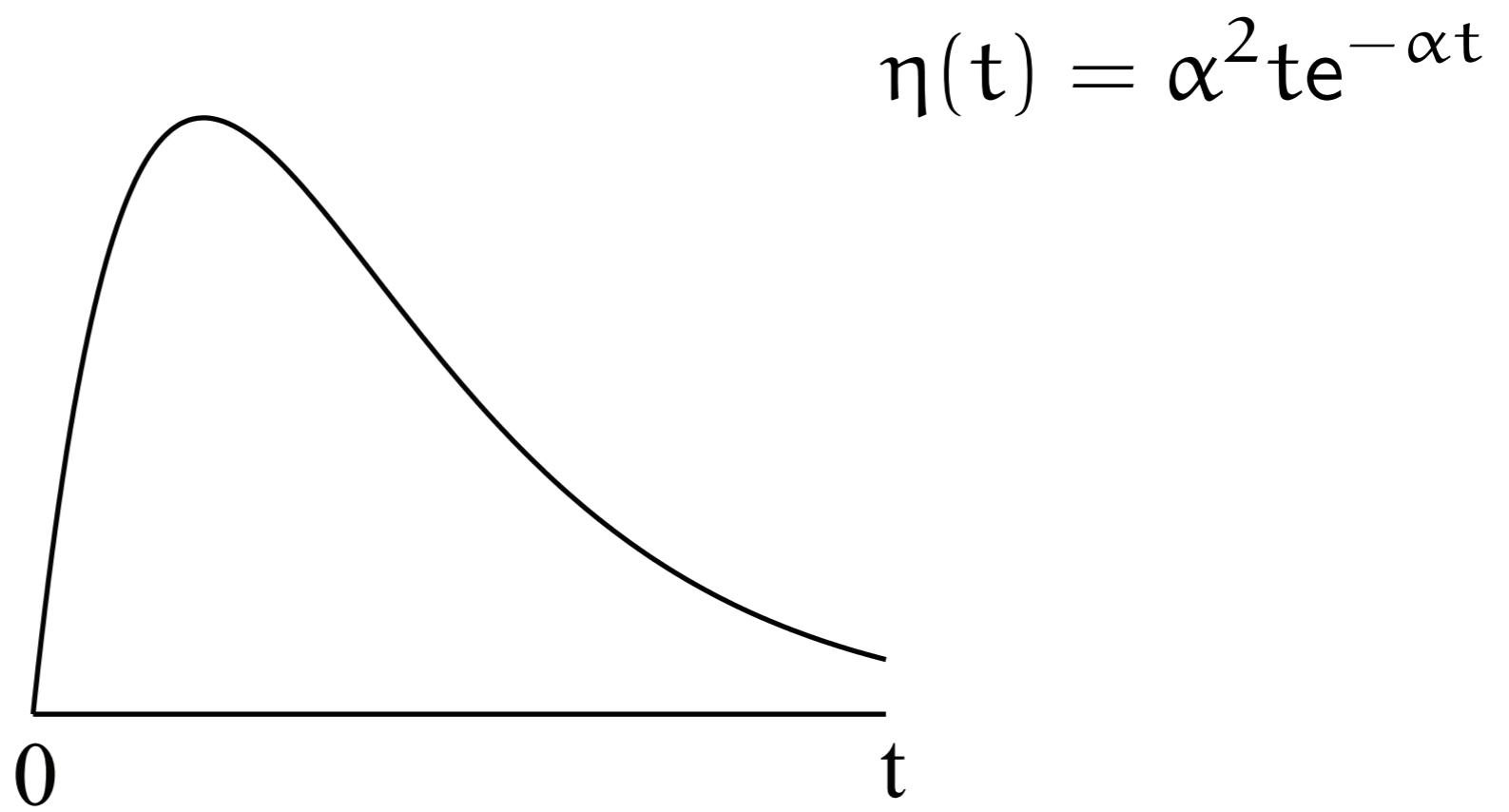
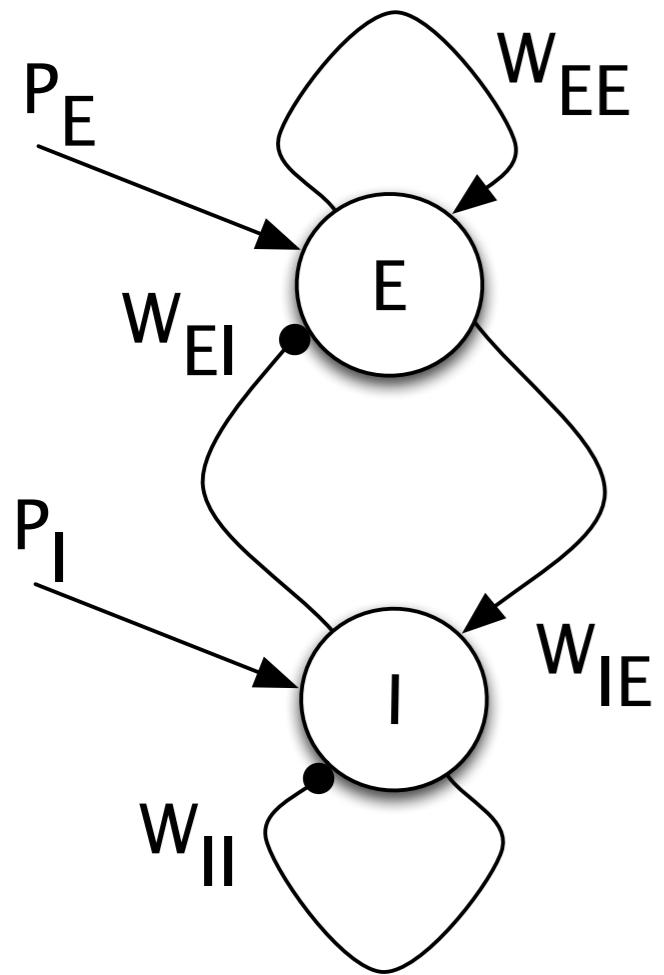
Firing rate activity  $f(E)$

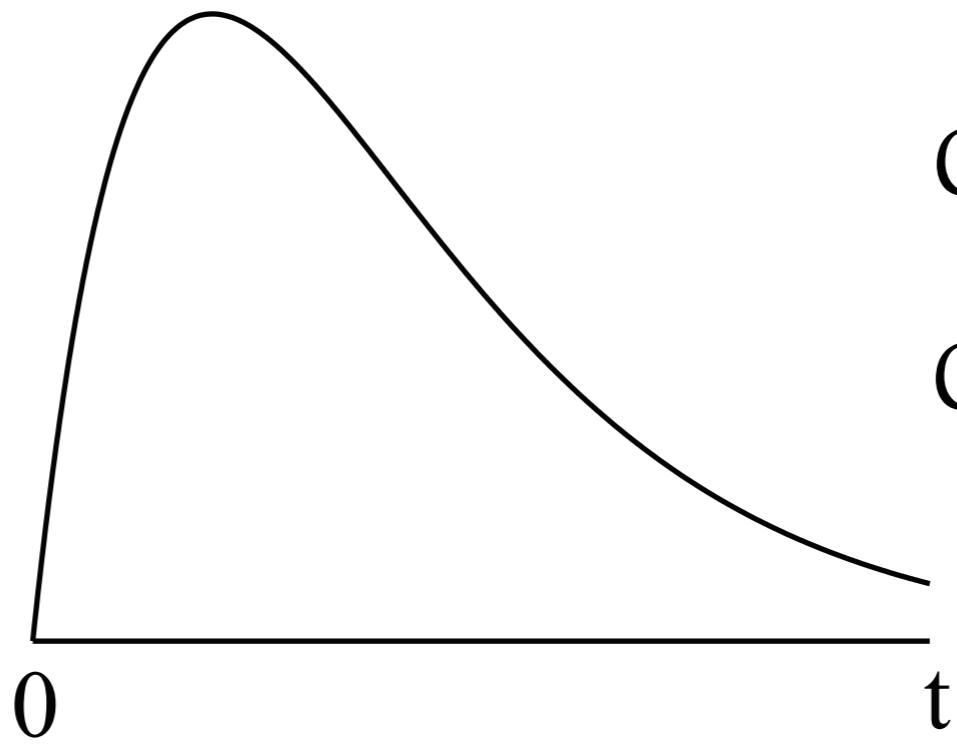
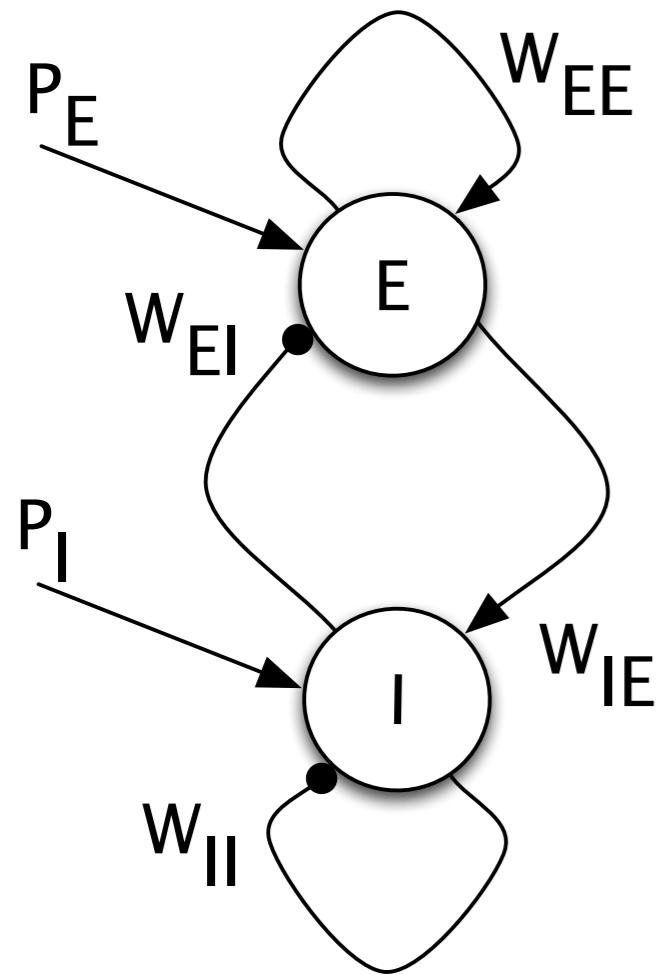


Firing rate activity  $f(I)$

$$\dot{E} = -\frac{E}{\tau_E} + W_{EE}g_{EE}(A^+ - E) + W_{EI}g_{EI}(A^- - E) + P_E$$

$$\dot{I} = -\frac{I}{\tau_I} + W_{II}g_{II}(A^- - I) + W_{IE}g_{IE}(A^+ - I) + P_I$$

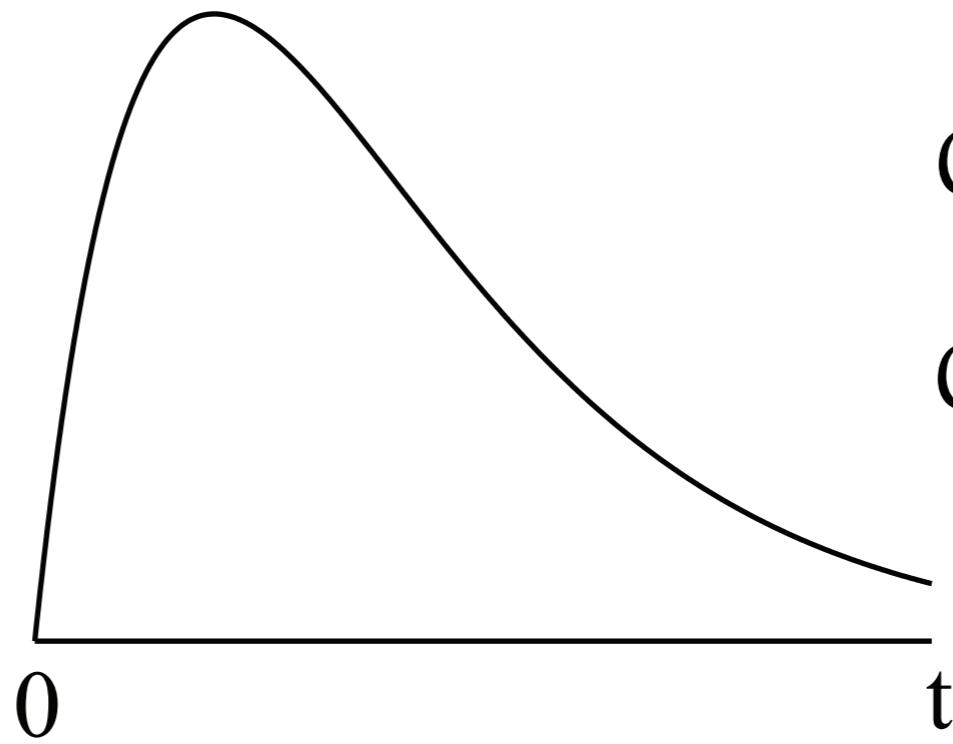
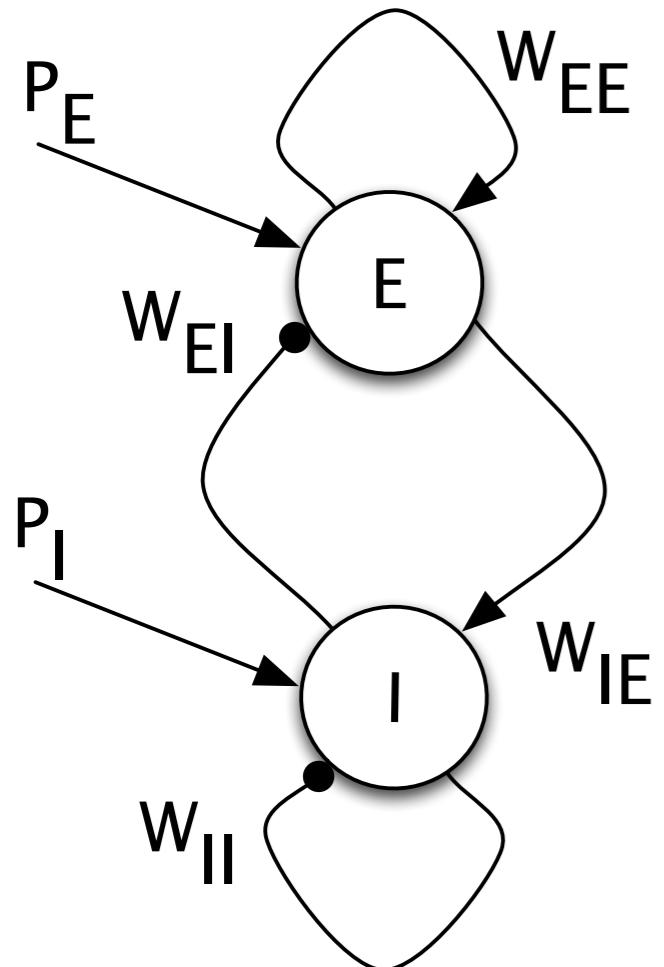




$$\eta(t) = \alpha^2 t e^{-\alpha t}$$

$$Q\eta = \delta$$

$$Q = \left( 1 + \frac{1}{\alpha} \frac{d}{dt} \right)^2$$



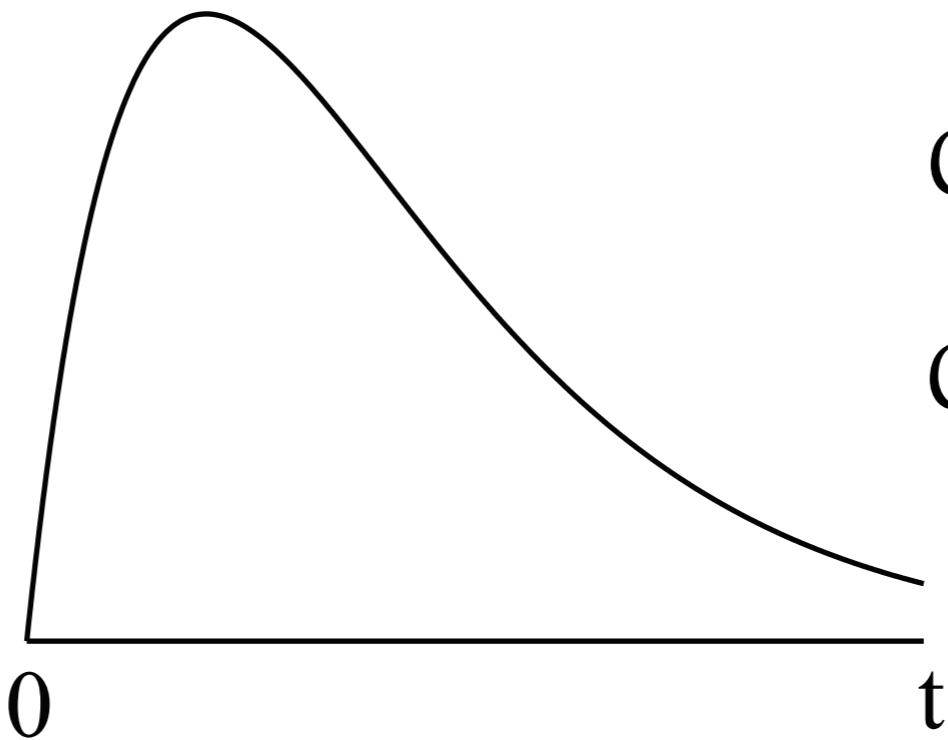
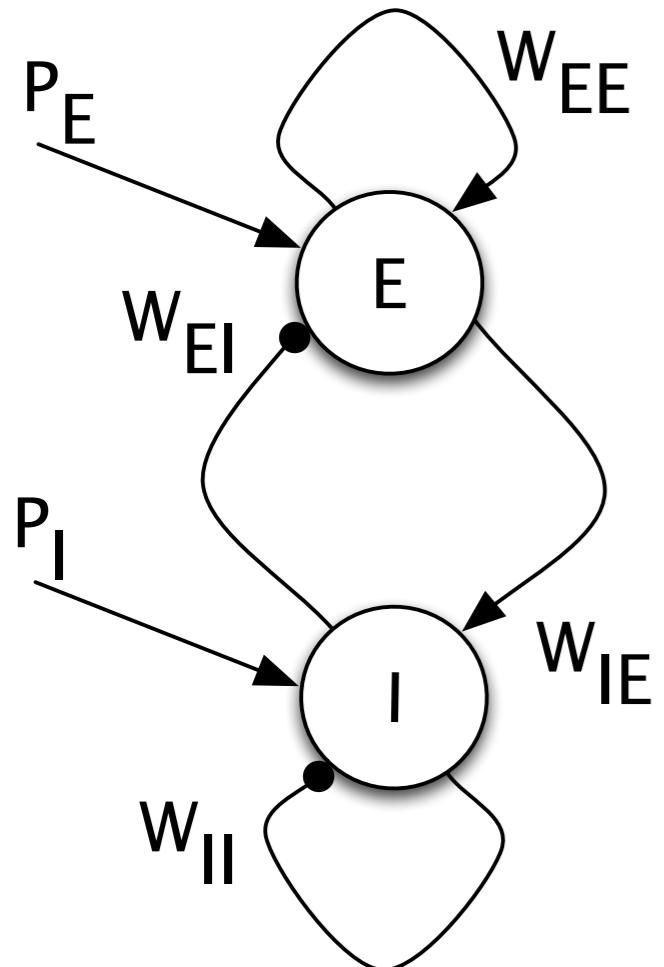
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$$Qg_{jE} = f(E)$$

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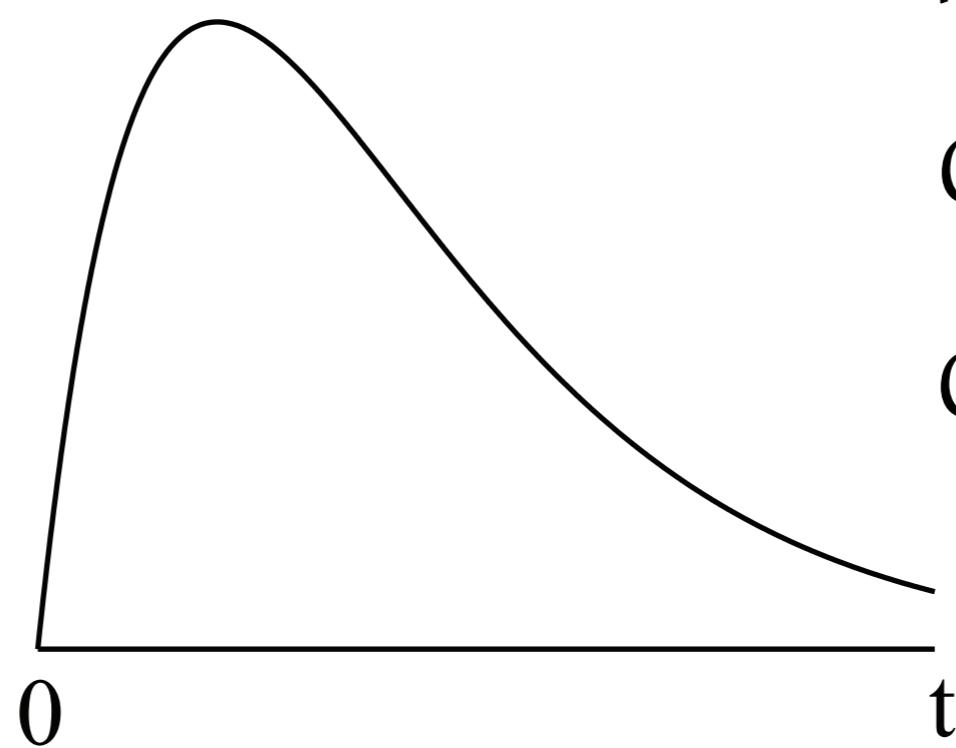
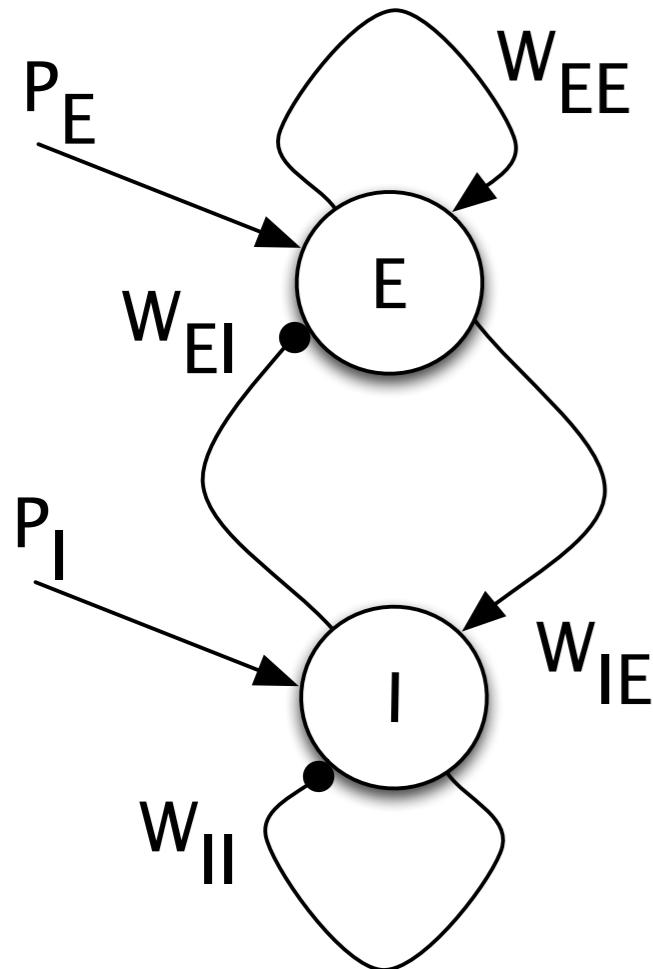
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**Steady state  
approximation**

$$I = I(g_{II}, g_{IE})$$

$$E = E(g_{EE}, g_{EI})$$



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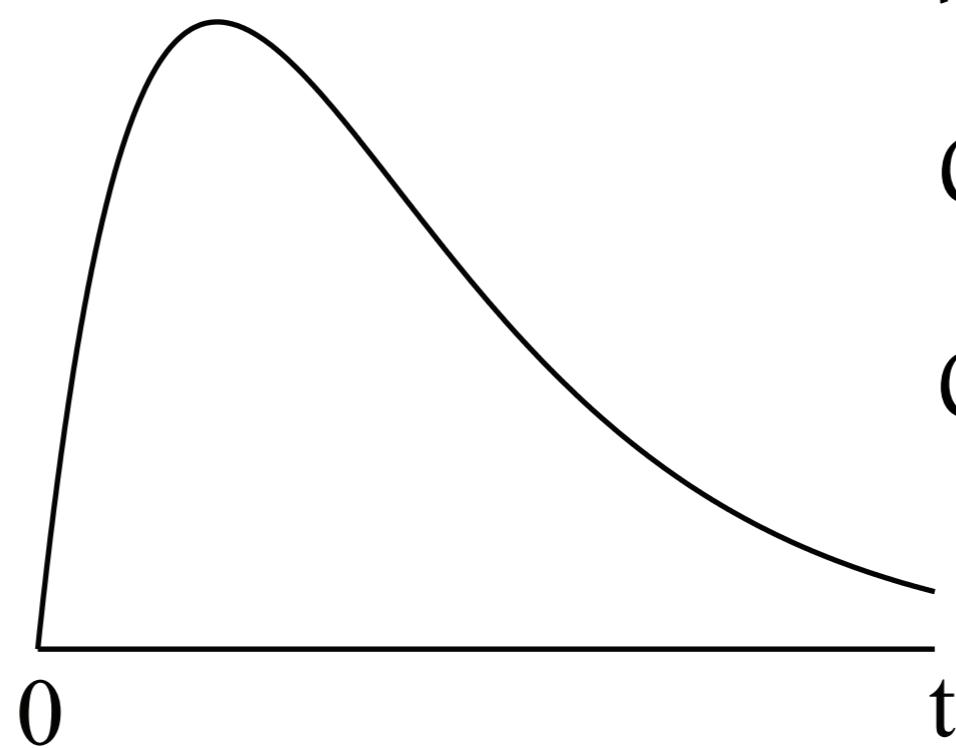
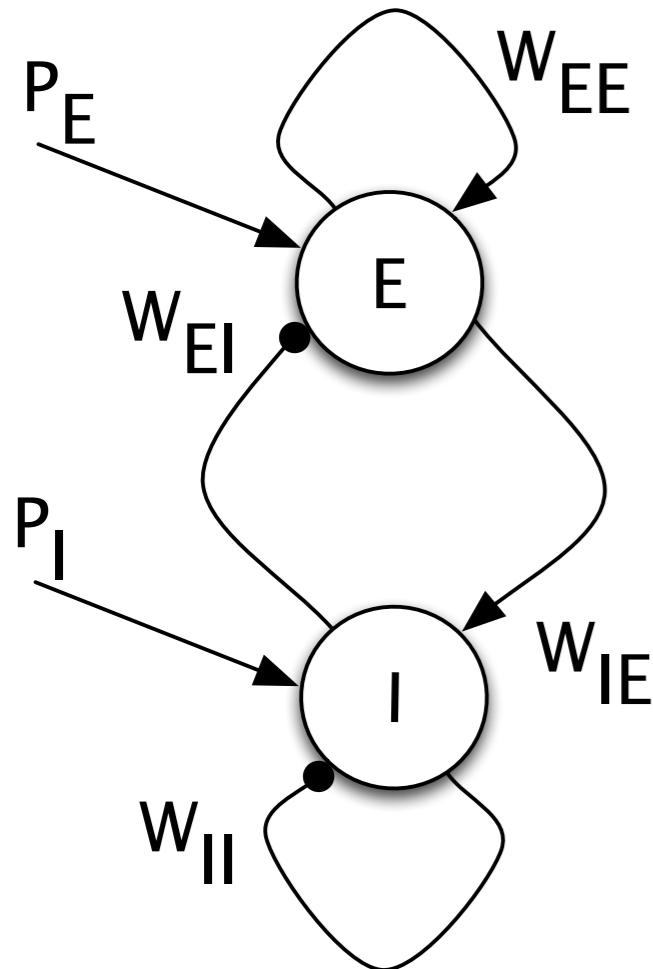
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$$Qg = f$$

$$f = f(\{g\})$$



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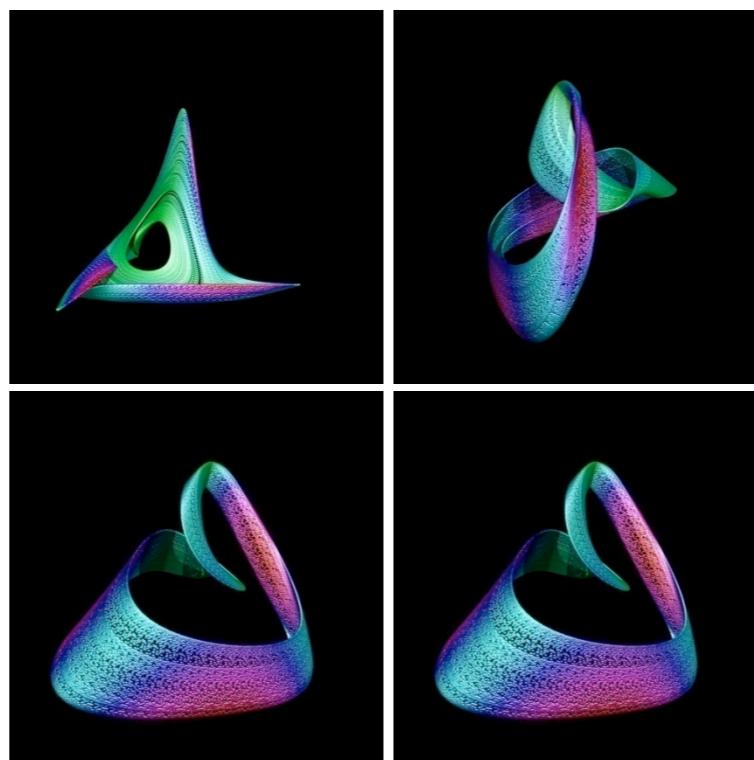
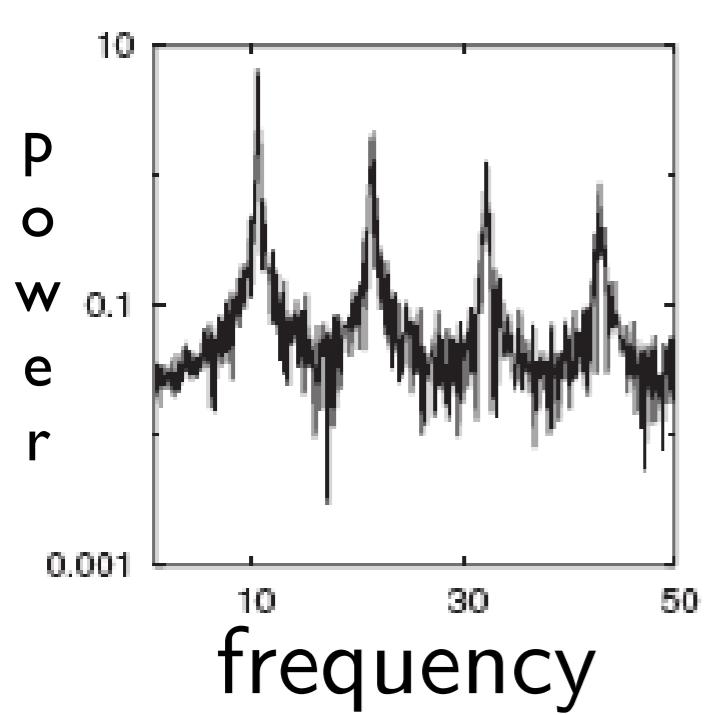
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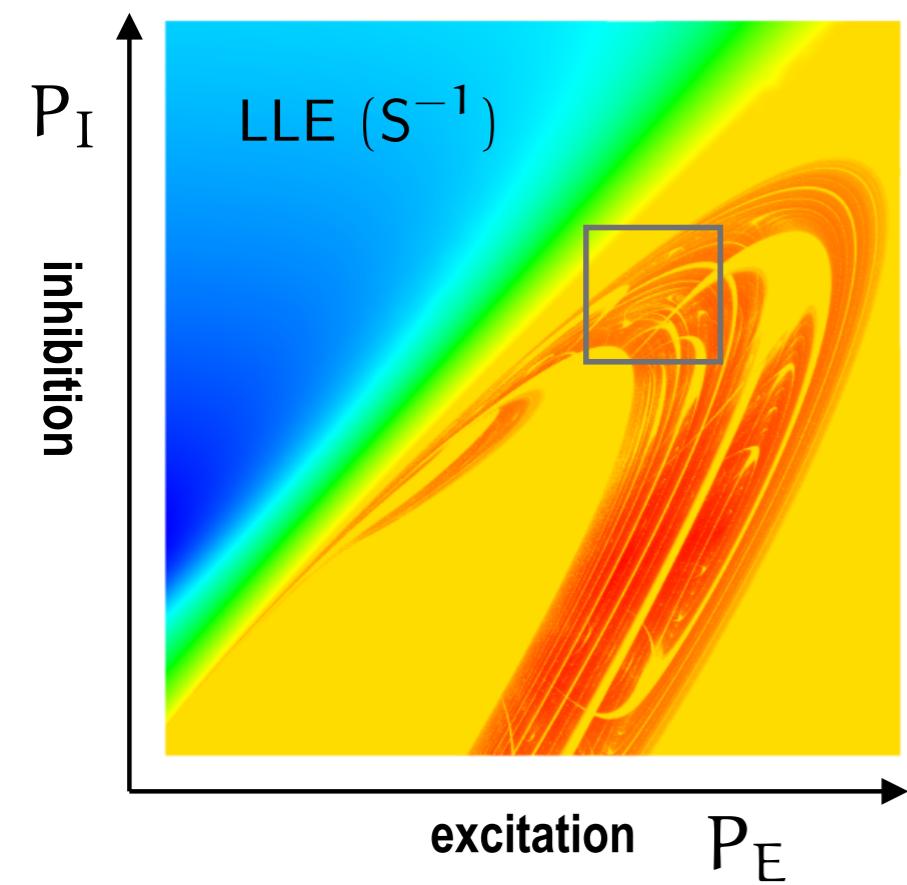
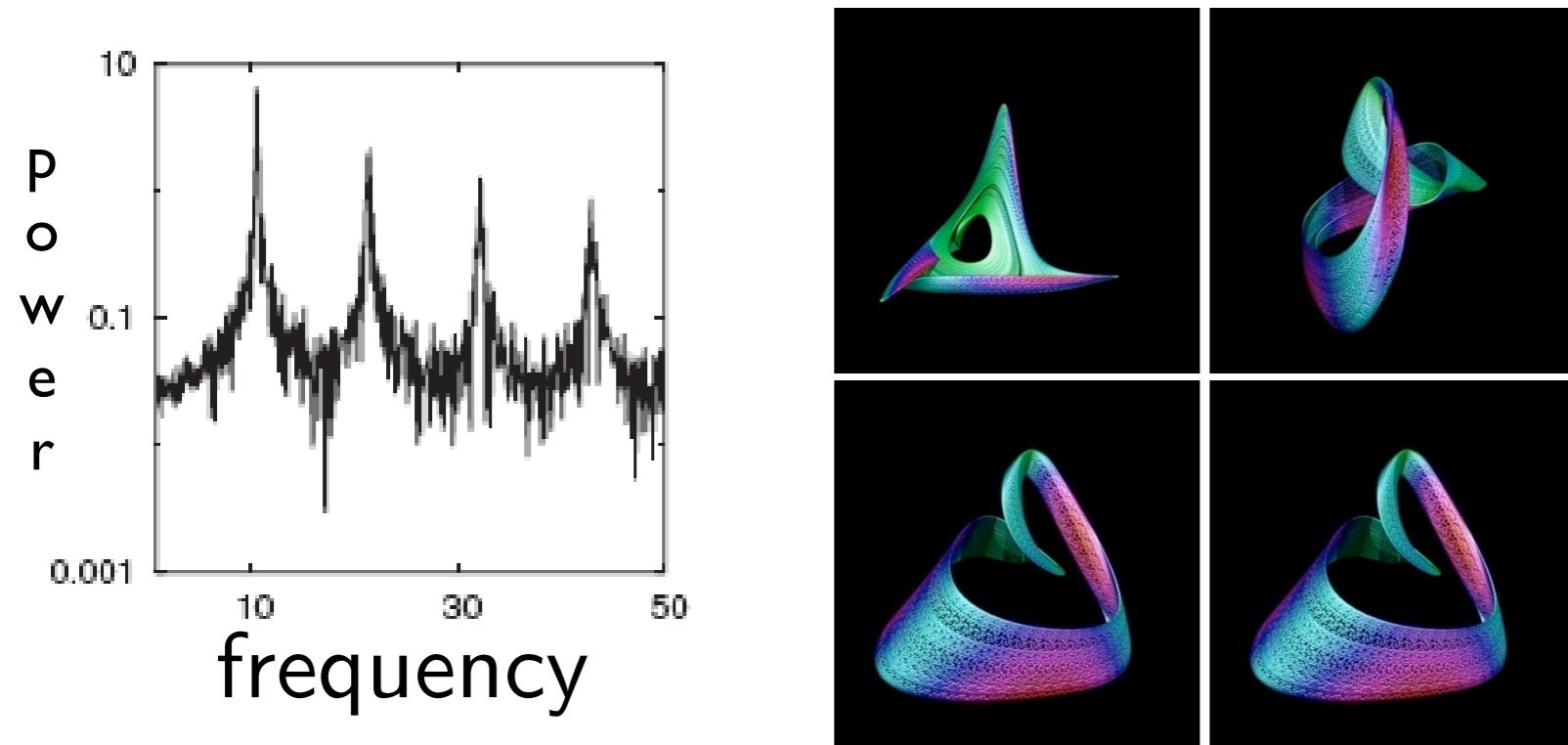
$$\begin{aligned} Qg &= f \\ f &= f(\{g\}) \end{aligned}$$

$$g = \eta * f$$

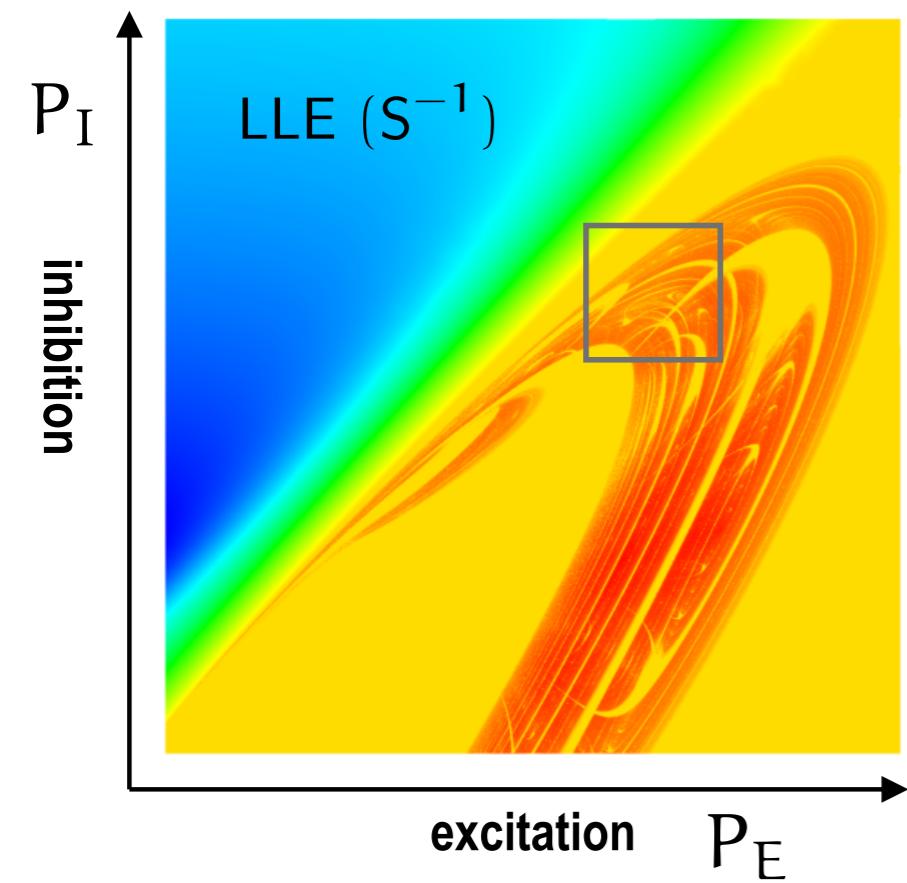
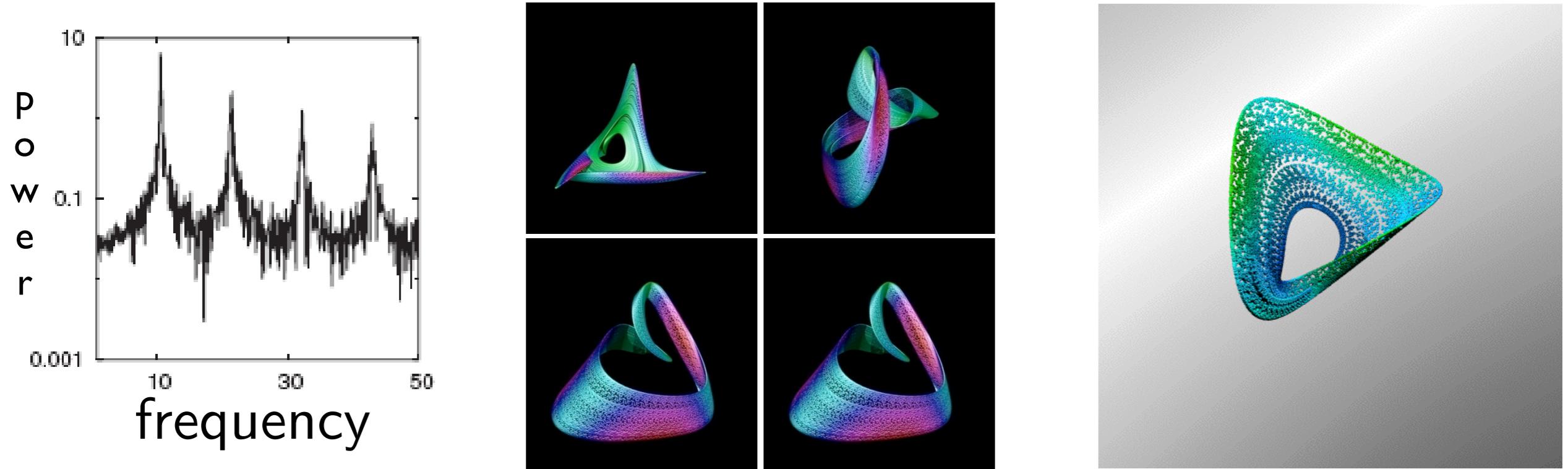
# Alphoid chaos



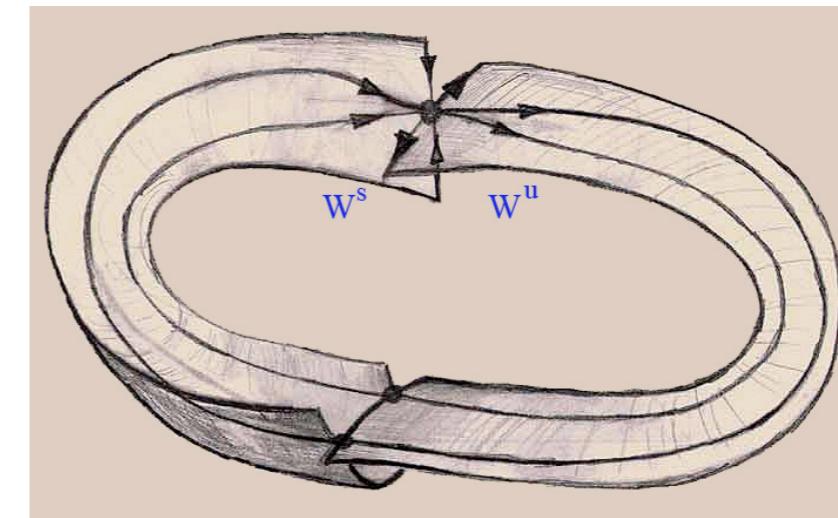
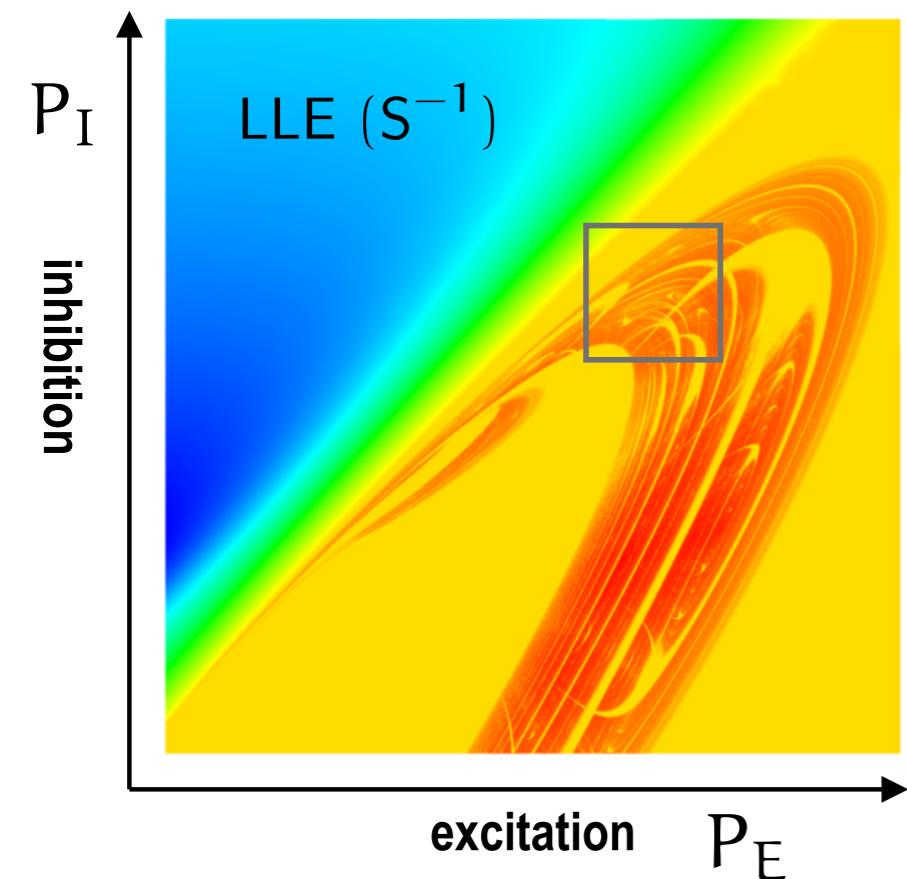
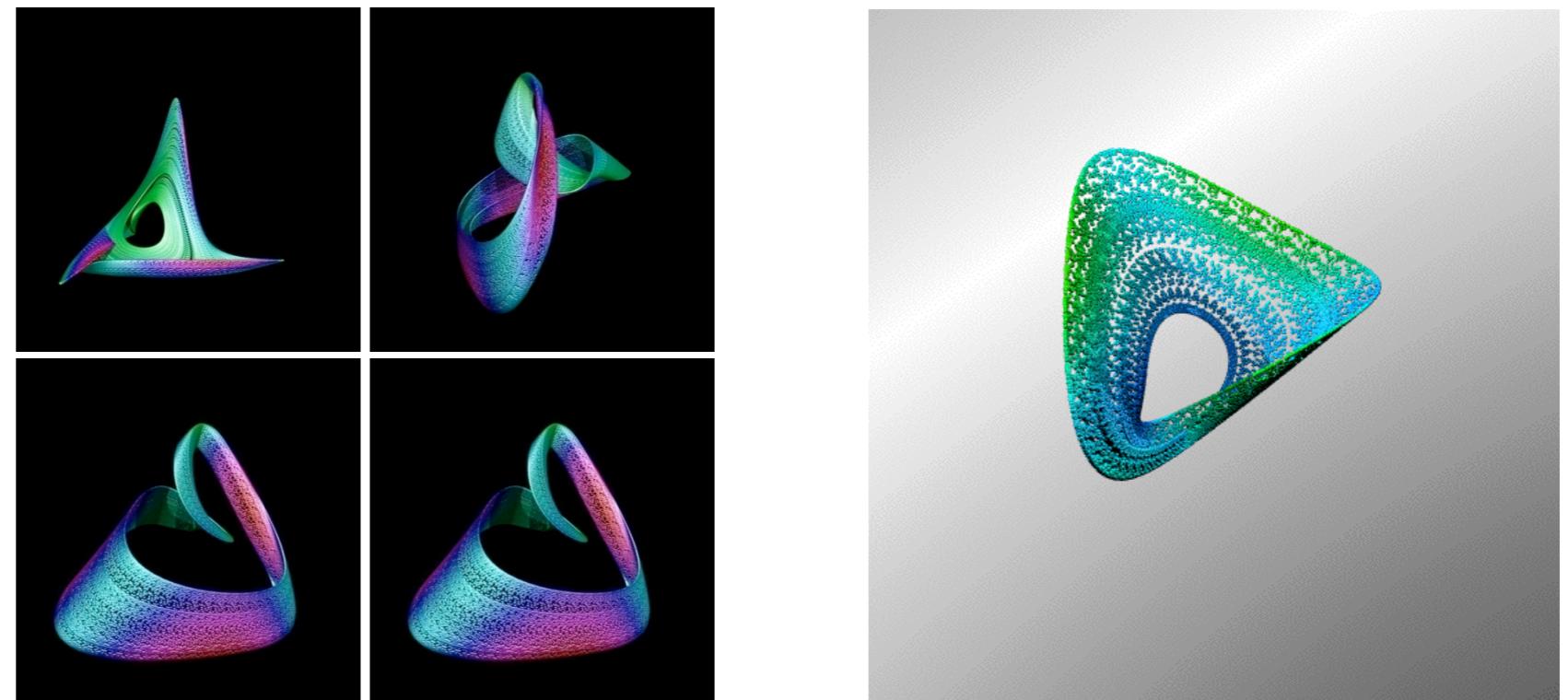
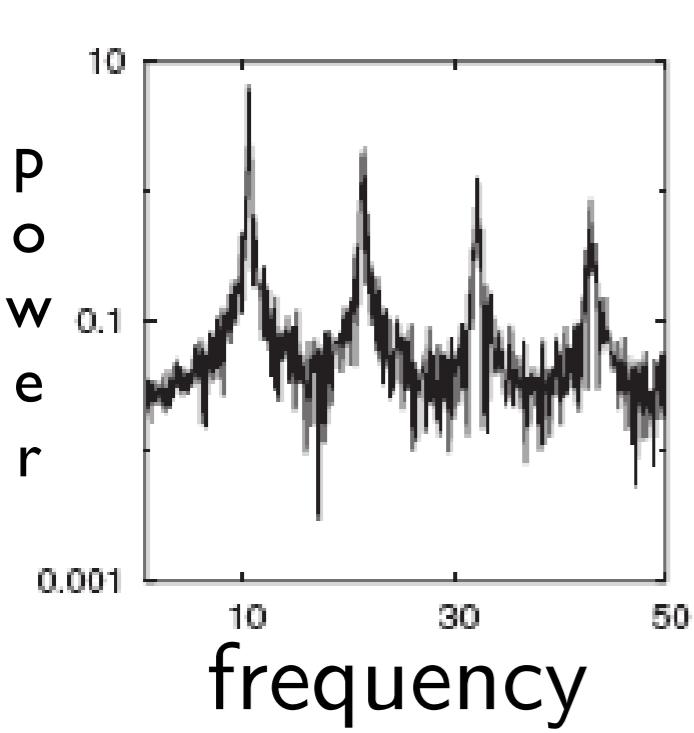
# Alphoid chaos



# Alphoid chaos



# Alphoid chaos



Shilnikov saddle-node route to chaos  
van Veen and Liley, PRL, **97**, 208101 (2006)

# Spatially extended models

$$g = w \otimes \eta * f$$

# Spatially extended models

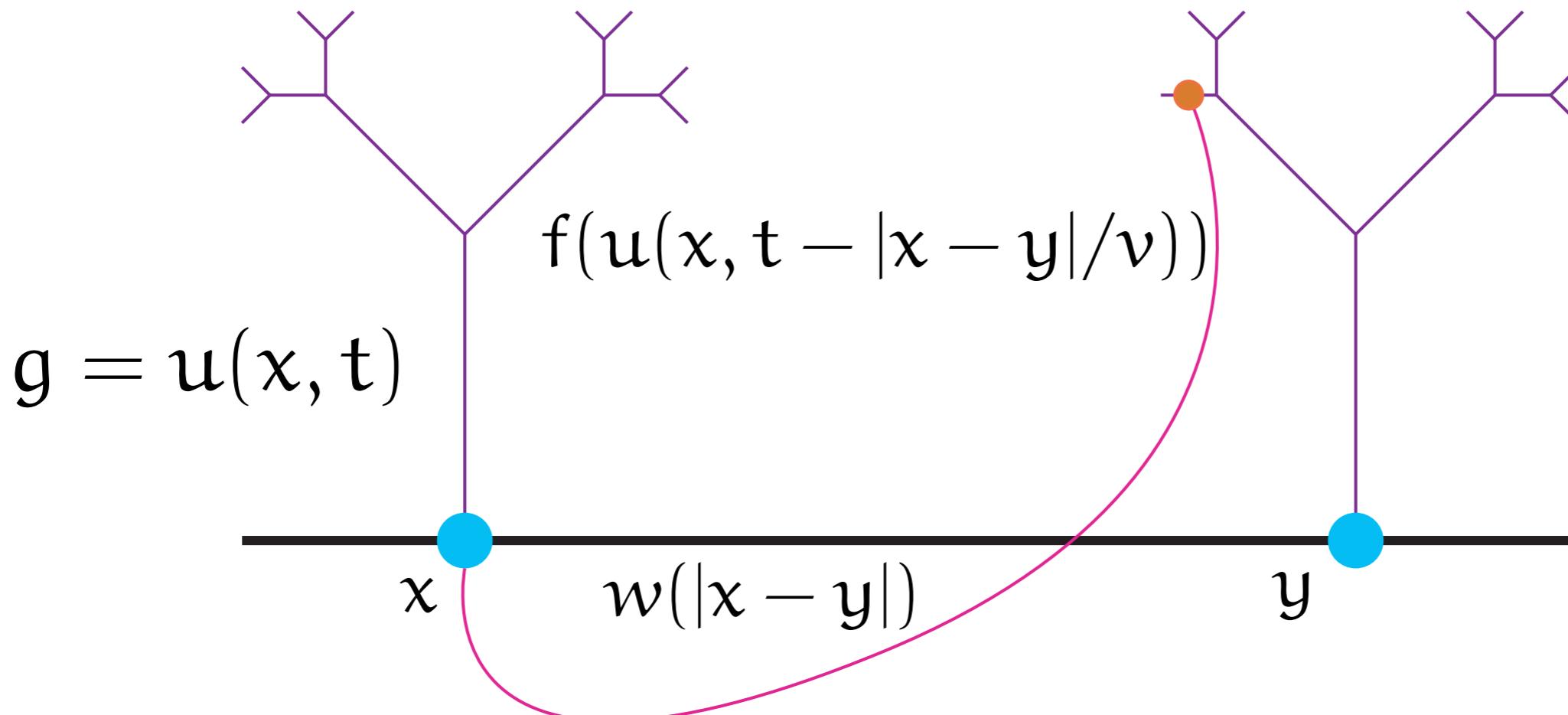
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Simplest neural field model: Wilson-Cowan ('72), Amari ('77)

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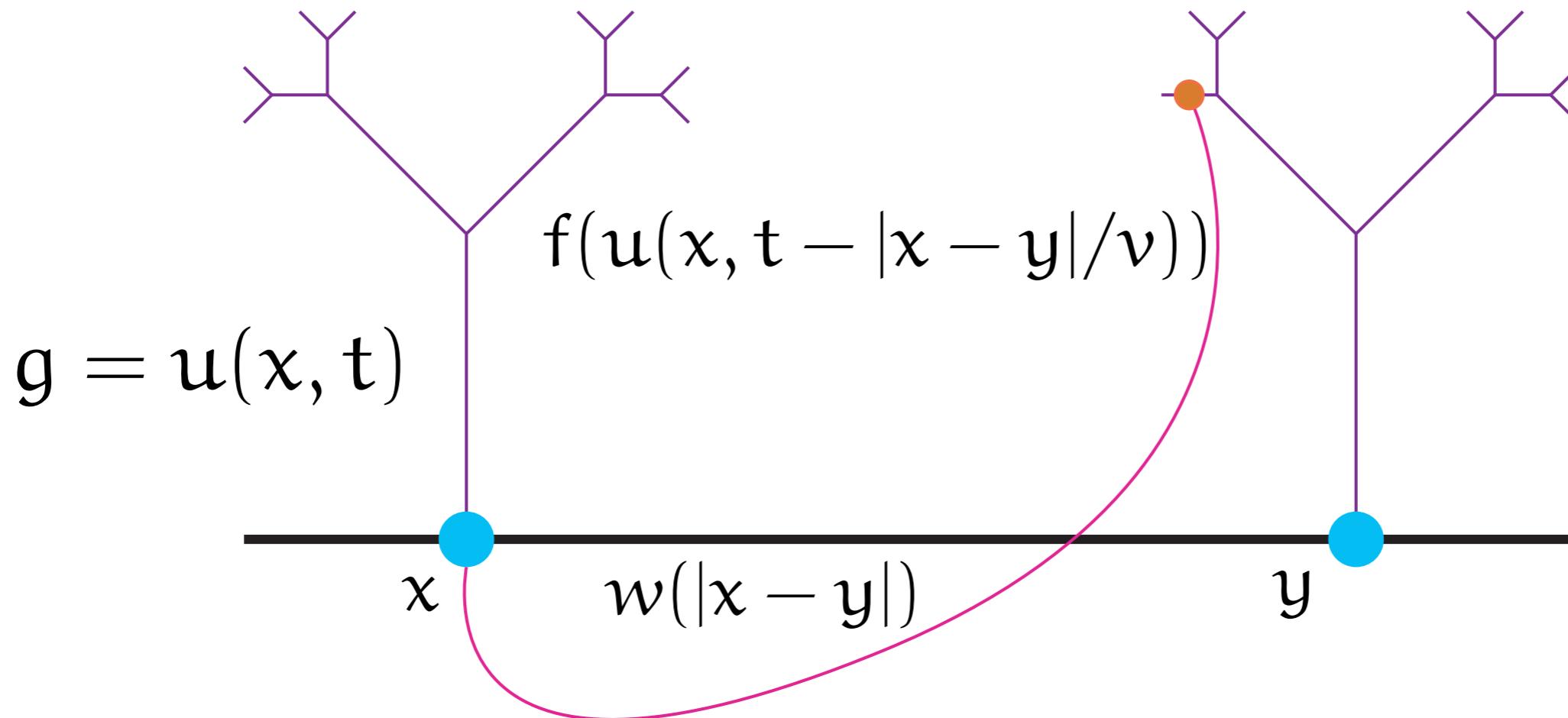
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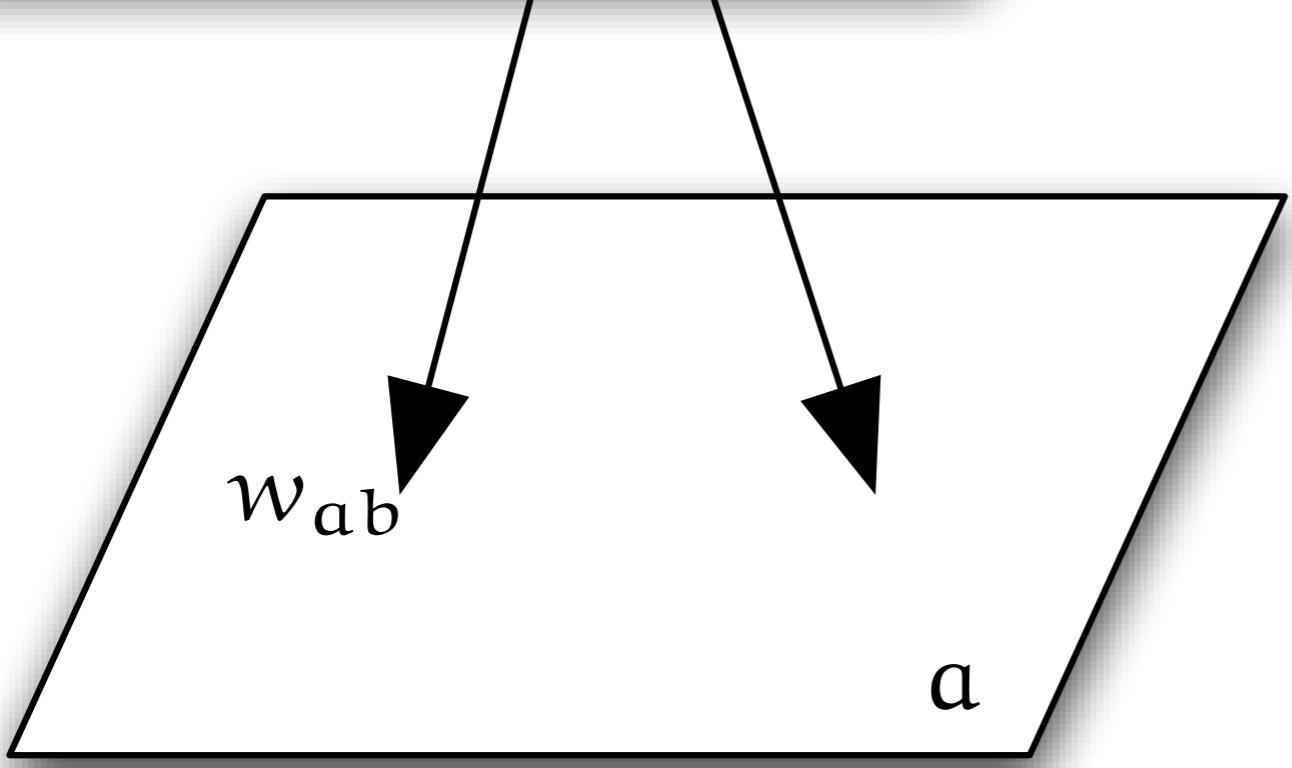
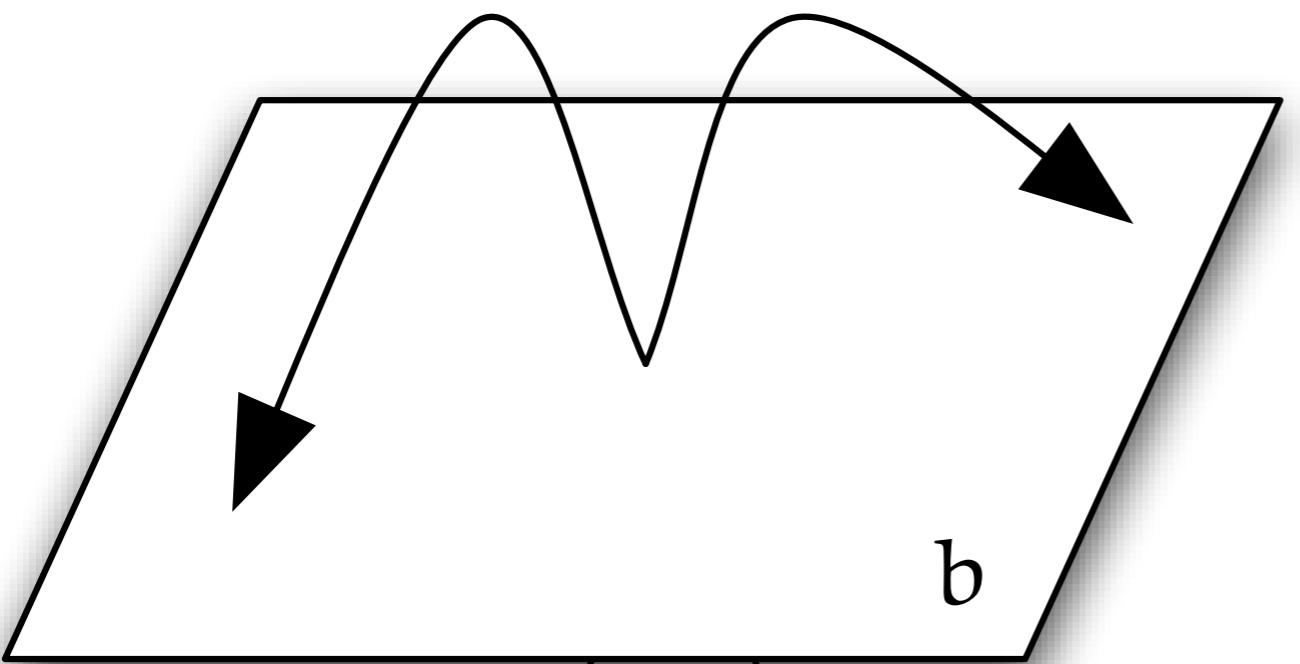
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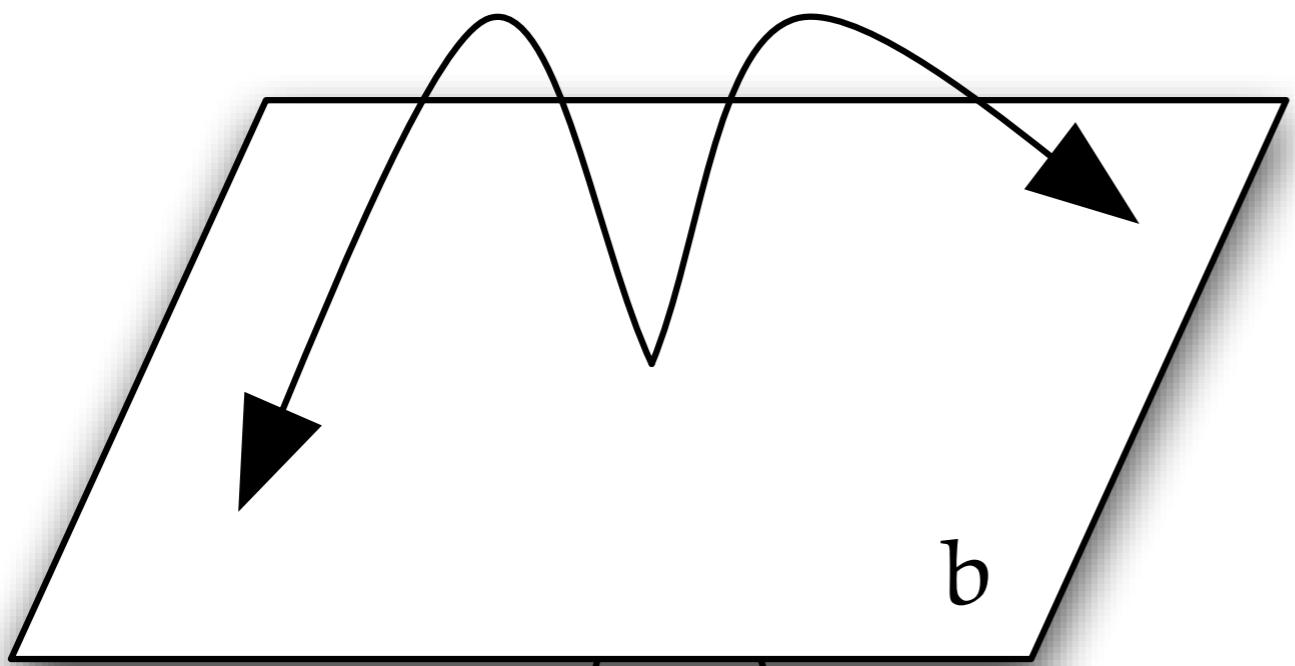
$$u(x, t) = \int_{-\infty}^{\infty} dy w(y) \int_0^{\infty} ds \eta(s) f(u(x - y, t - s - |y|/v))$$

# 2D layers

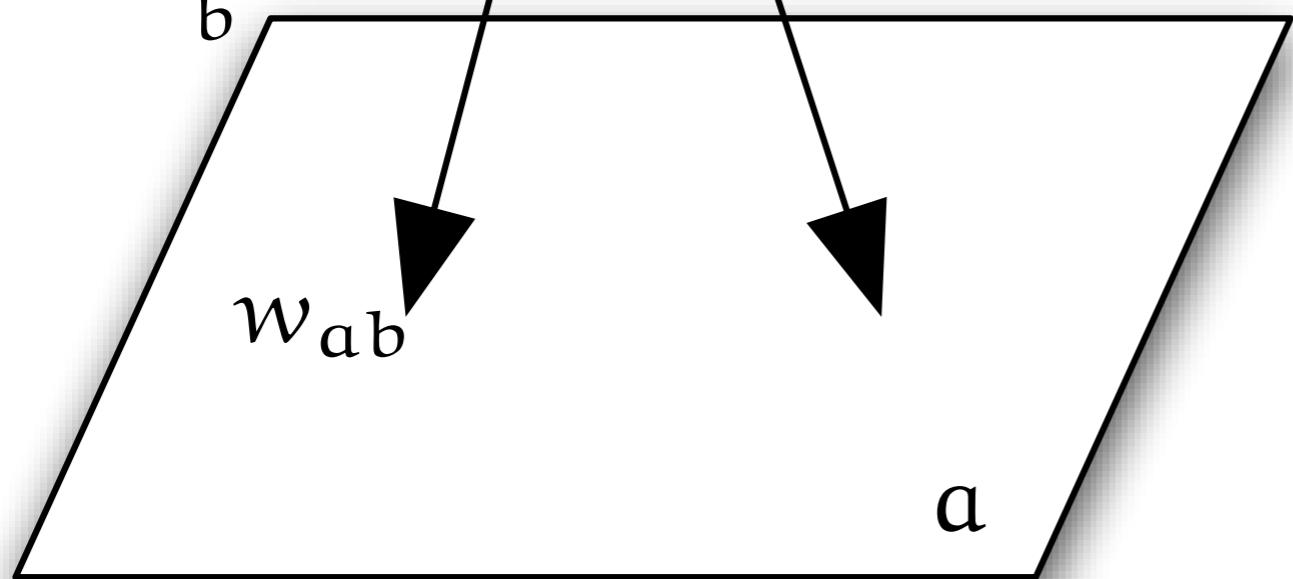


# 2D layers

$$u_{ab} = \eta_{ab} * \psi_{ab}$$



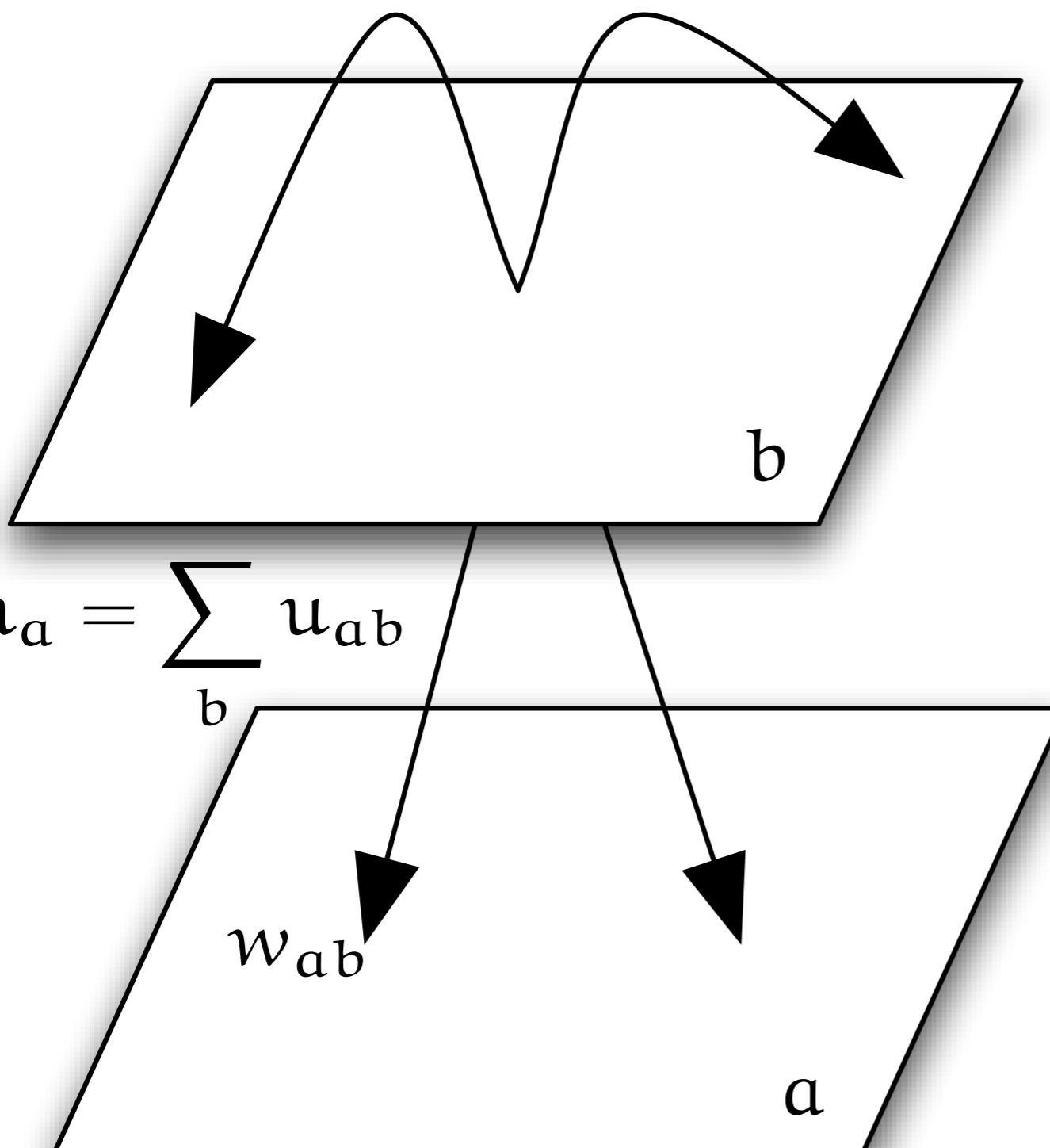
$$h_a = \sum_b u_{ab}$$



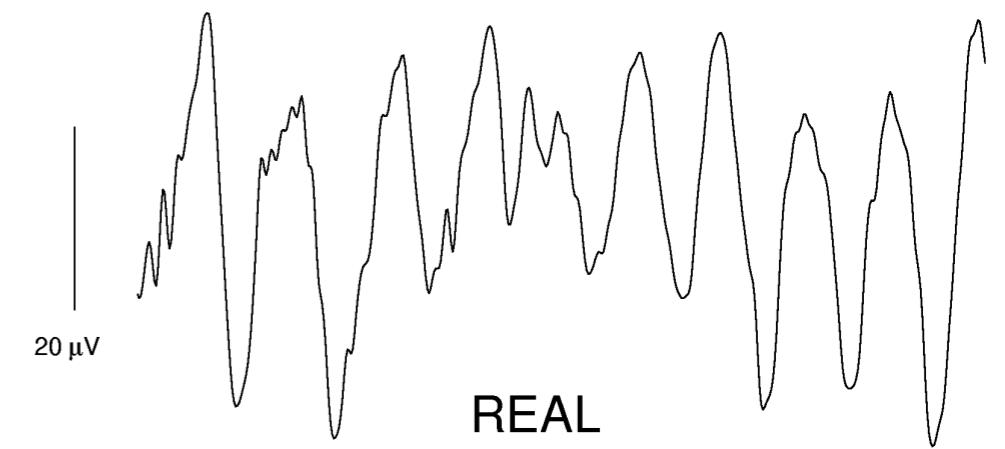
$$\psi_{ab}(\mathbf{r}, t) = \int_{\mathbb{R}^2} d\mathbf{r}' w_{ab}(\mathbf{r}, \mathbf{r}') f_b \circ h_b (\mathbf{r}', t - |\mathbf{r} - \mathbf{r}'|/\nu_{ab})$$

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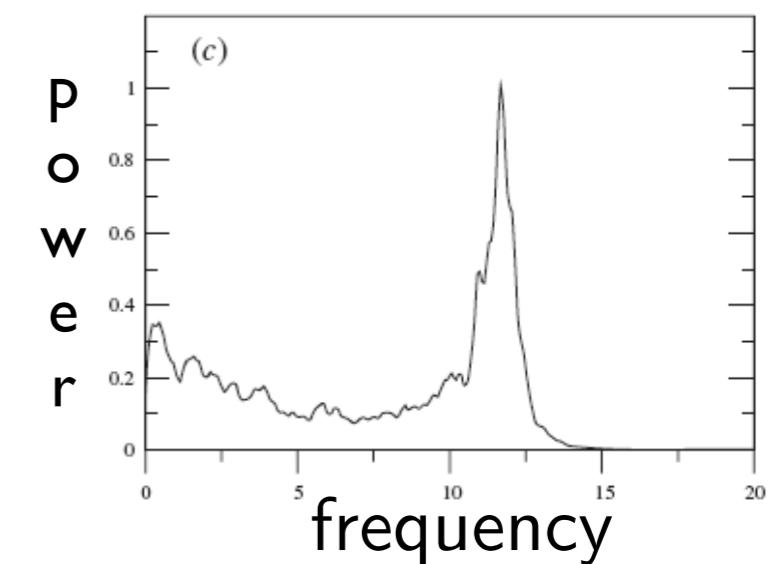
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REAL  
(scalp)



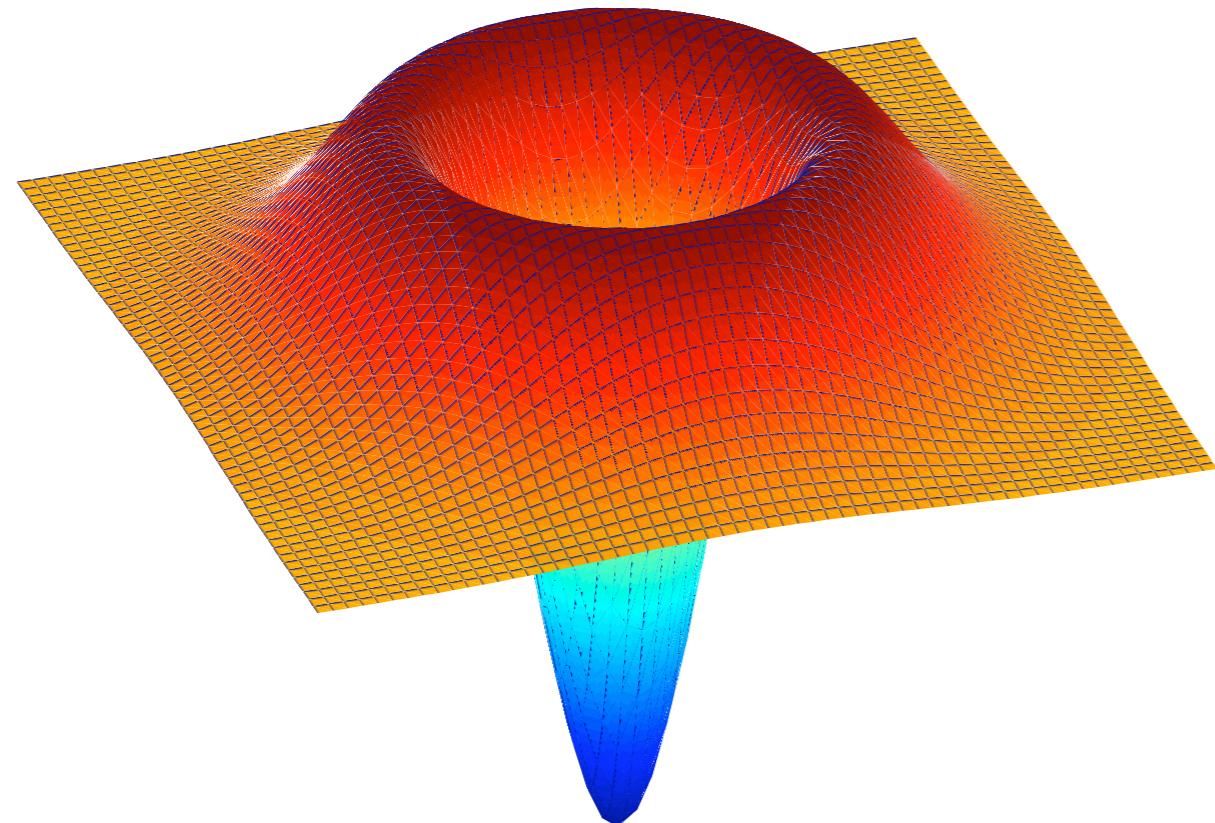
SIMULATED  
(cortex)



(c)

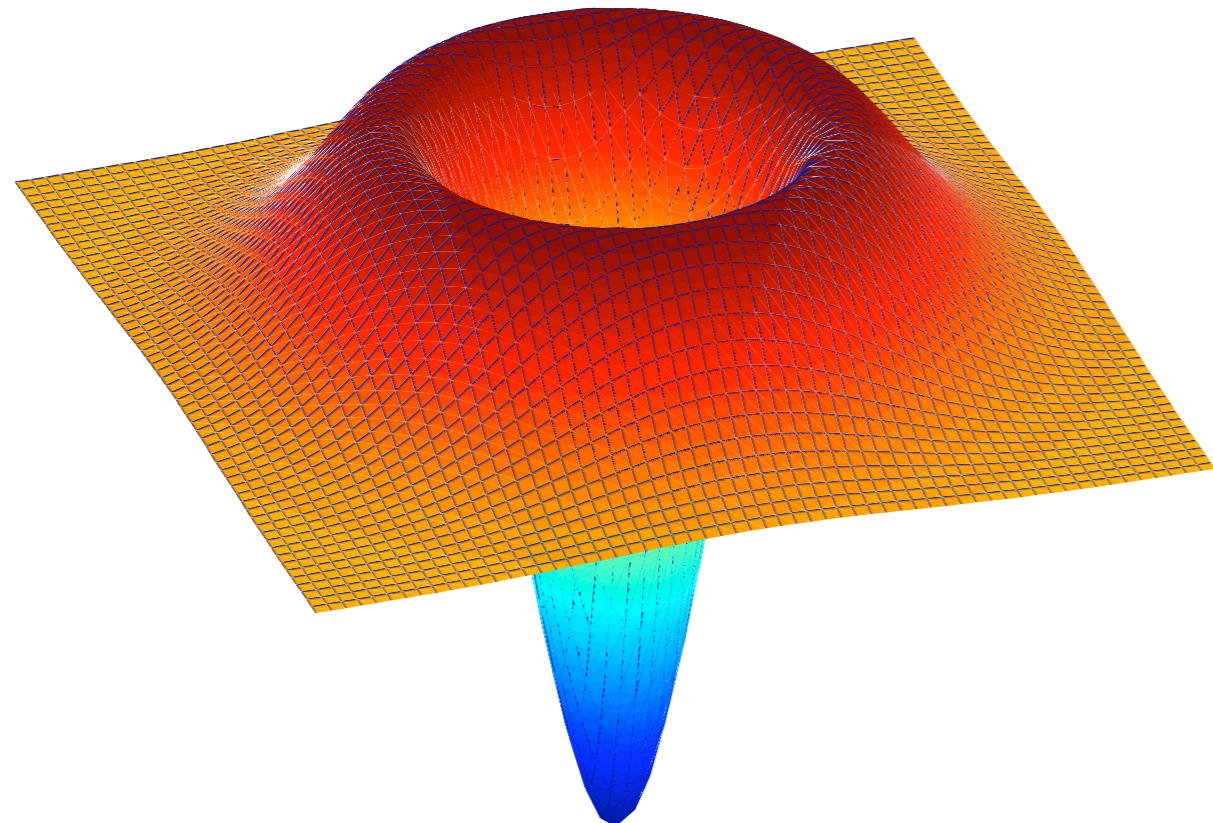
# Turing instability analysis

E layer and I layer



# Turing instability analysis

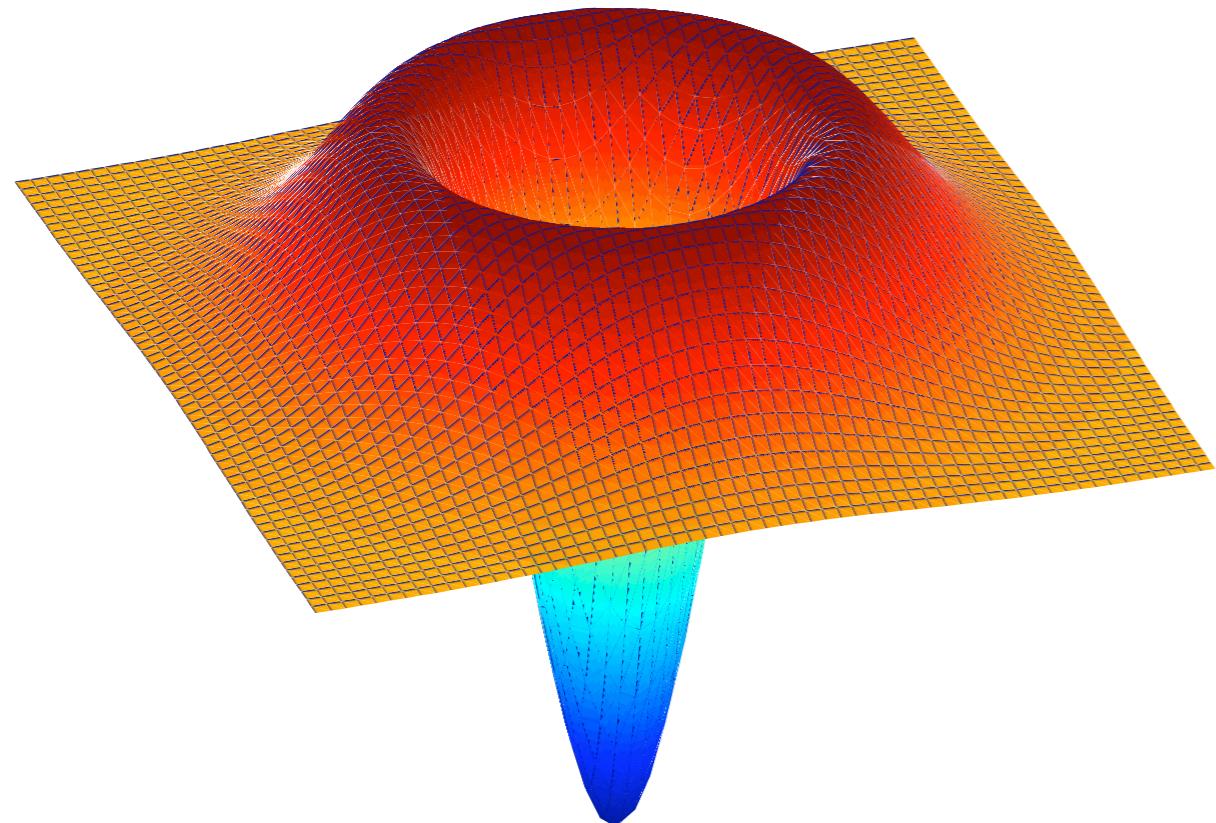
E layer and I layer



$$e^{ik \cdot r} e^{\lambda t}$$

# Turing instability analysis

E layer and I layer



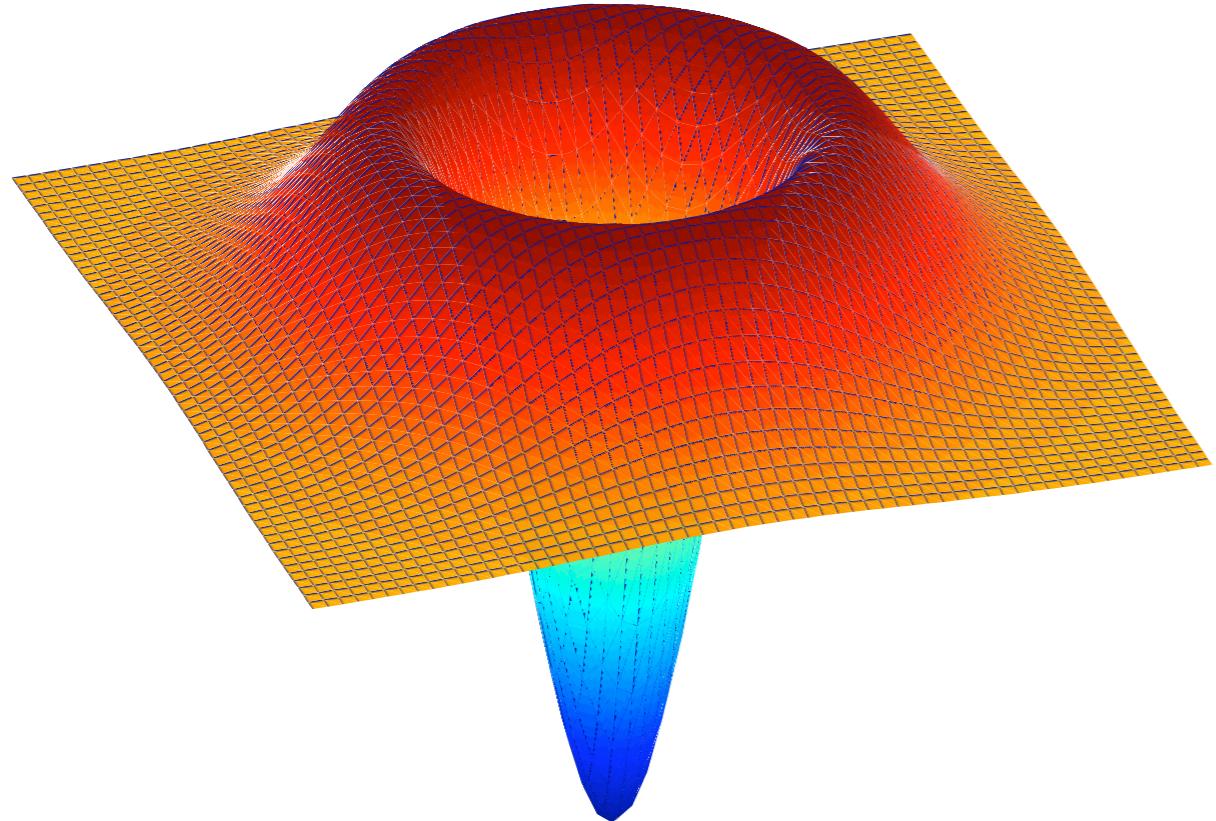
$$e^{ik \cdot r} e^{\lambda t}$$

Continuous spectrum

$$\det(\mathcal{D}(k, \lambda) - I) = 0$$

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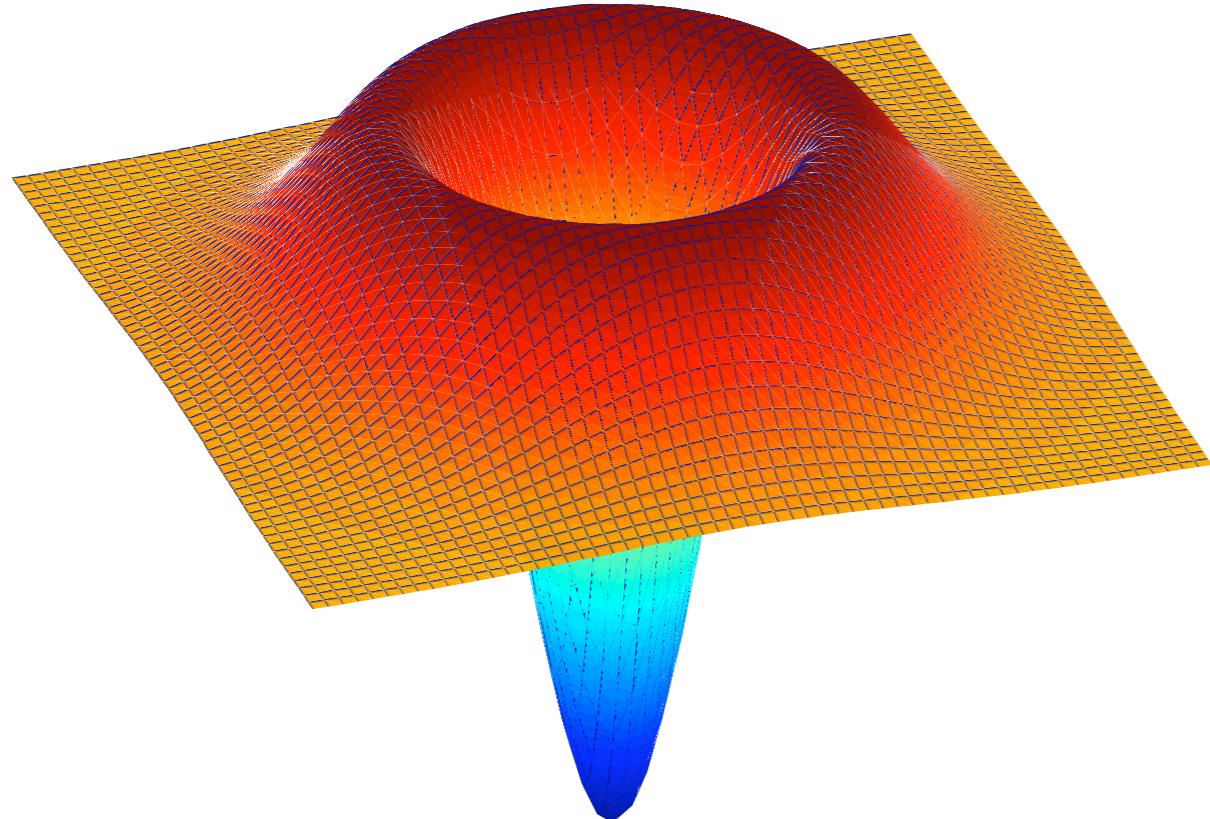
Continuous spectrum

$$\det(\mathcal{D}(k, \lambda) - I) = 0$$

$$[\mathcal{D}(k, \lambda)]_{ab} = \tilde{\eta}_{ab}(\lambda) G_{ab}(k, -i\lambda) \gamma_b$$

# Turing instability analysis

E layer and I layer



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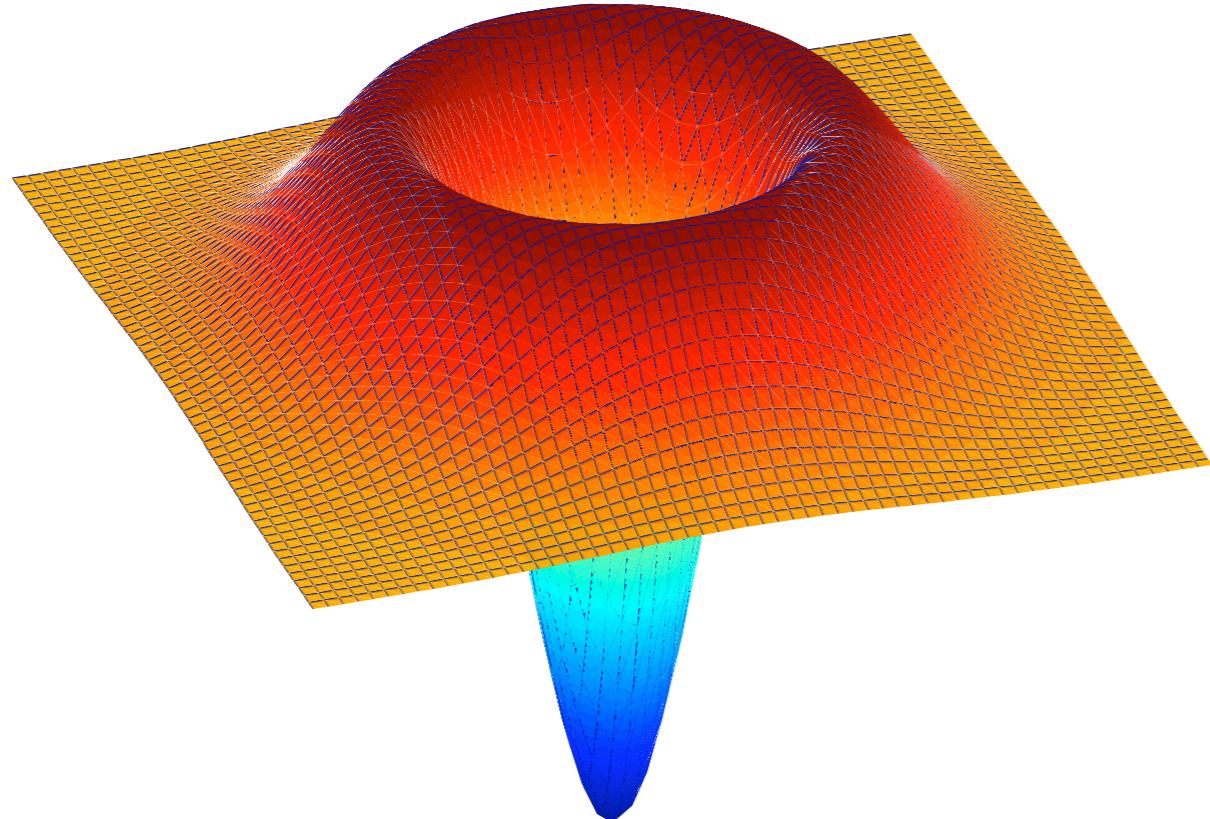
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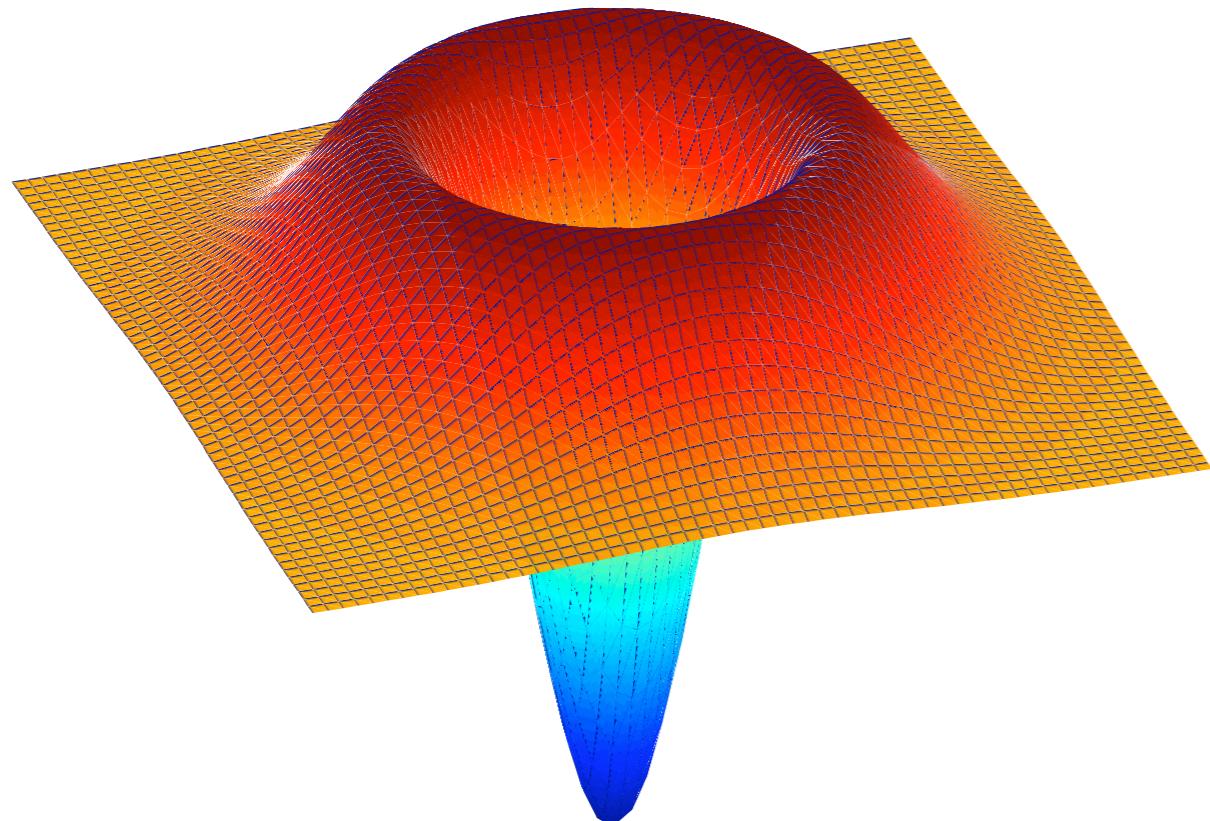
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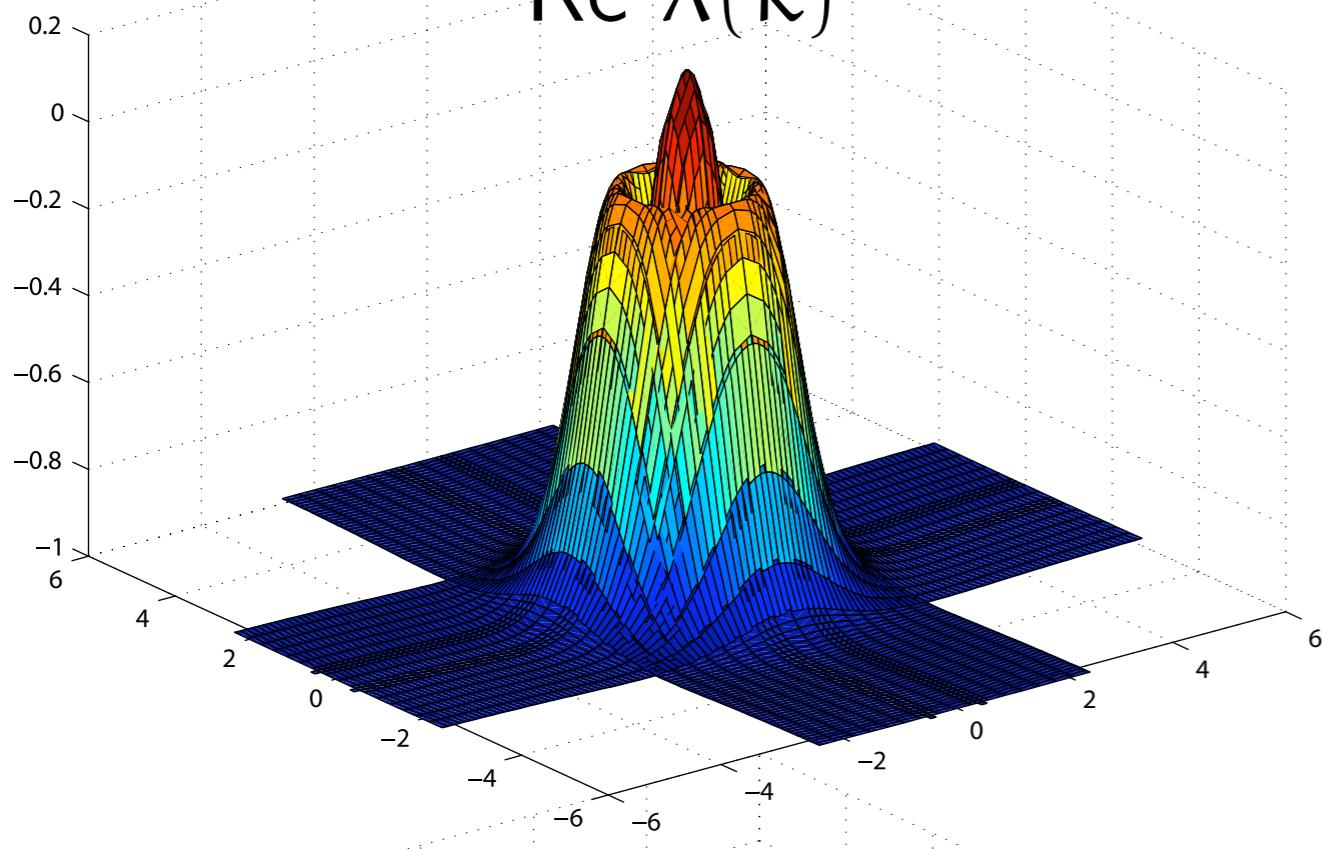
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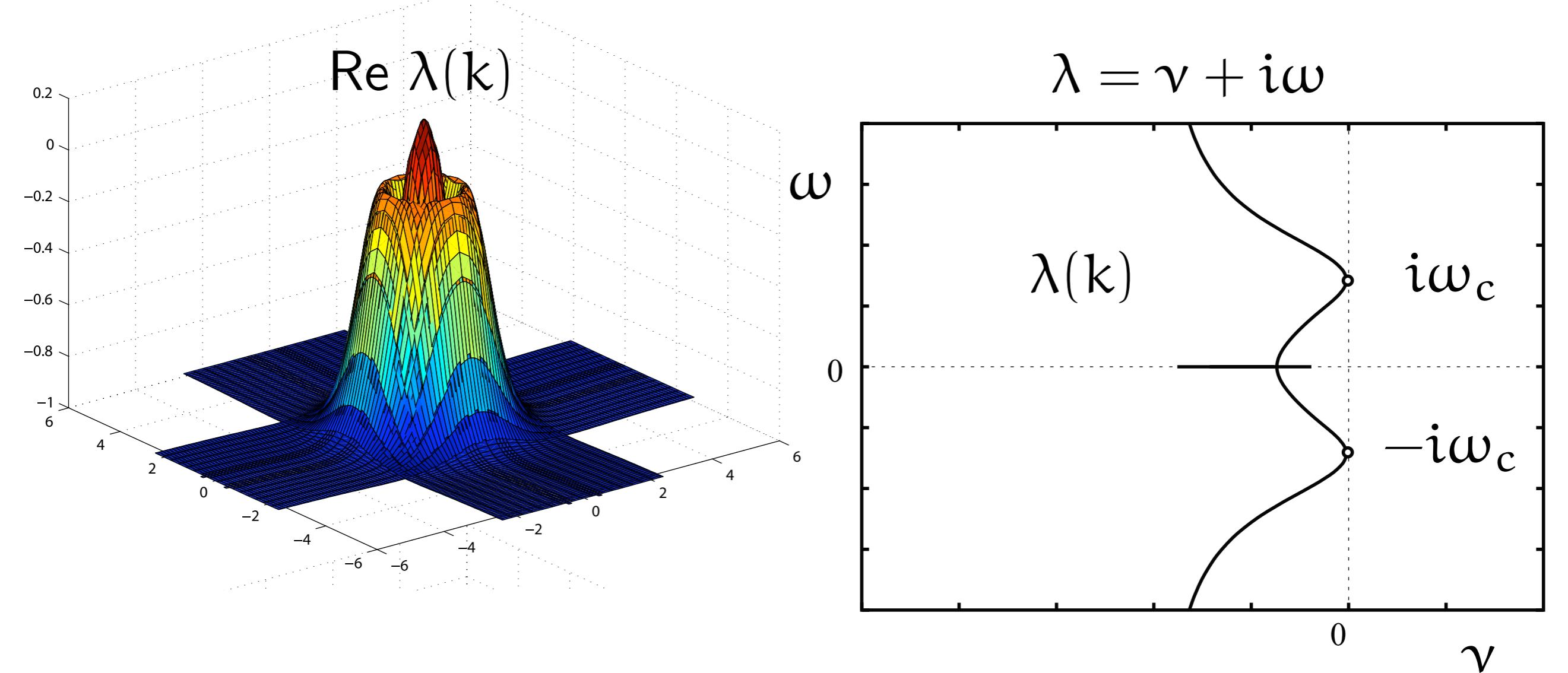
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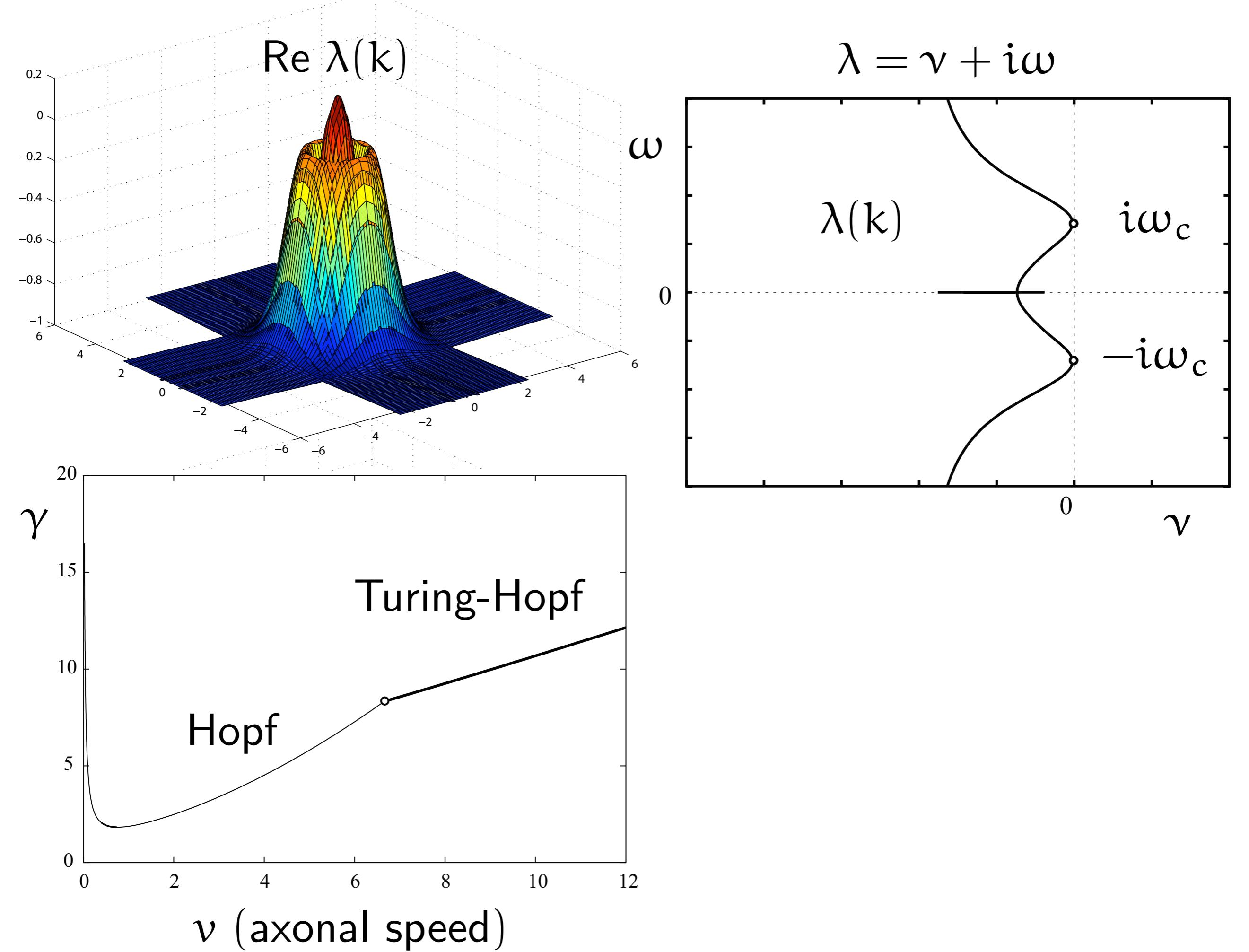
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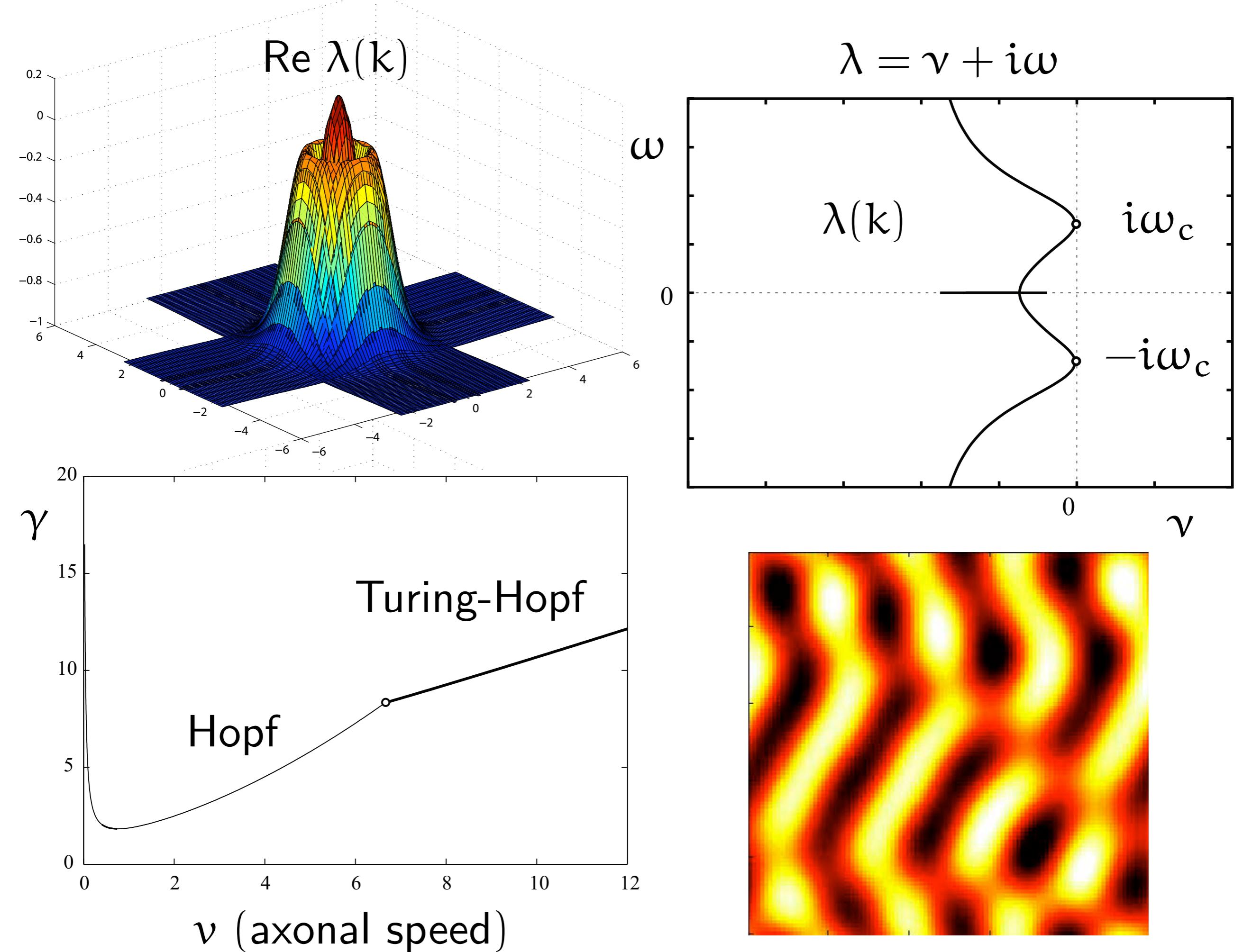
$$\gamma = f'(ss)$$

$\operatorname{Re} \lambda(k)$









# Amplitude Equations (one D)

Coupled mean-field Ginzburg–Landau equations describing a Turing–Hopf bifurcation with modulation group velocity of  $O(1)$ .

$$\frac{\partial A_1}{\partial \tau} = A_1(a + b|A_1|^2 + c\langle |A_2|^2 \rangle) + d \frac{\partial^2 A_1}{\partial \xi_+^2}$$

$$\frac{\partial A_2}{\partial \tau} = A_2(a + b|A_2|^2 + c\langle |A_1|^2 \rangle) + d \frac{\partial^2 A_2}{\partial \xi_-^2}$$

Coefficients in terms of integral transforms of  $w$  and  $\eta$ .

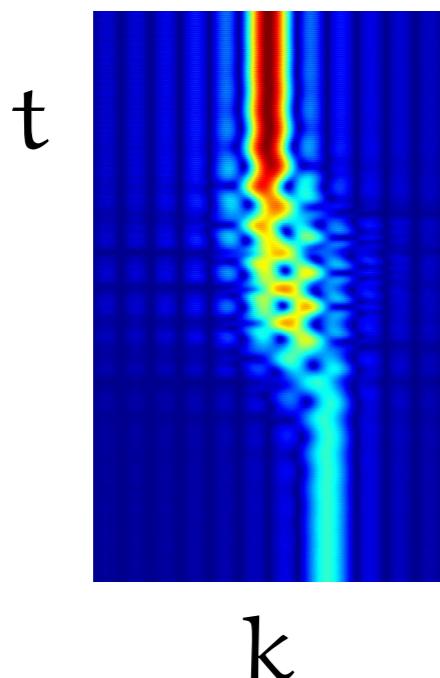
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Benjamin–Feir (BF)



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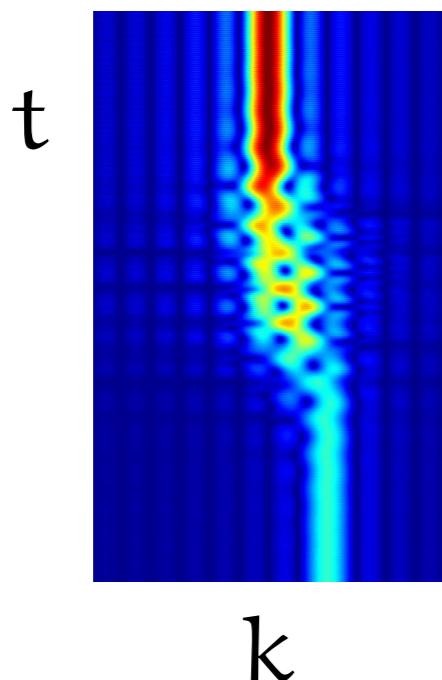
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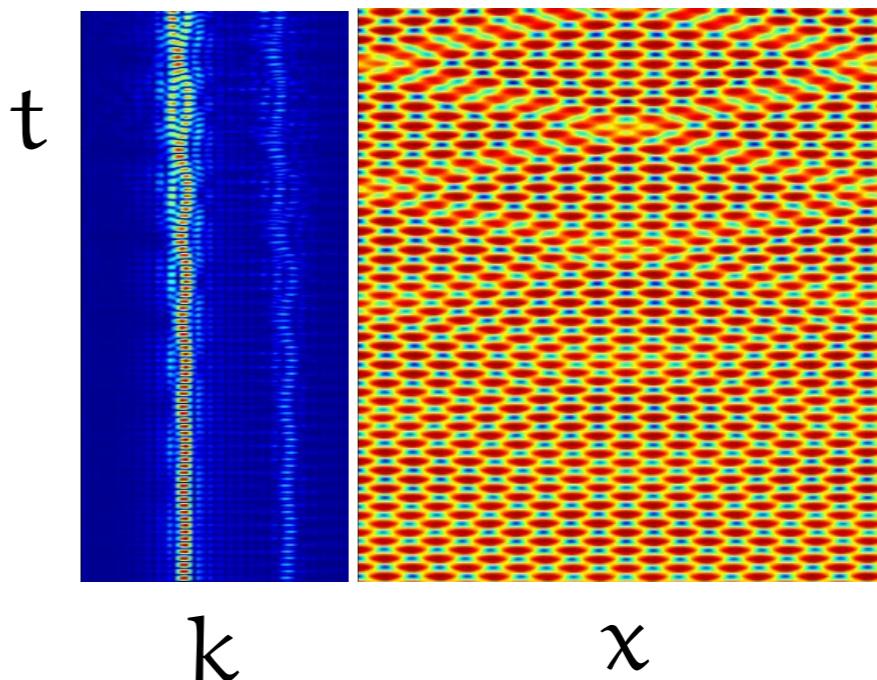
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Benjamin–Feir (BF)



BF-Eckhaus instability



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# Time independent localised solutions

$$w \otimes \eta * f \rightarrow w \otimes f$$

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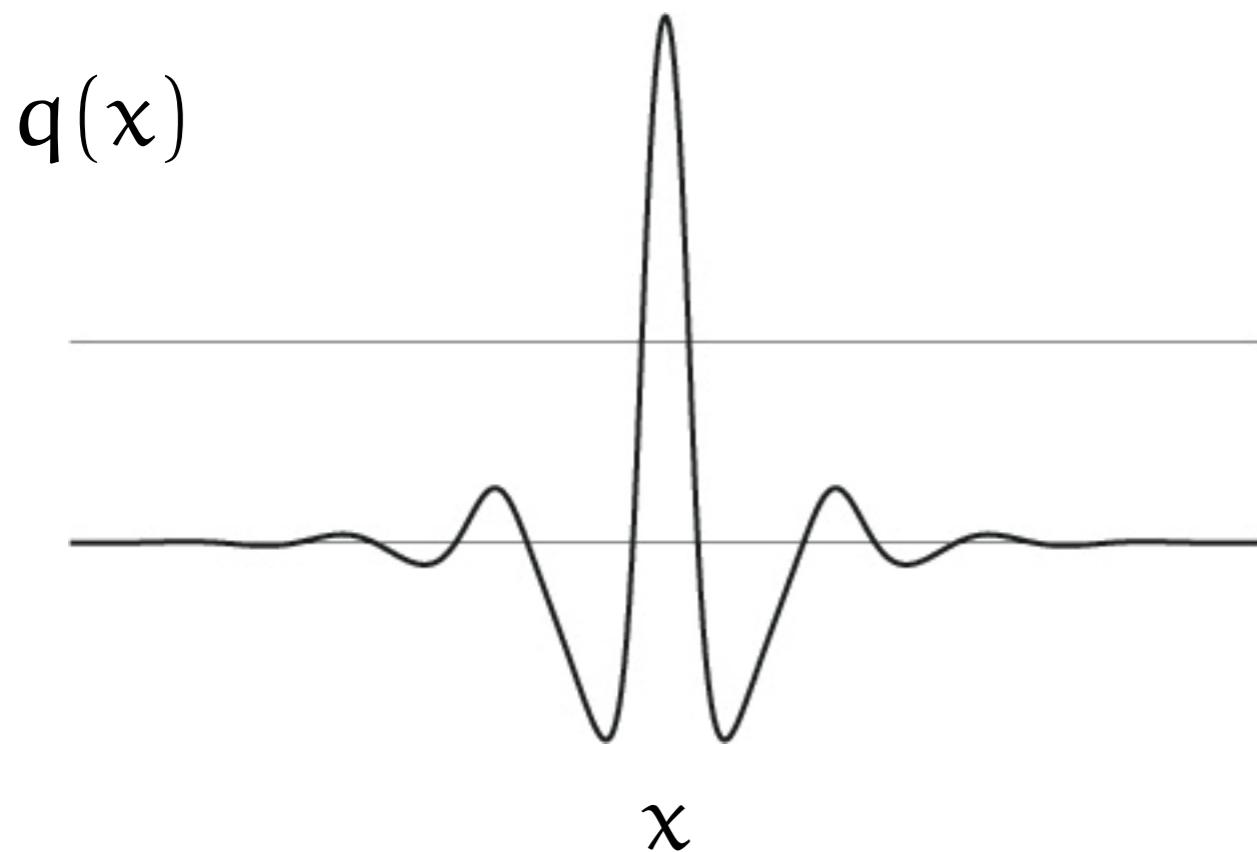
$$q(x)$$

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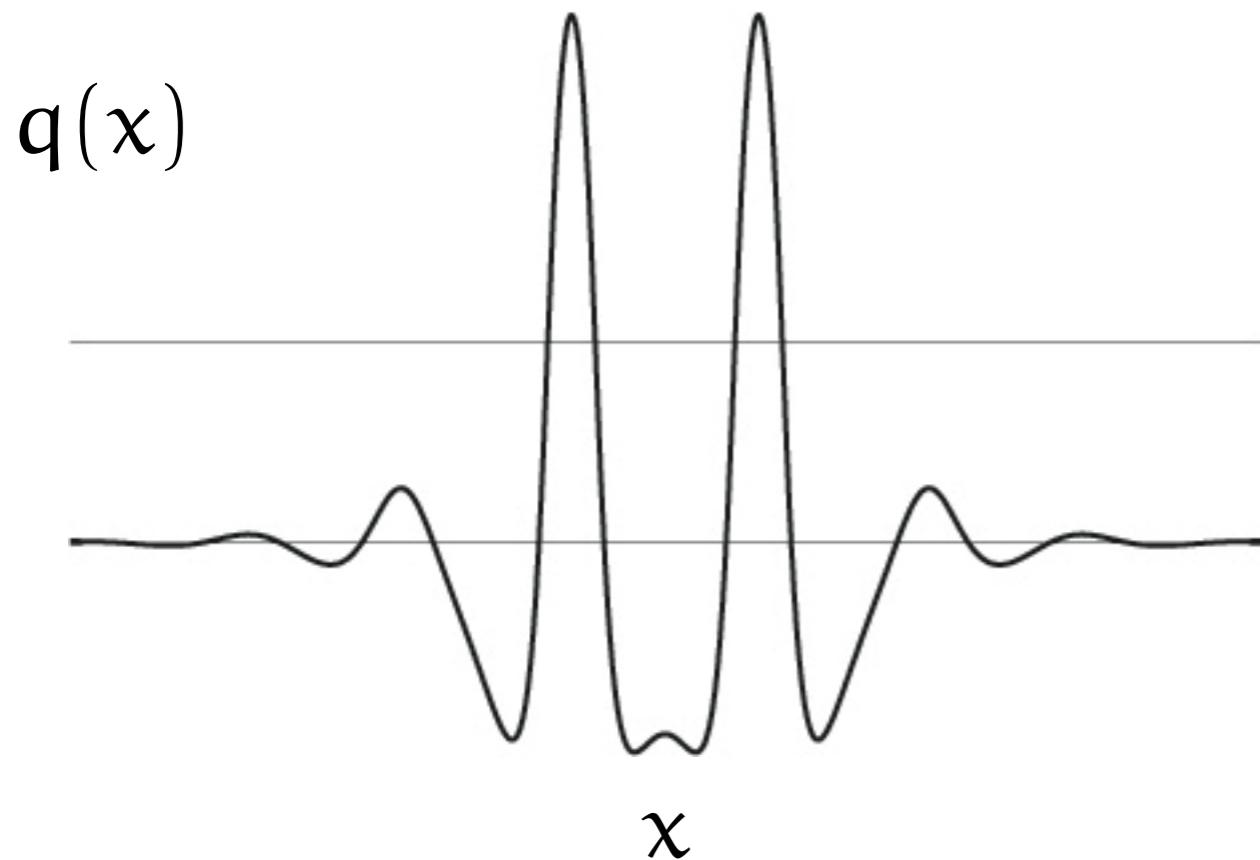


# Time independent localised solutions

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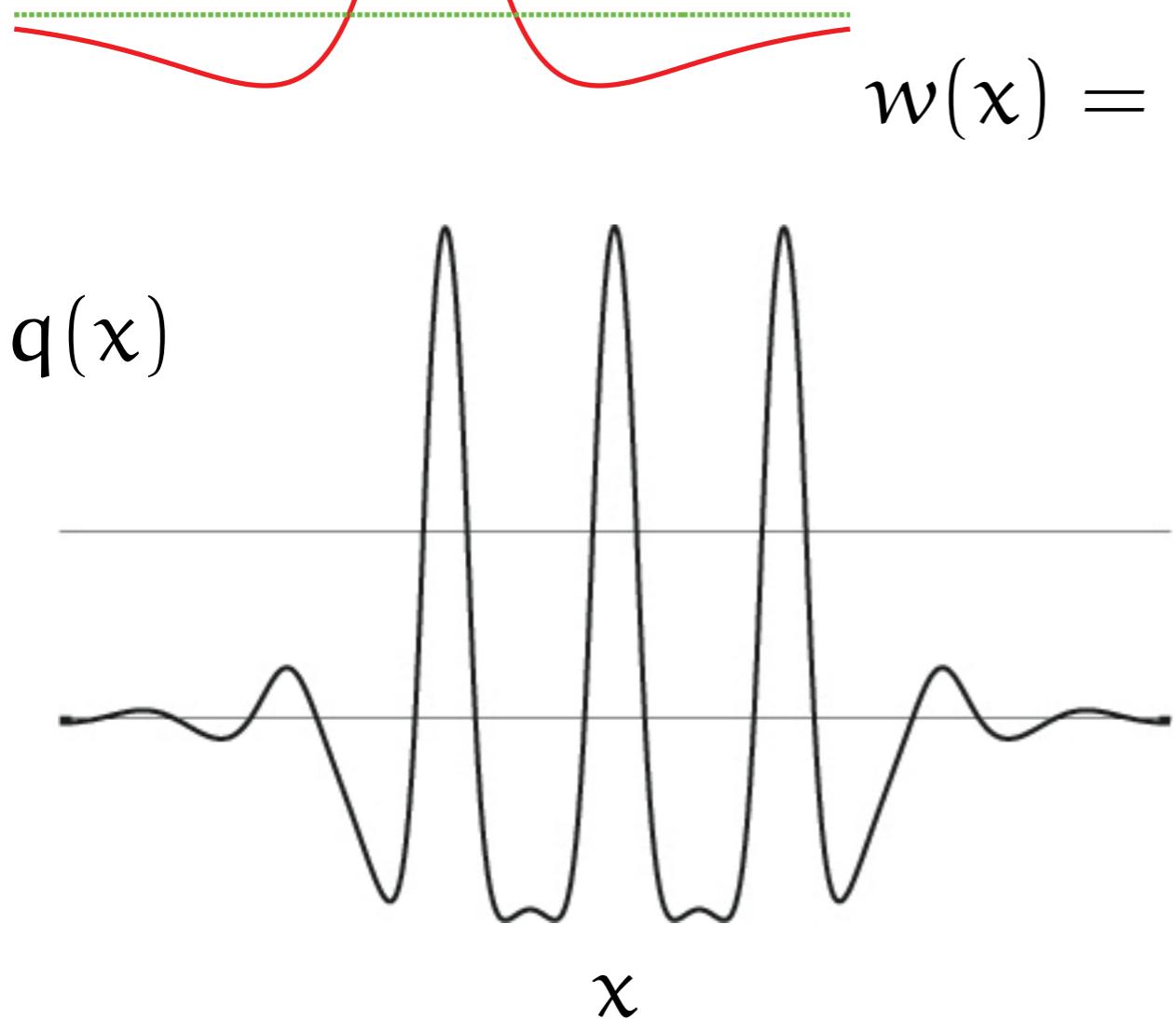

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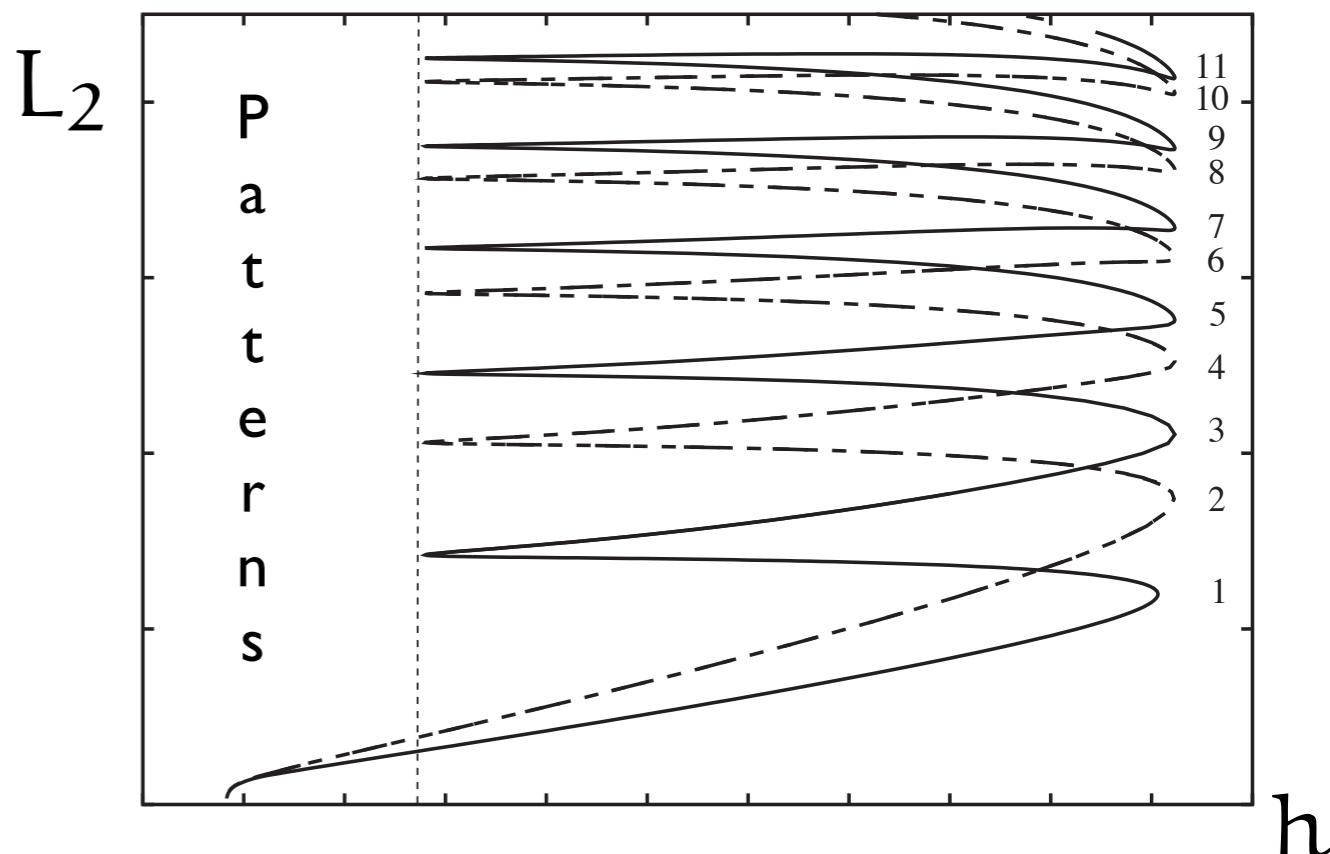
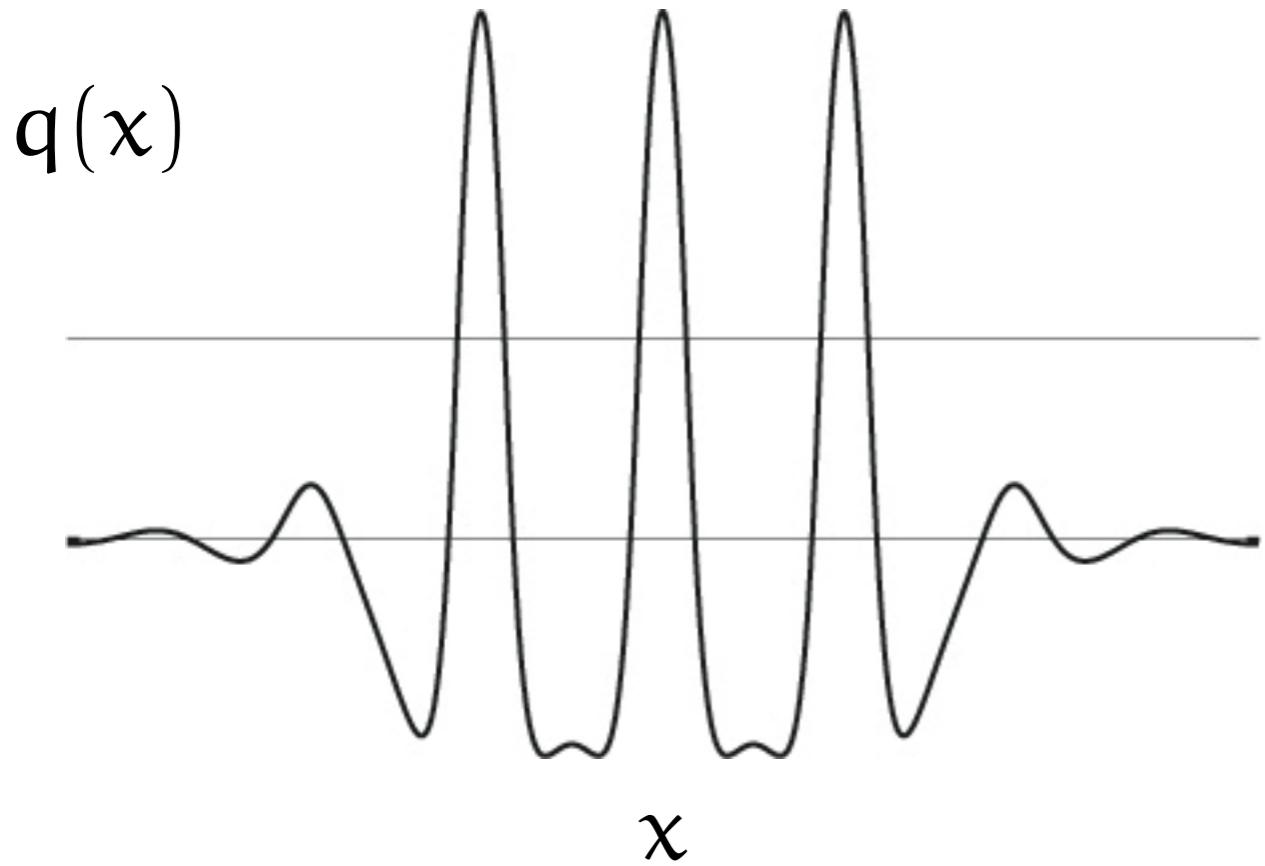
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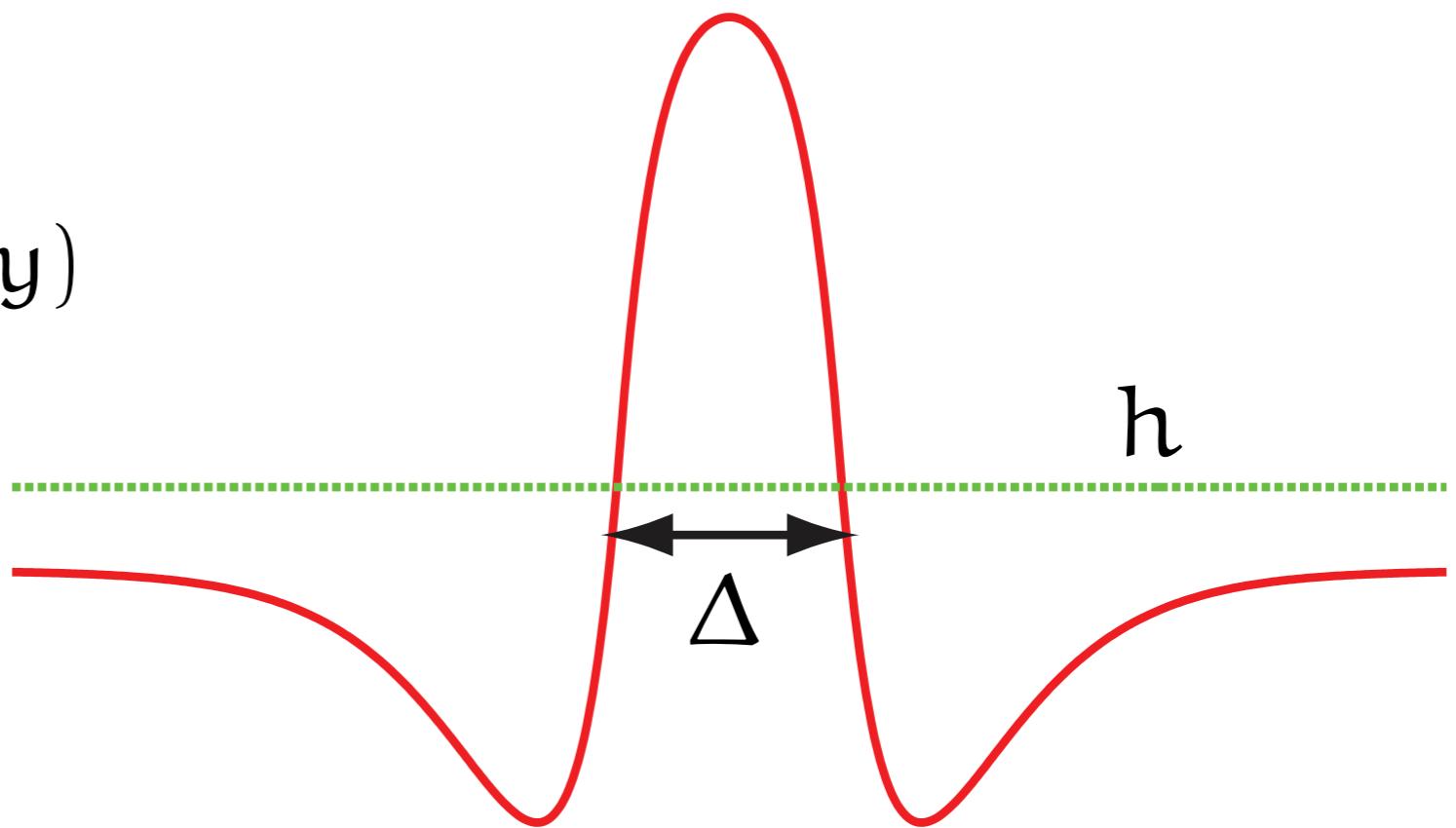


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Exact result for 1-bump:  $f(u) = H(u - h)$

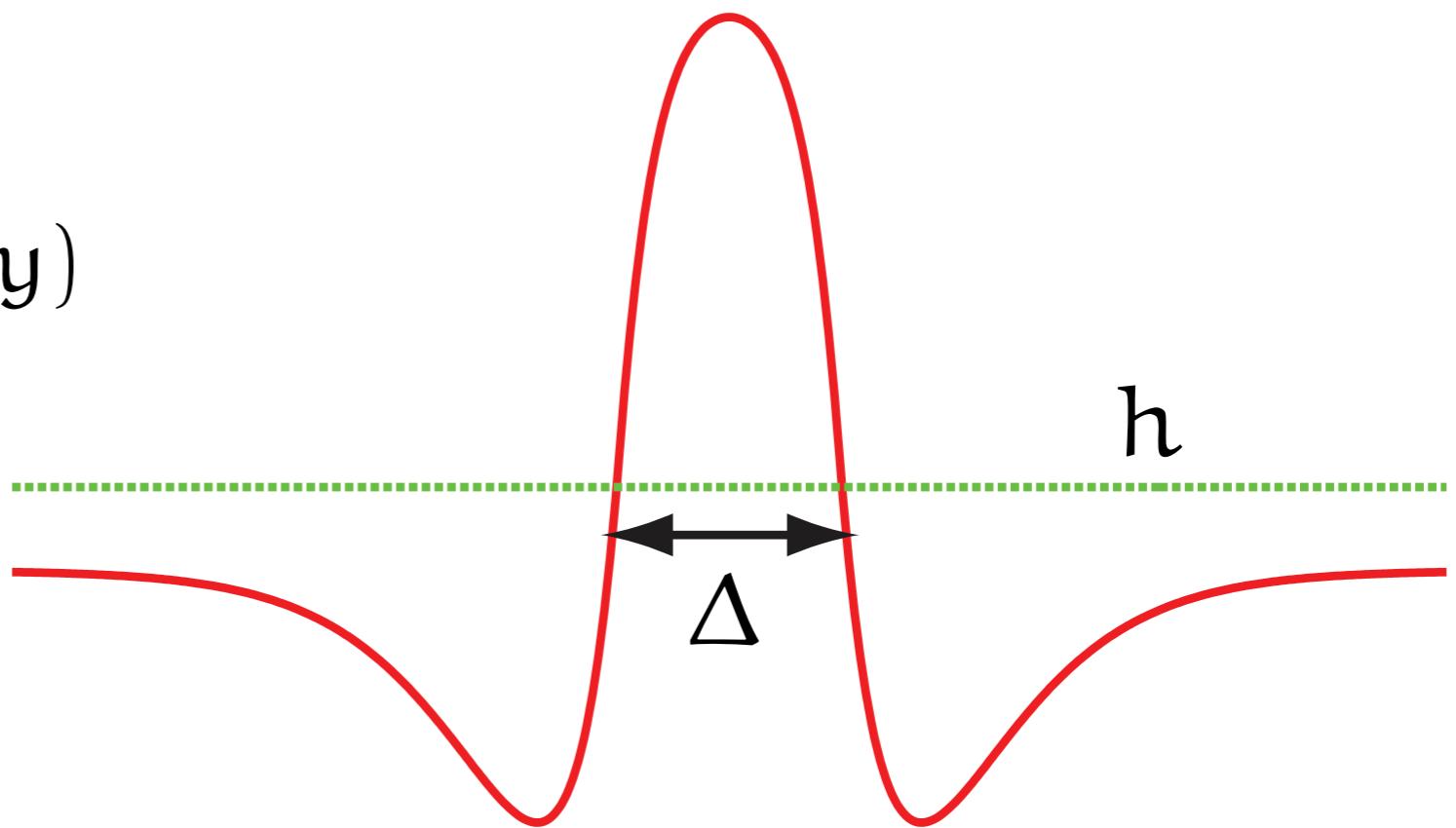
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working memory

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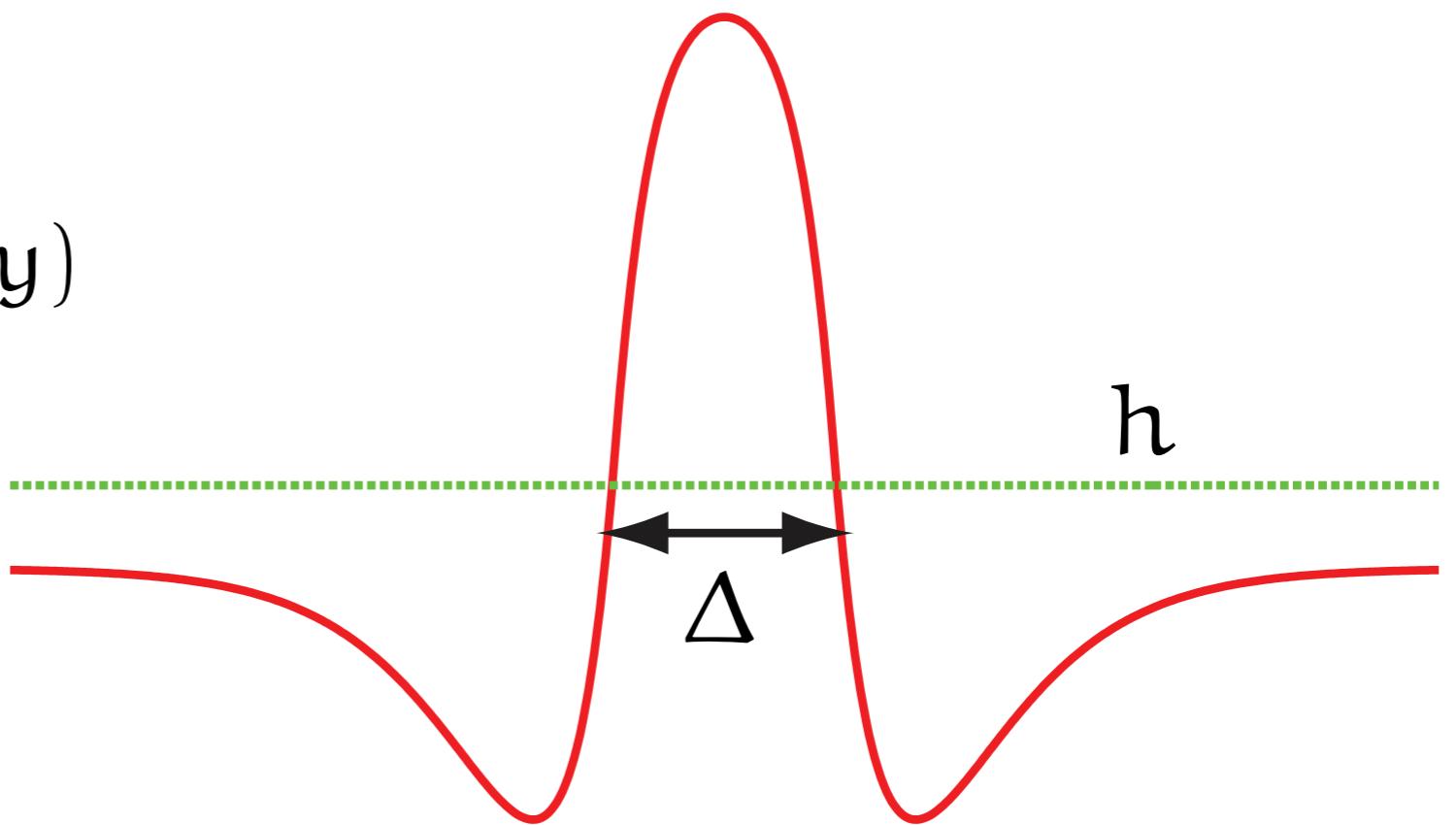


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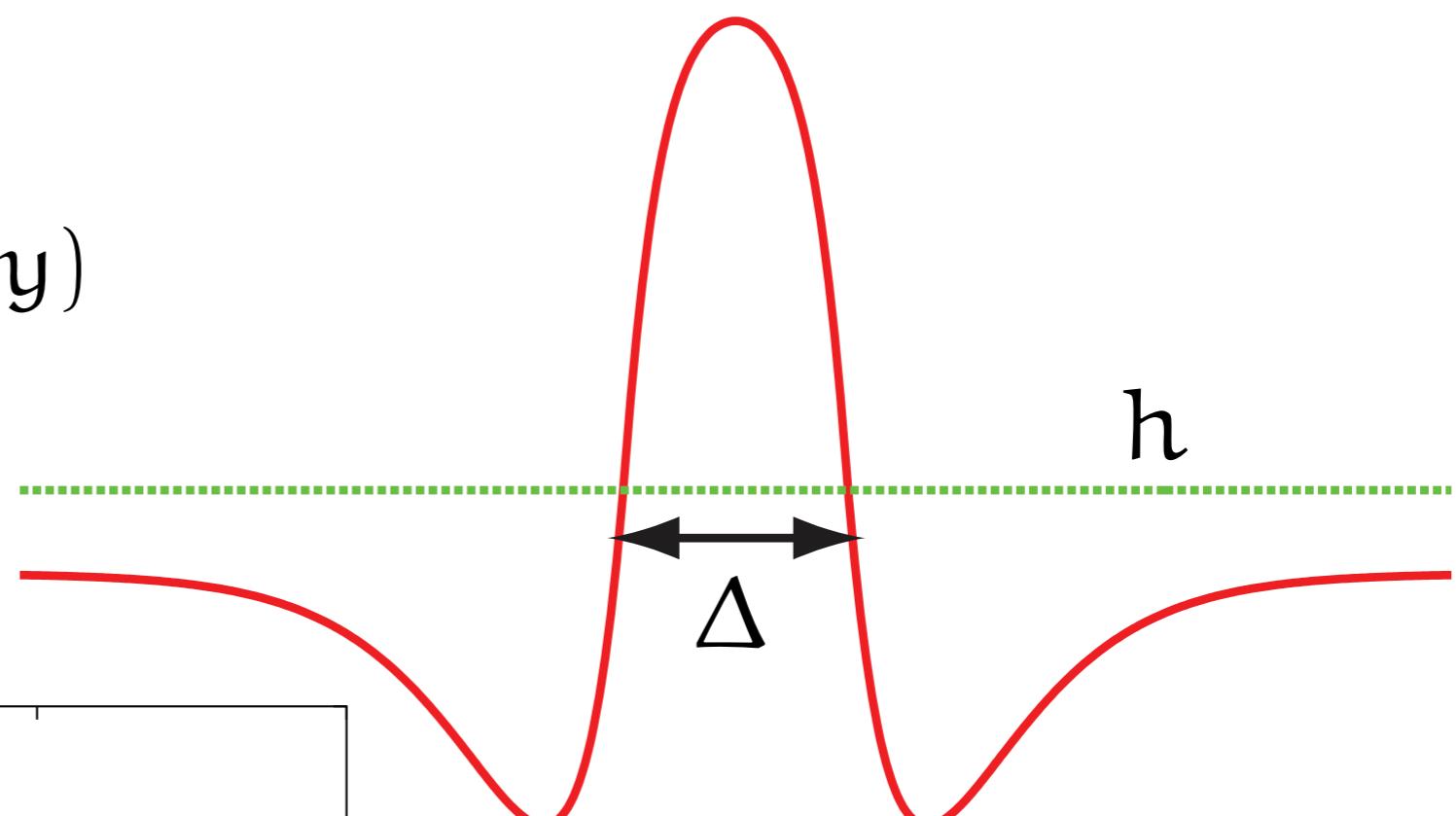
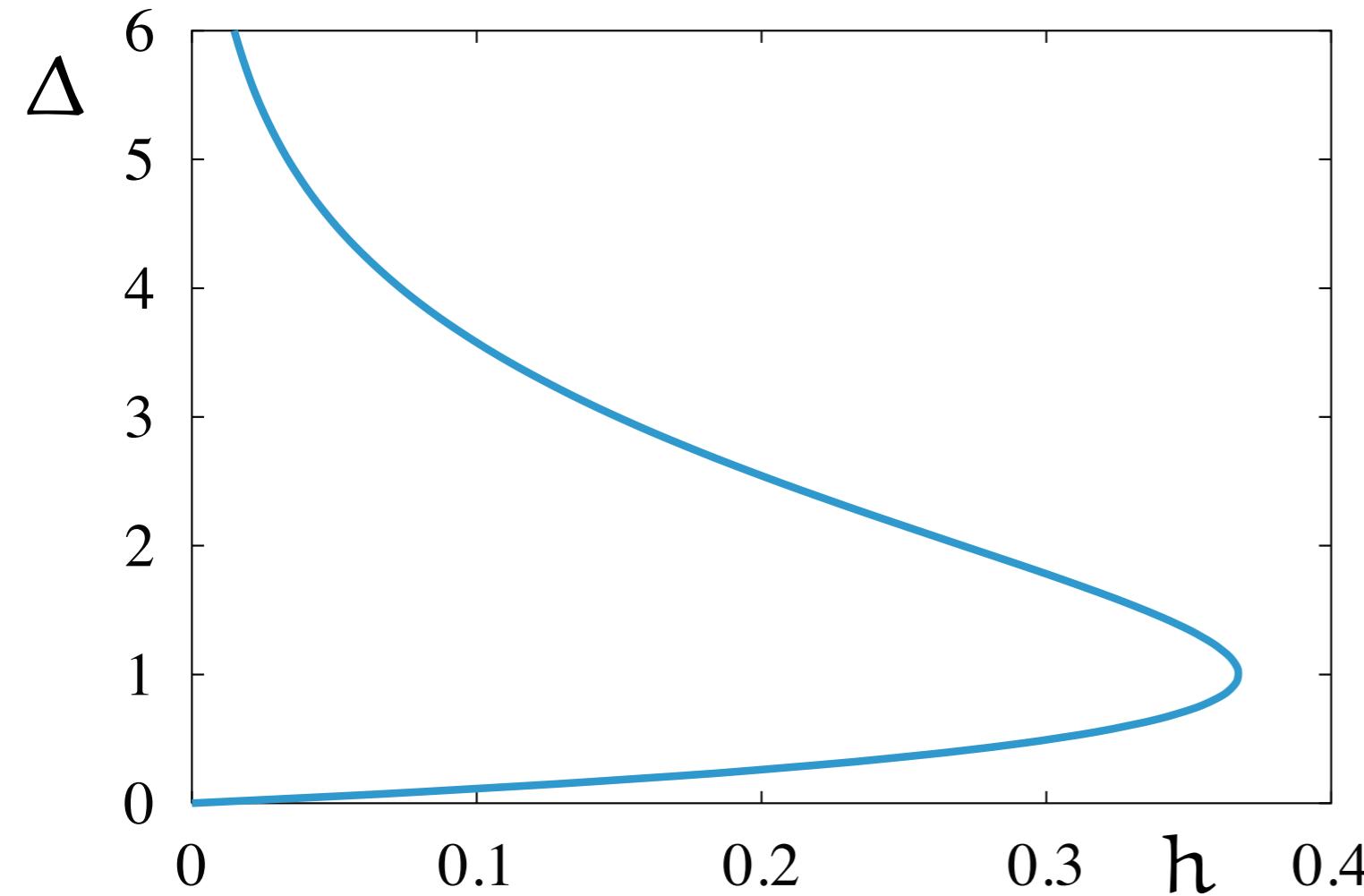
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For Heaviside firing rate

$$f'(q(x)) = \frac{\delta(x)}{|q'(0)|} + \frac{\delta(x - \Delta)}{|q'(\Delta)|}$$

so

$$u(x) = \frac{\tilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} [w(x)u(0) + w(x - \Delta)u(\Delta)]$$

# System of linear equations for perturbations at threshold

$$\begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix} = \mathcal{A}(\lambda) \begin{bmatrix} u(0) \\ u(\Delta) \end{bmatrix}, \quad \mathcal{A}(\lambda) = \frac{\tilde{\eta}(\lambda)}{|w(0) - w(\Delta)|} \begin{bmatrix} w(0) & w(\Delta) \\ w(\Delta) & w(0) \end{bmatrix}$$

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Non trivial solution if

$$\mathcal{E}(\lambda) = \det(\mathcal{A}(\lambda) - I) = 0$$

Evans function for integral neural field equation

S Coombes and M R Owen (2004) Evans functions for integral neural field equations with Heaviside firing rate function, SIAM Journal on Applied Dynamical Systems, Vol 34, 574-600.

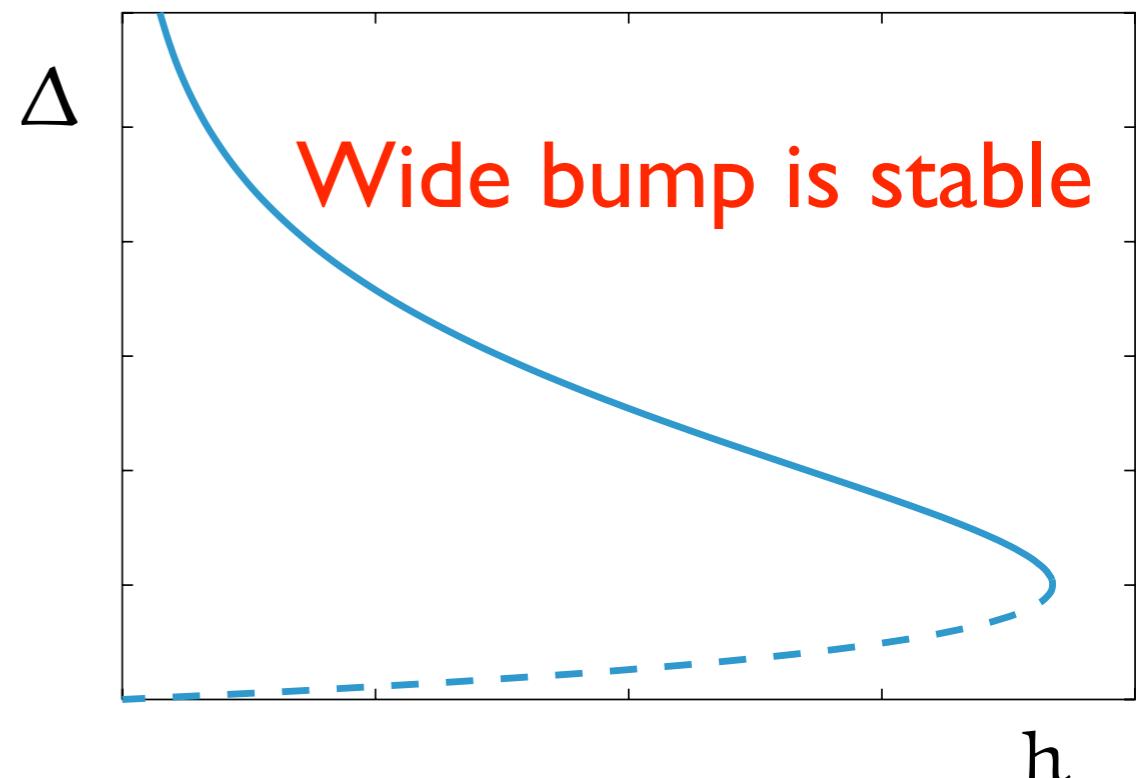
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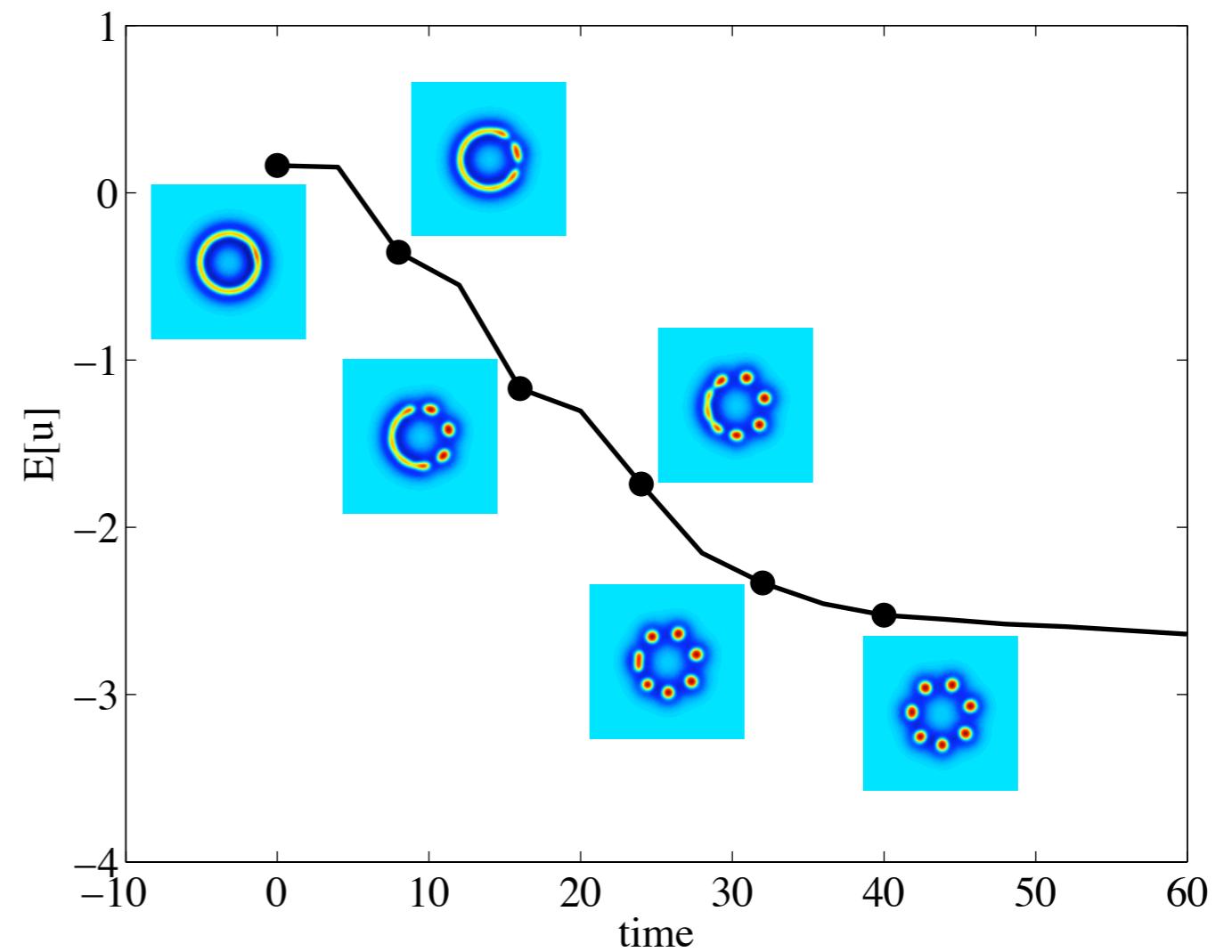
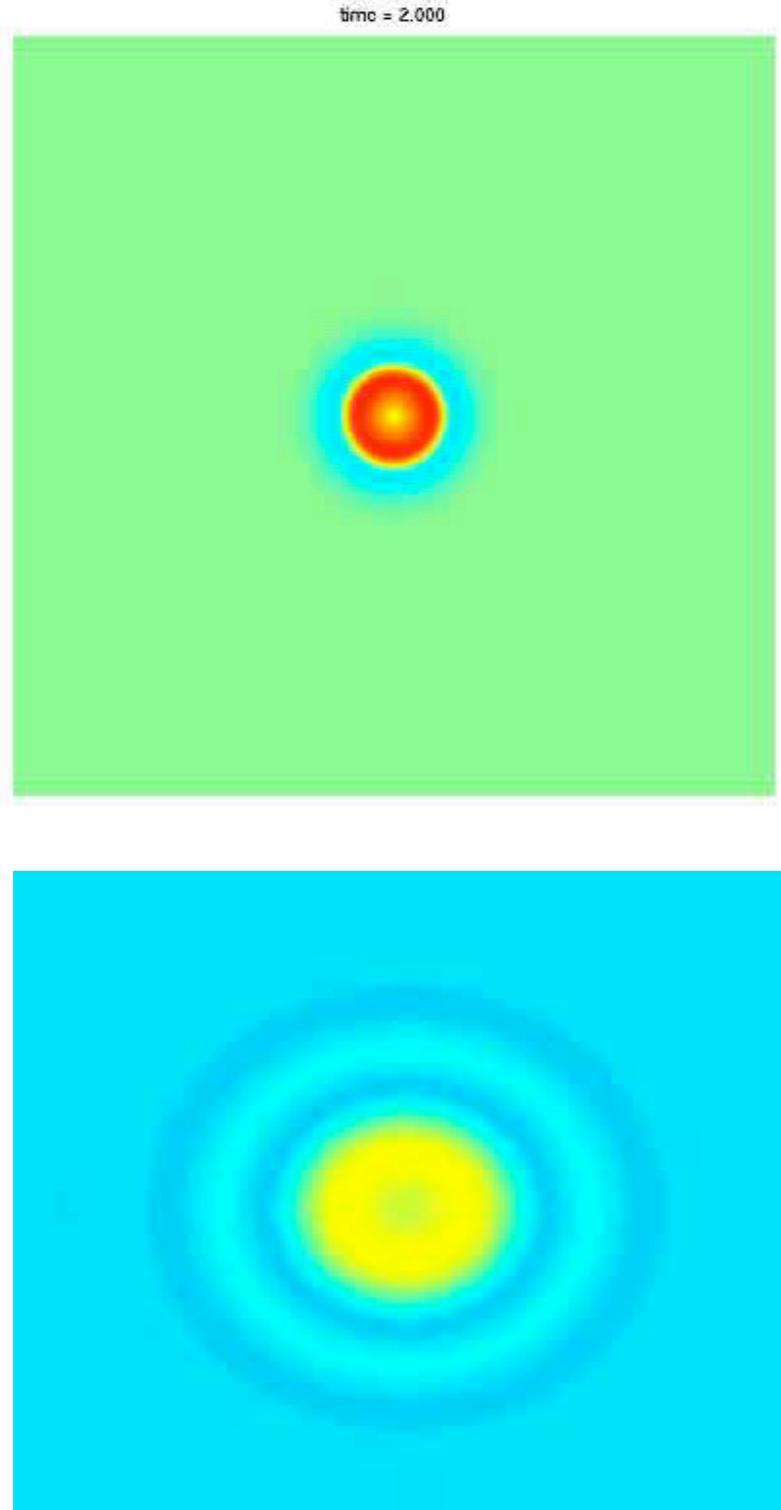
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Solutions stable if  $\text{Re } \lambda < 0$



## Evans function for integral neural field equation

# Predictions of Evans function



# Threshold accommodation

Hill (1936), "... the threshold rises when the *local potential* is maintained ... and reverts gradually to its original value when the nerve is allowed to rest."

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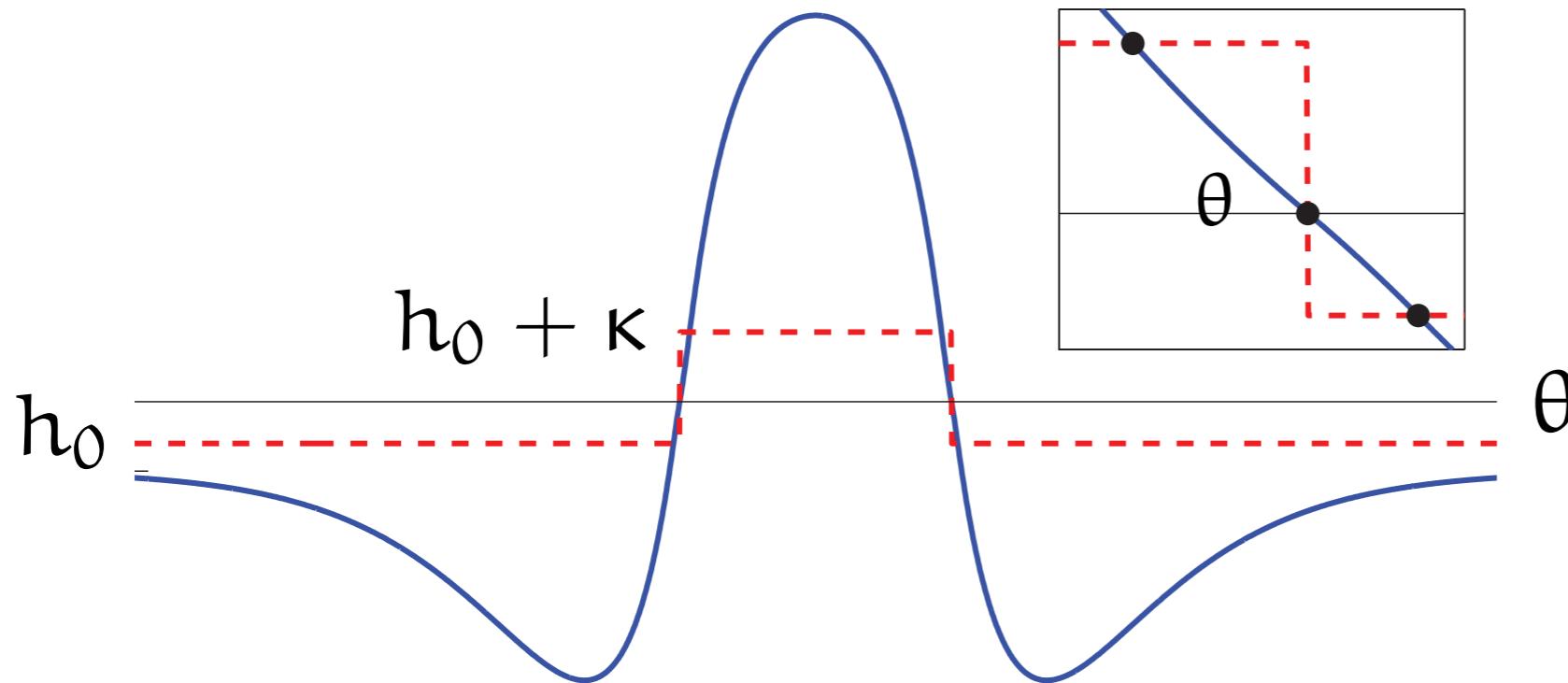
Hill (1936), "... the threshold rises when the *local potential* is maintained ... and reverts gradually to its original value when the nerve is allowed to rest."

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One bump  $(u, h) = (q(x), p(x))$

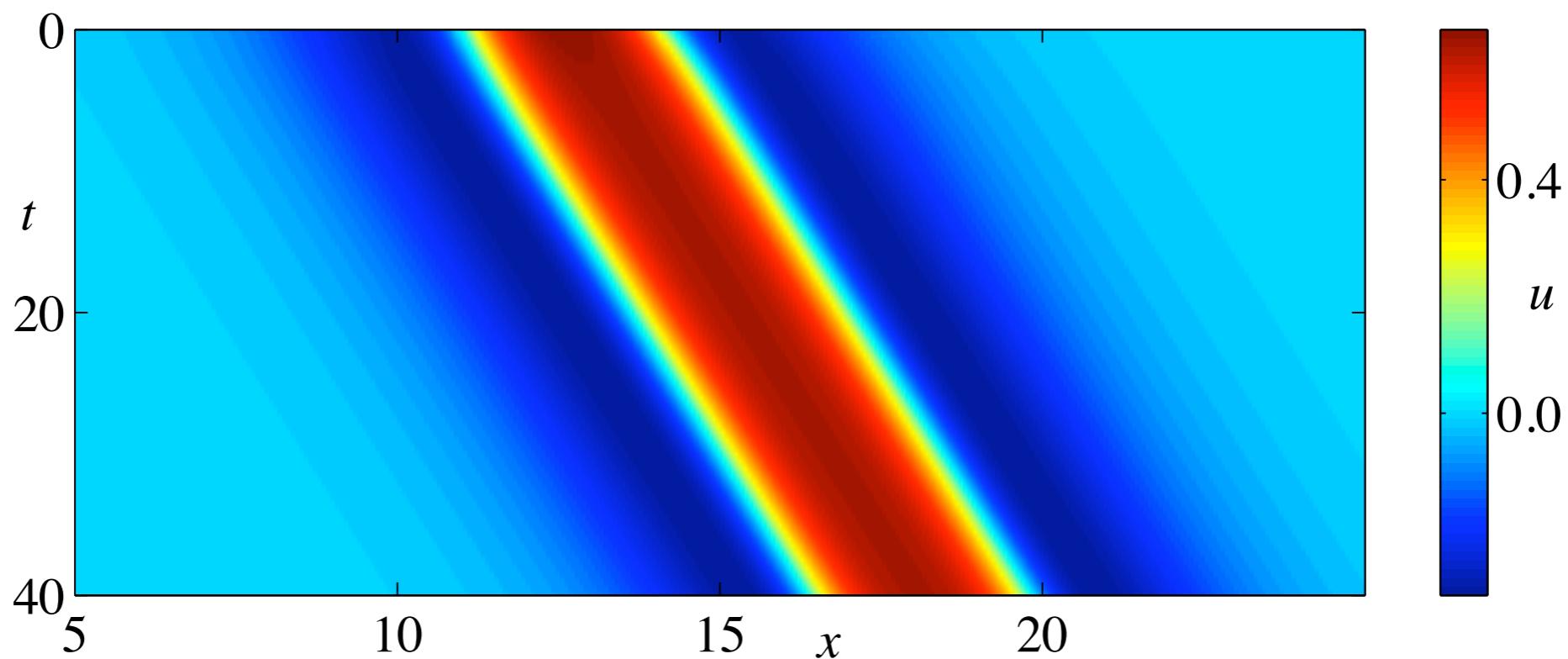
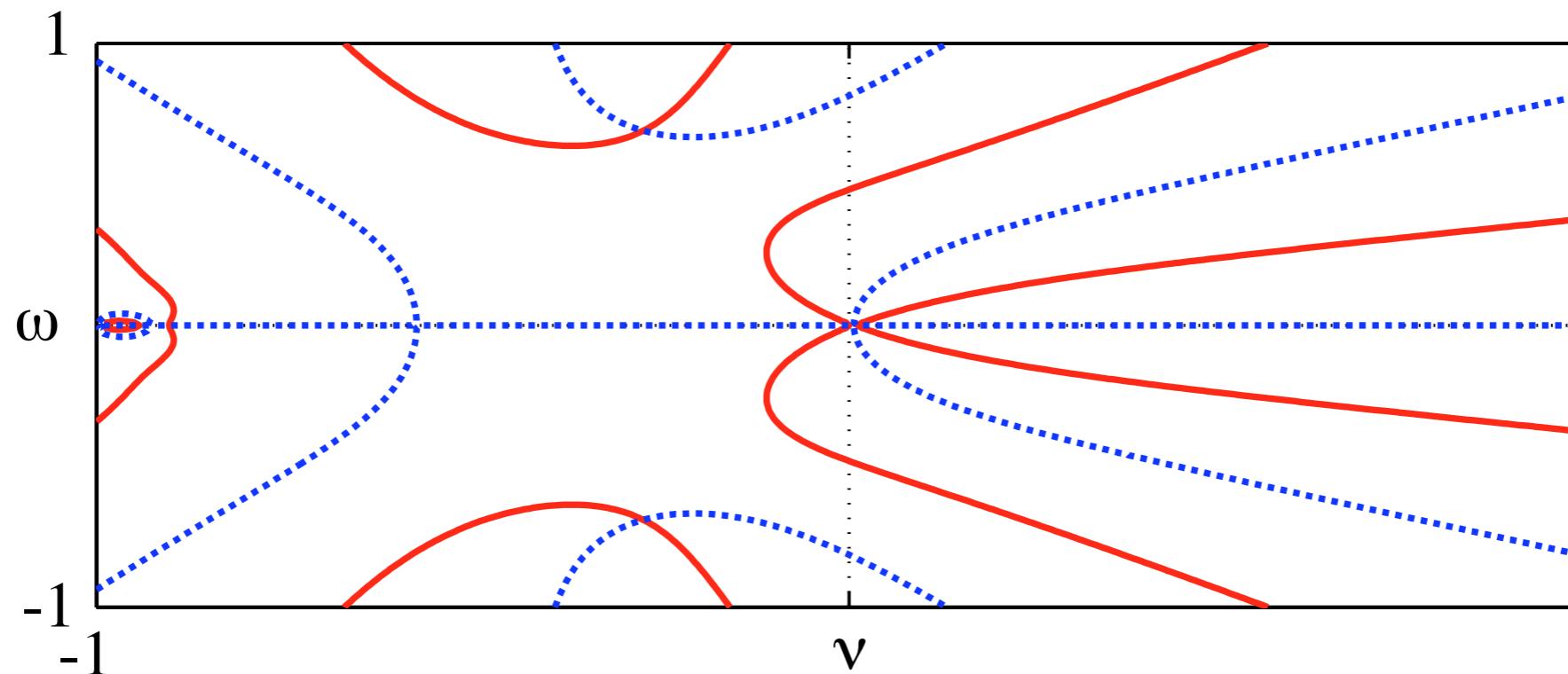
$$q = w \otimes H(q - p)$$

$$p = \begin{cases} h_0 & q < \theta \\ h_0 + \kappa & q \geq \theta \end{cases}$$



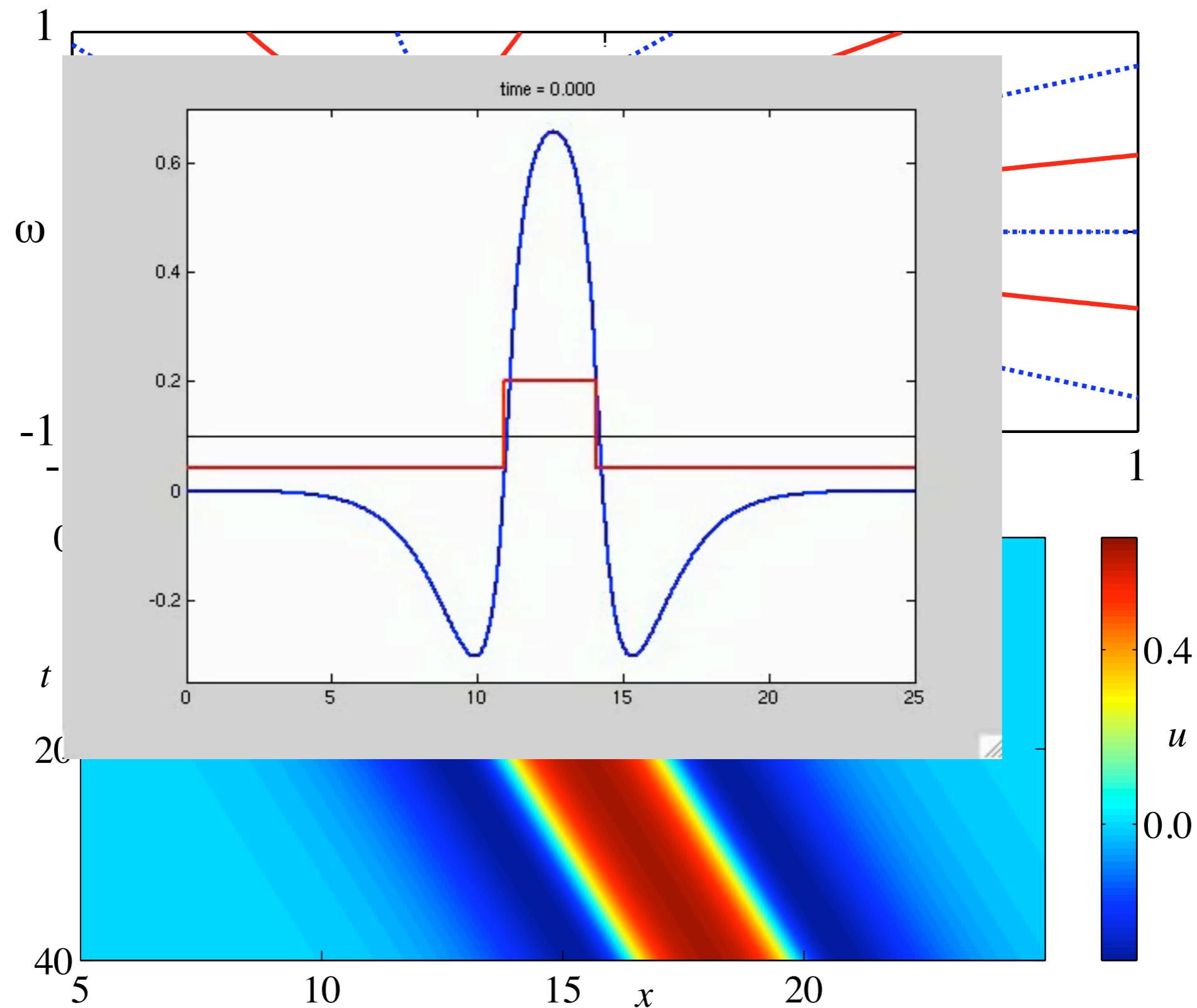
# Bump Stability I: $\eta(t) = \alpha^2 t e^{-\alpha t}$

Low  $\kappa$  instability on Re axis (increasing  $\alpha$ )



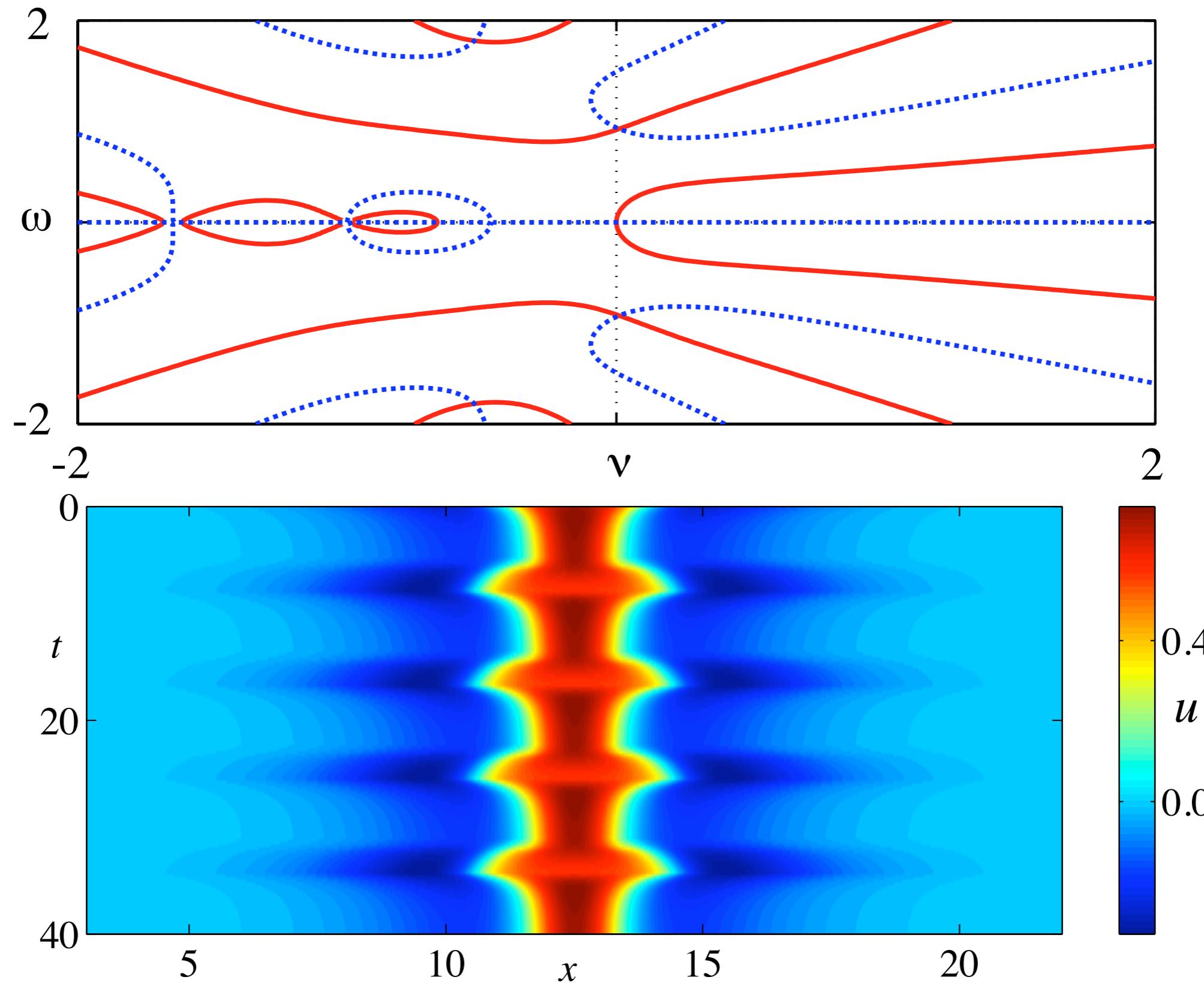
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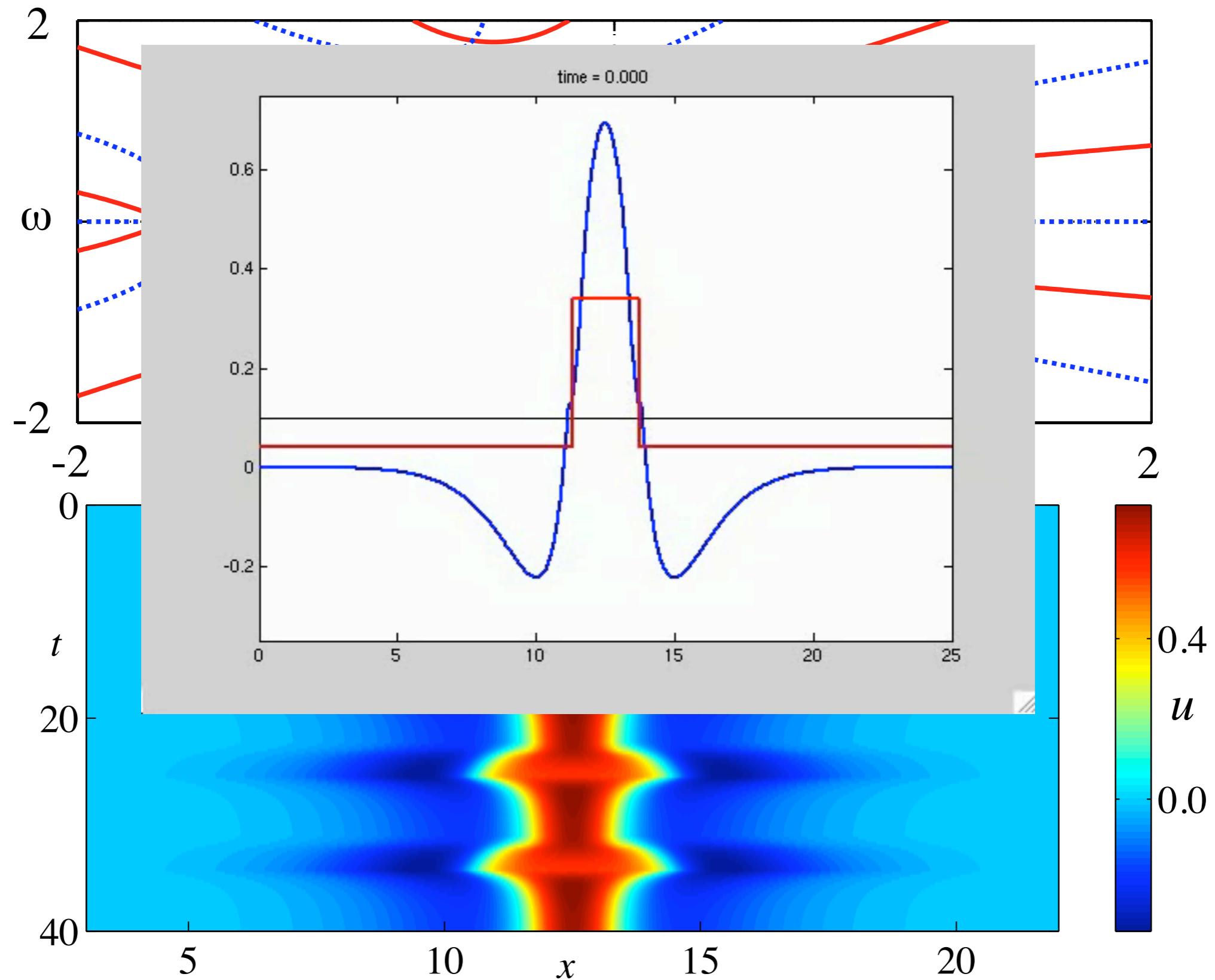
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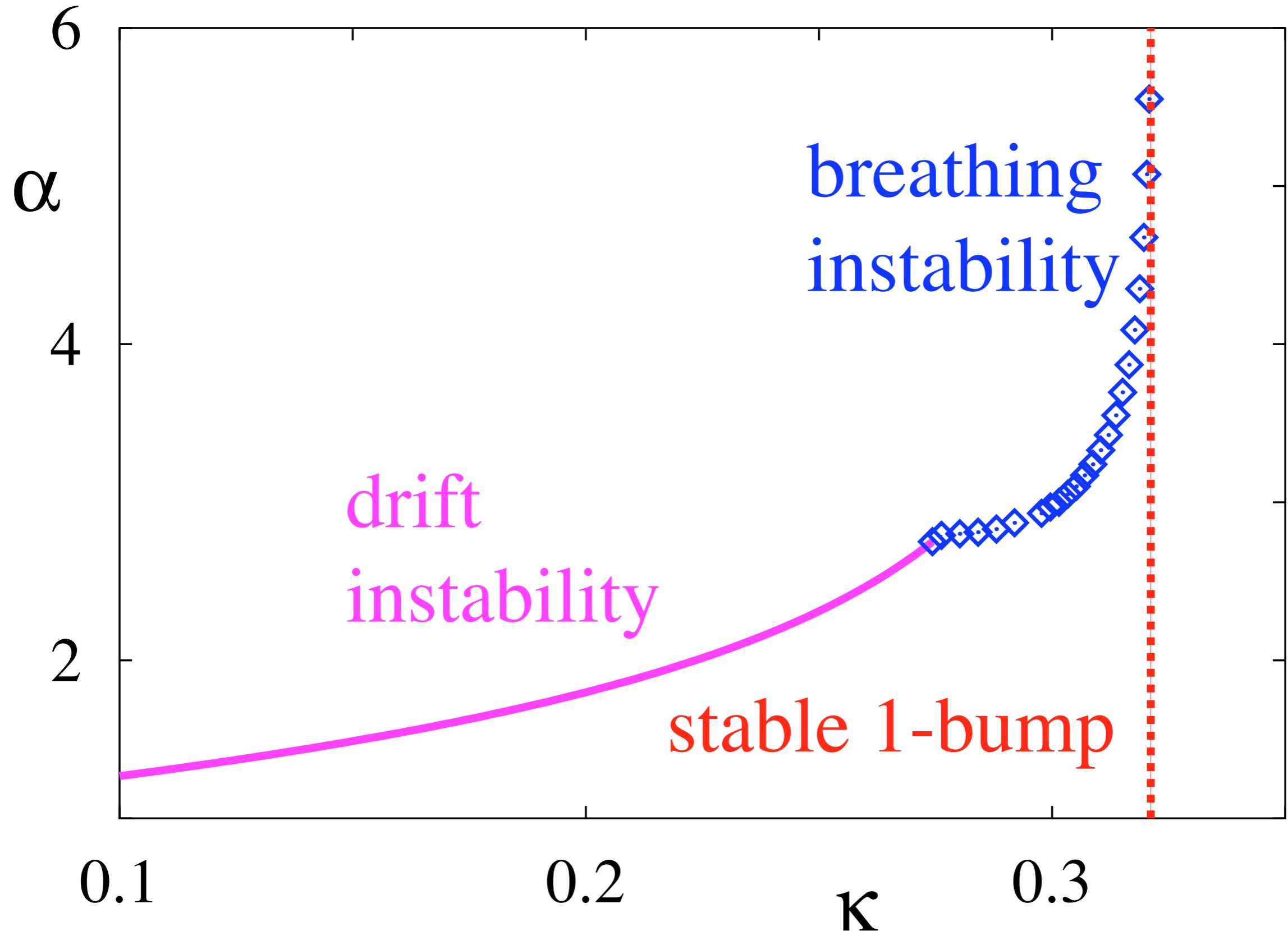


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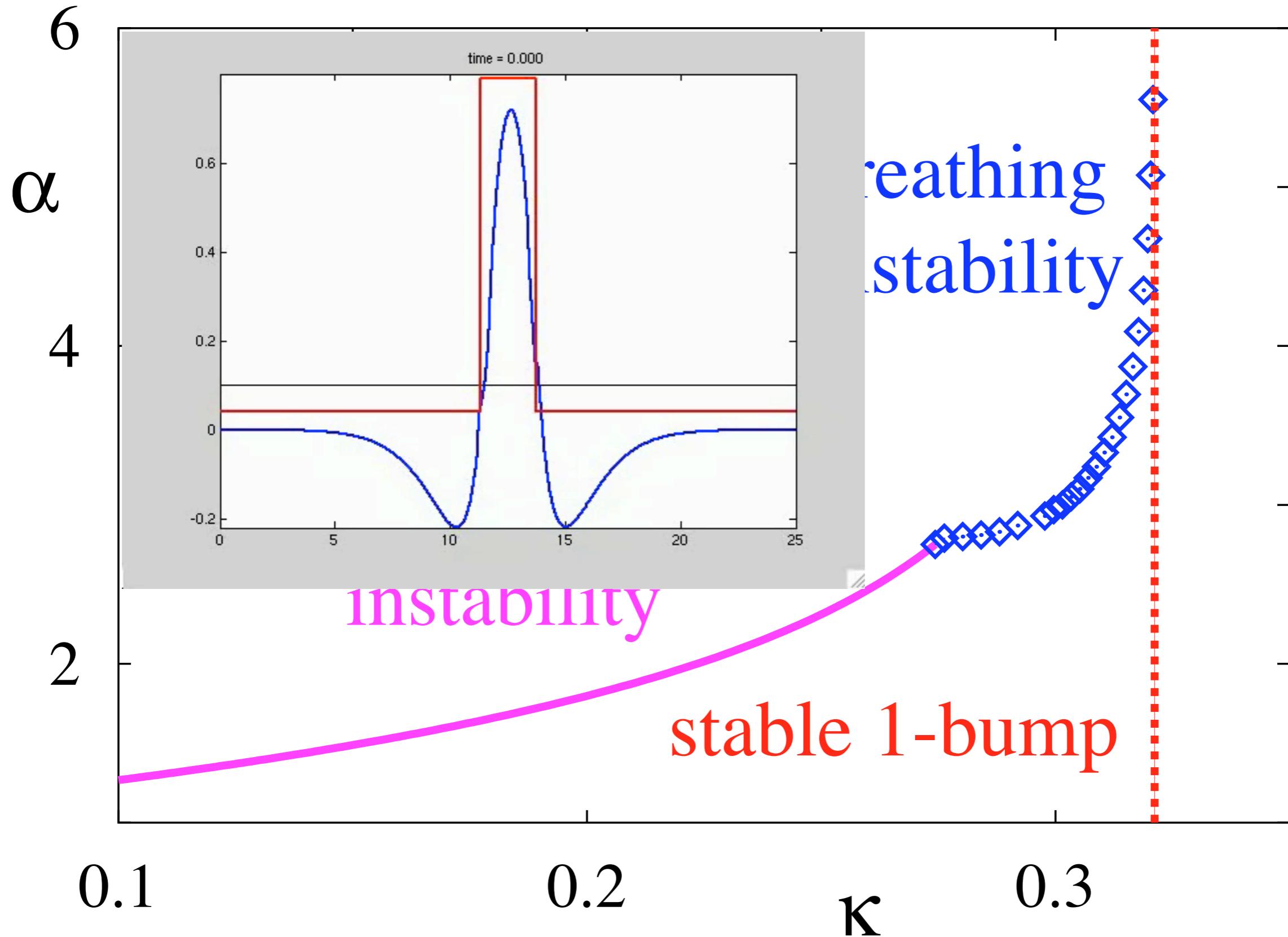
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# Summary of Bump instabilities

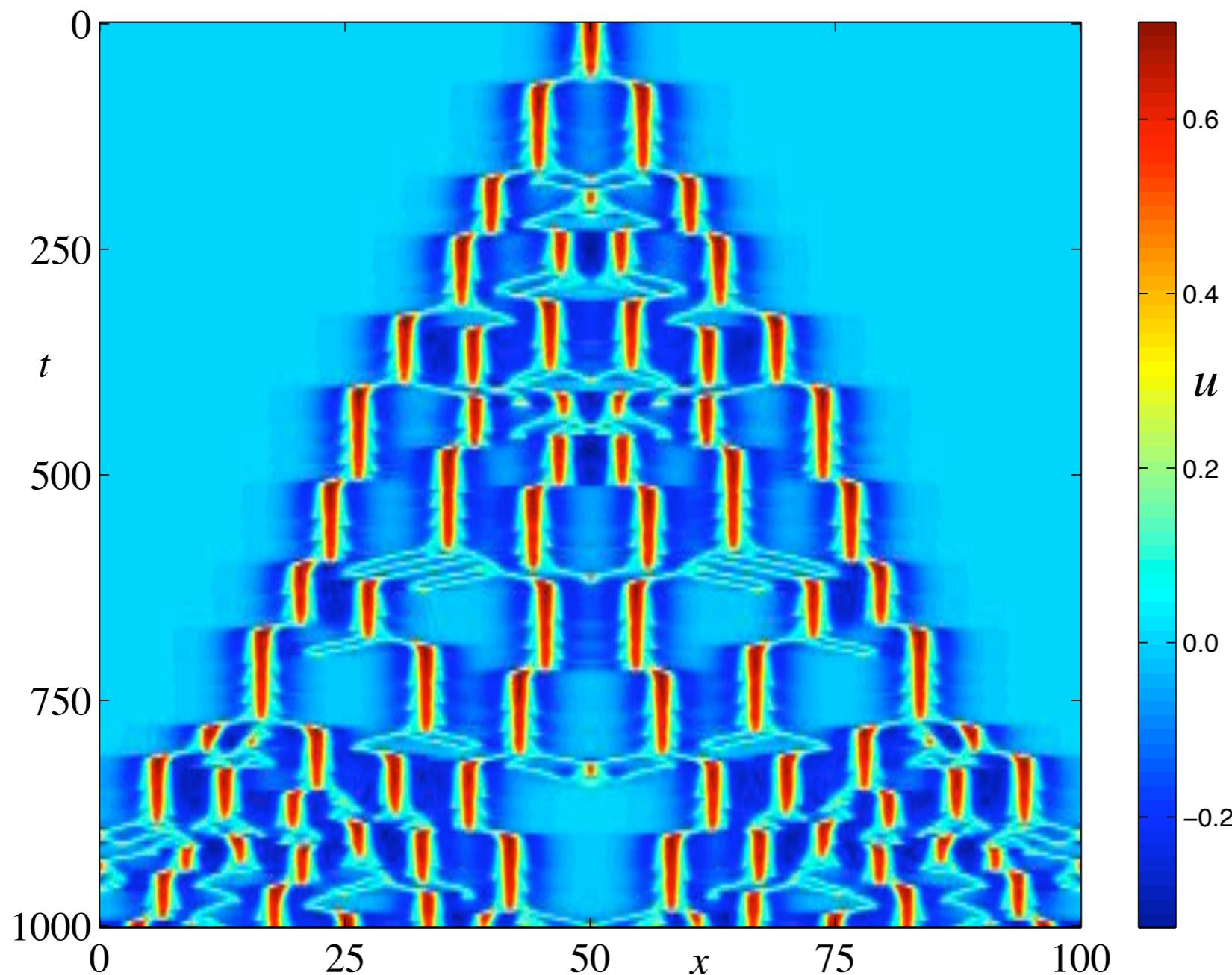


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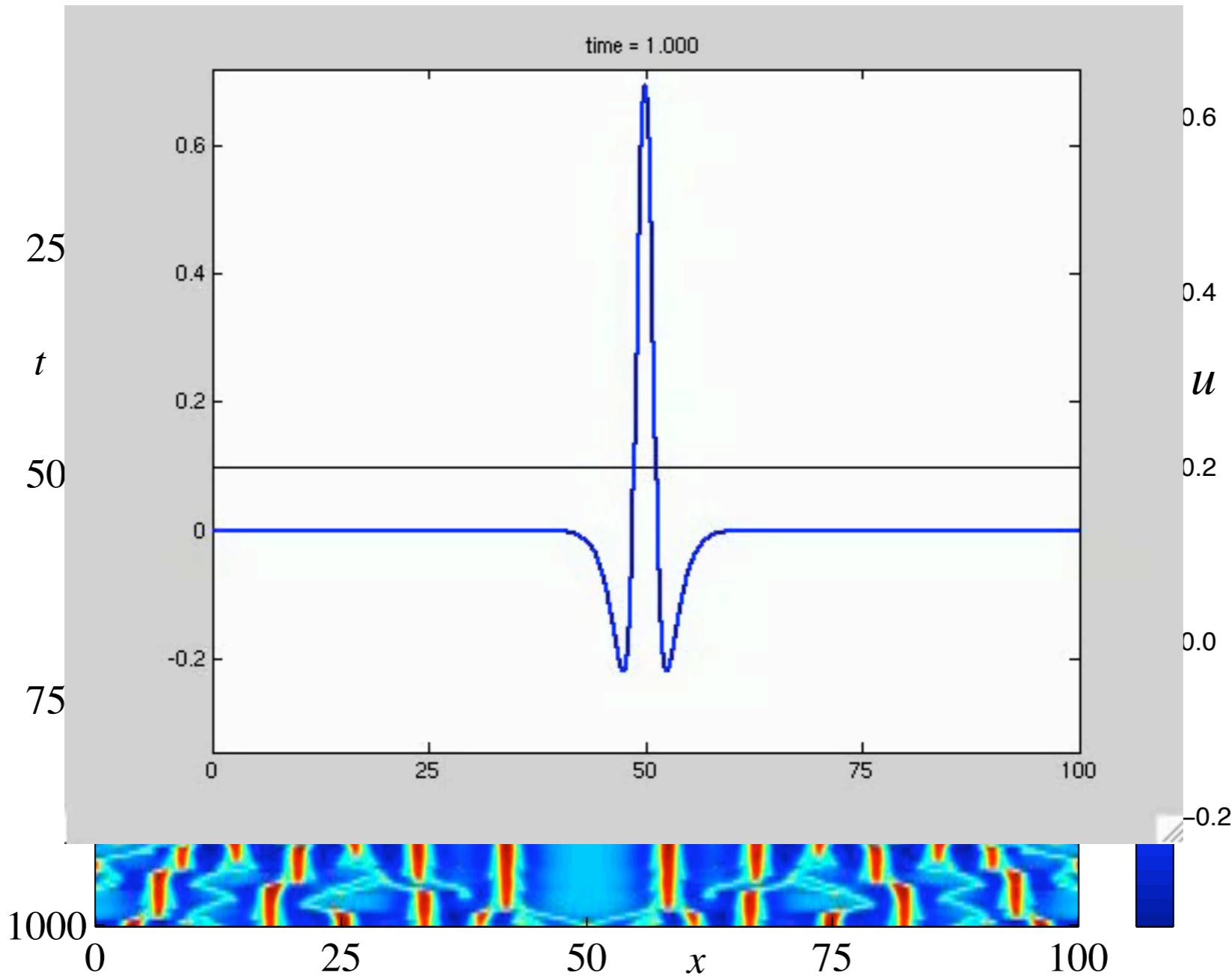
# Exotic Dynamics

... including asymmetric breathers, multiple bumps, multiple pulses, periodic traveling waves, and bump-splitting instabilities that appear to lead to spatio-temporal chaos.

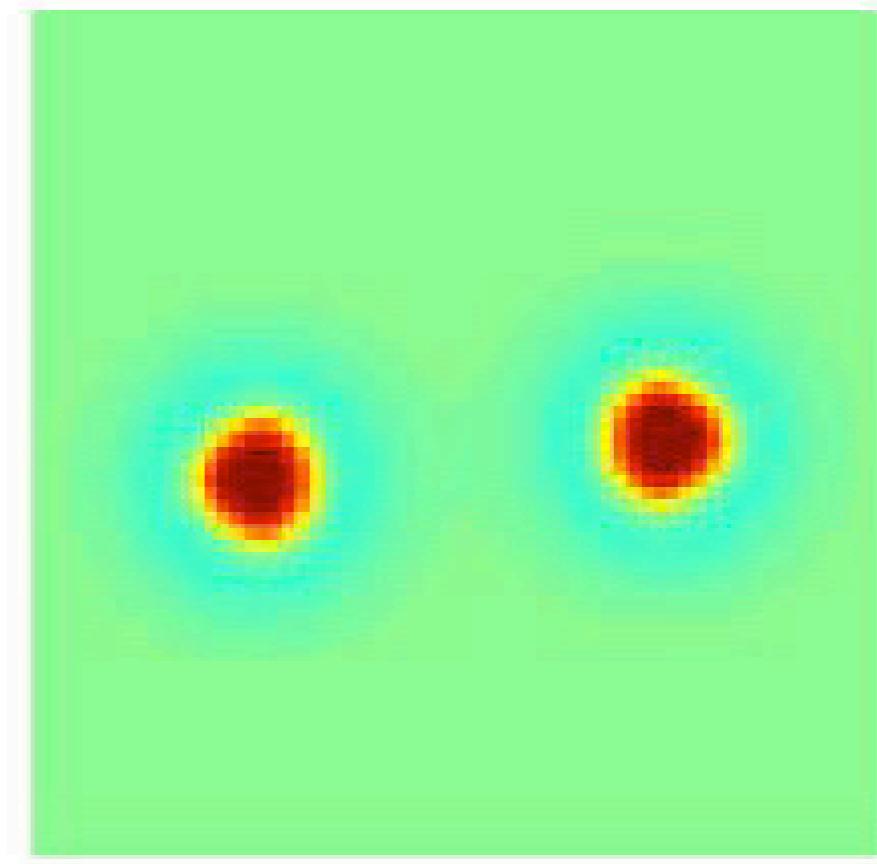
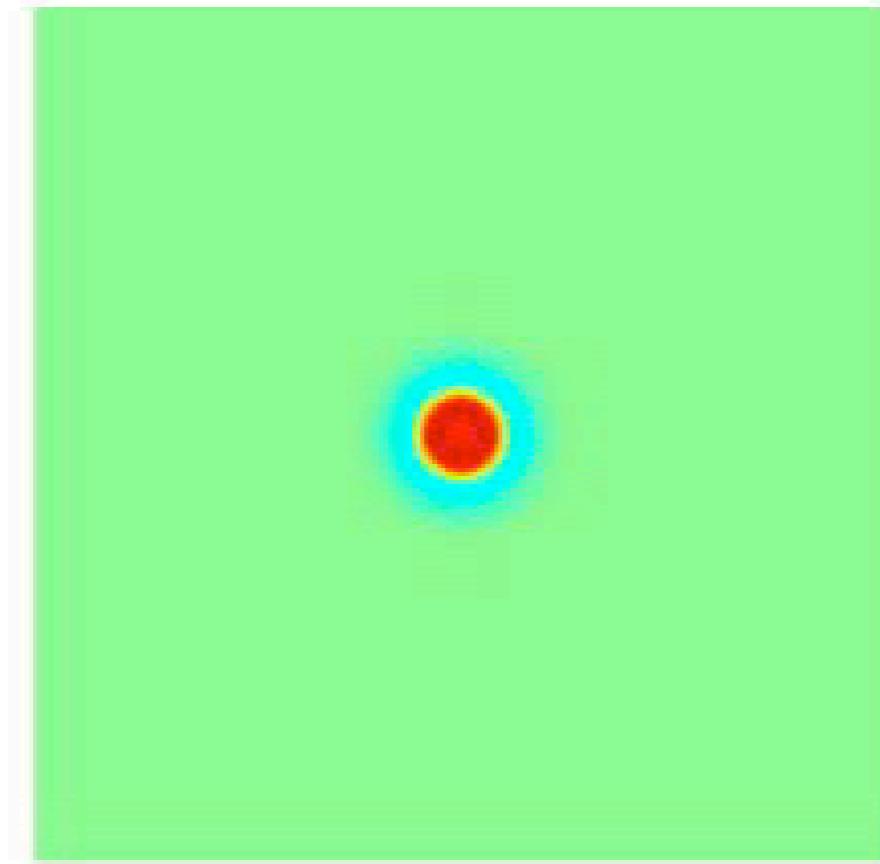


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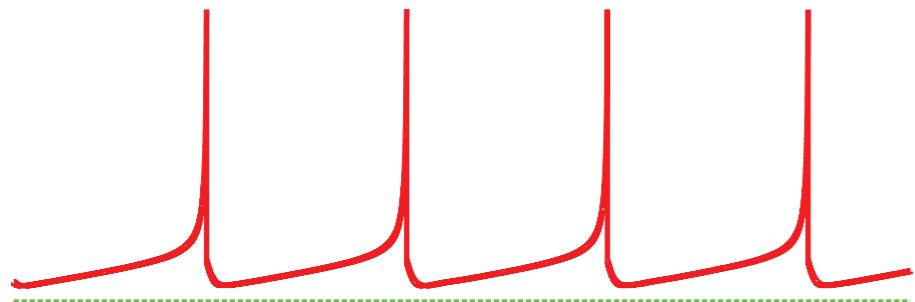


# Splitting and scattering



Auto/dispersive solitons as seen in coupled cubic complex Ginzburg-Landau systems and three component reaction-diffusion systems.

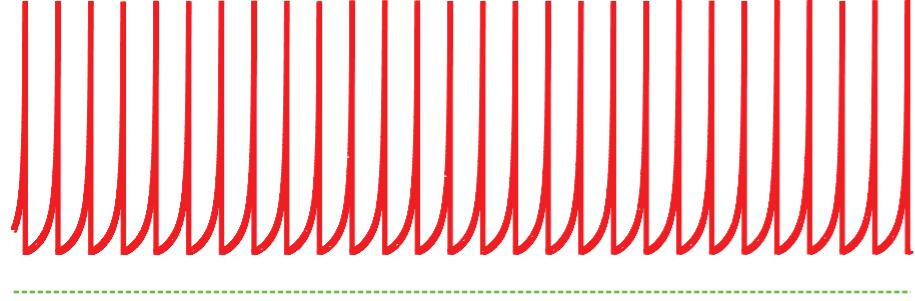
**Regular spiking**



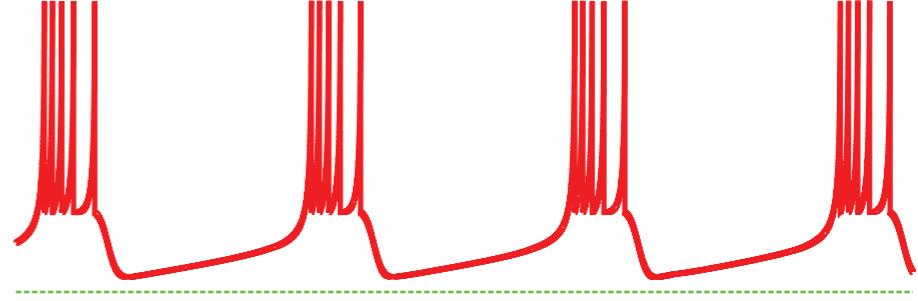
**Chattering**



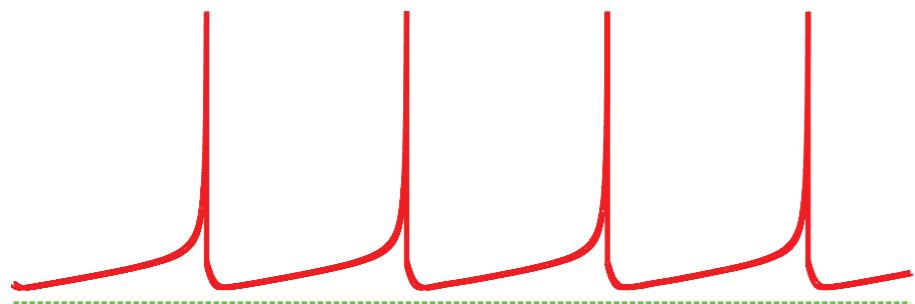
**Fast spiking**



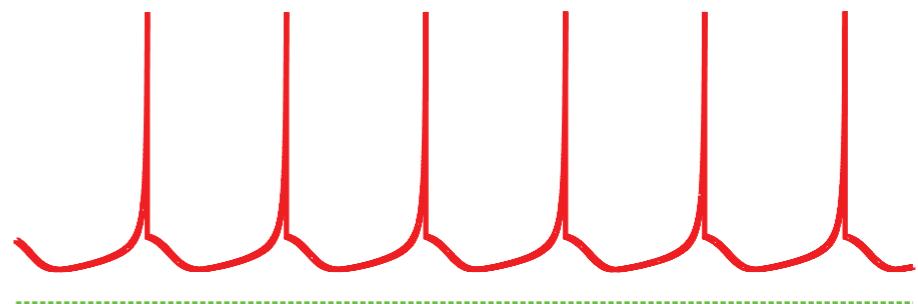
**Intrinsically bursting**



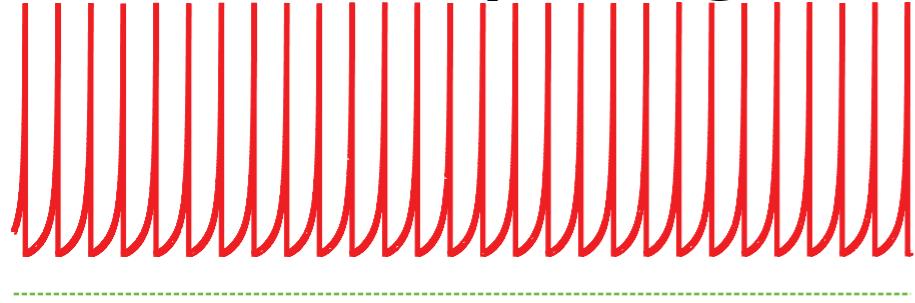
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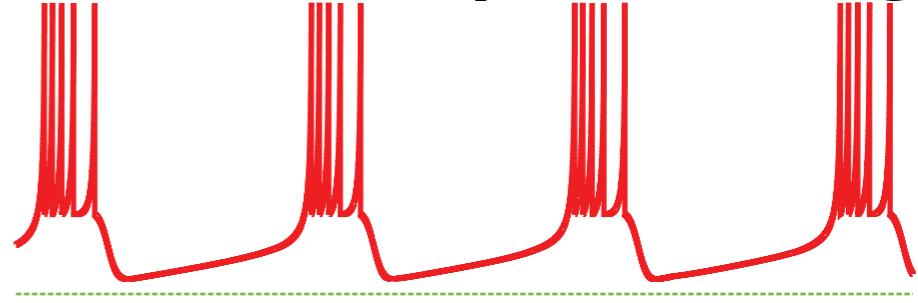
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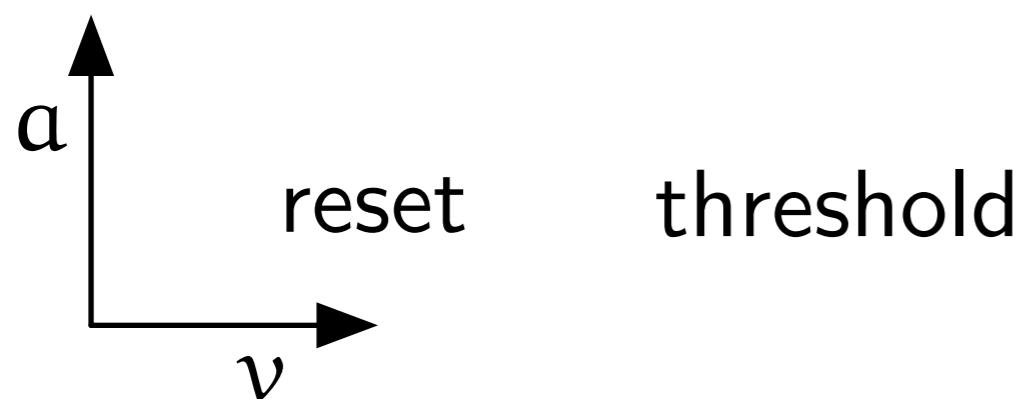


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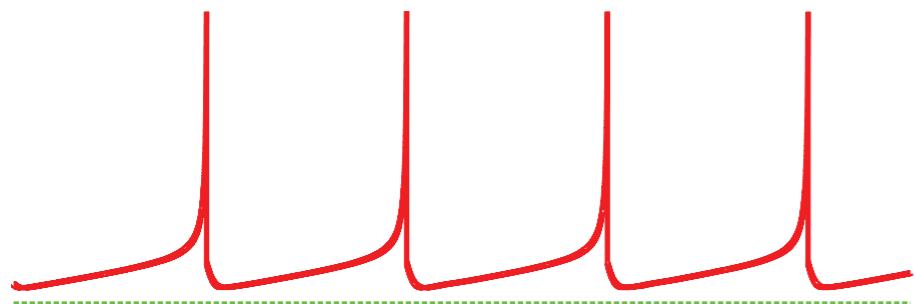


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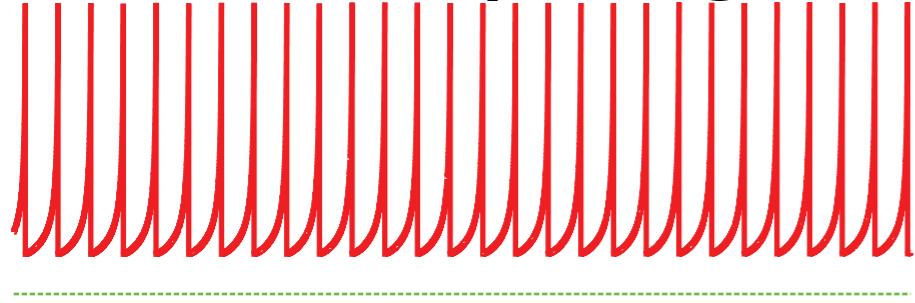
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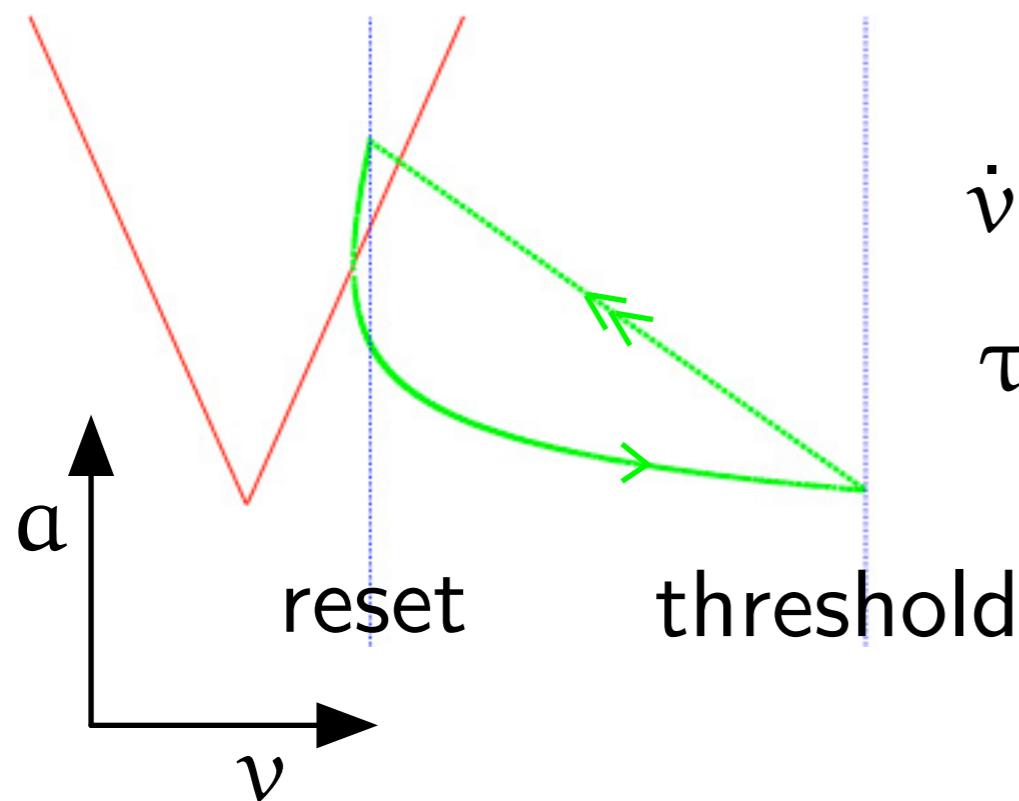
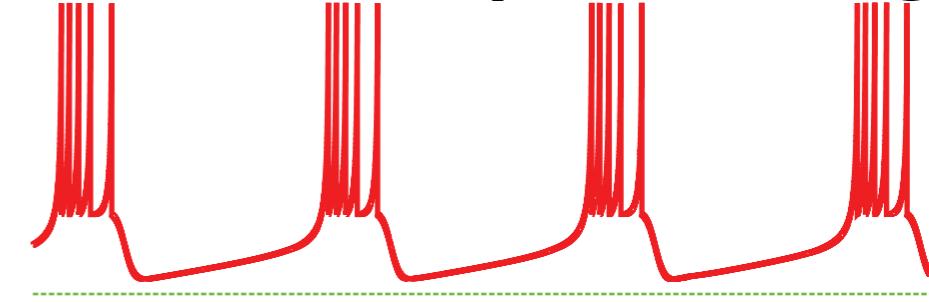
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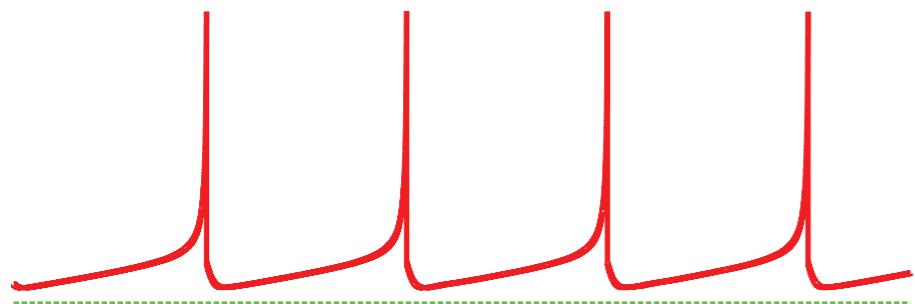
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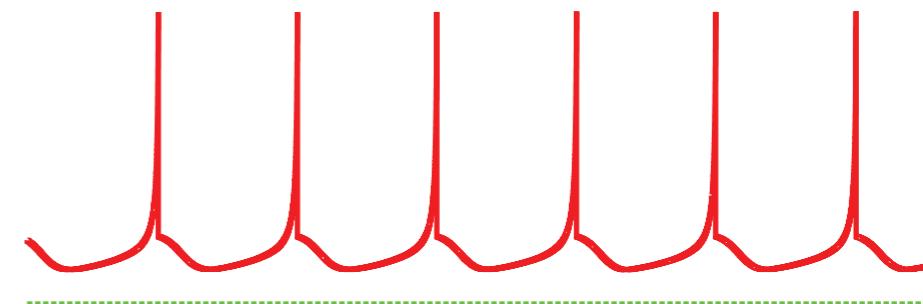
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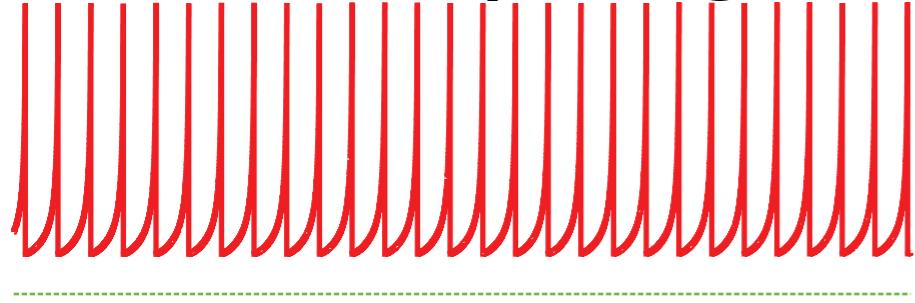
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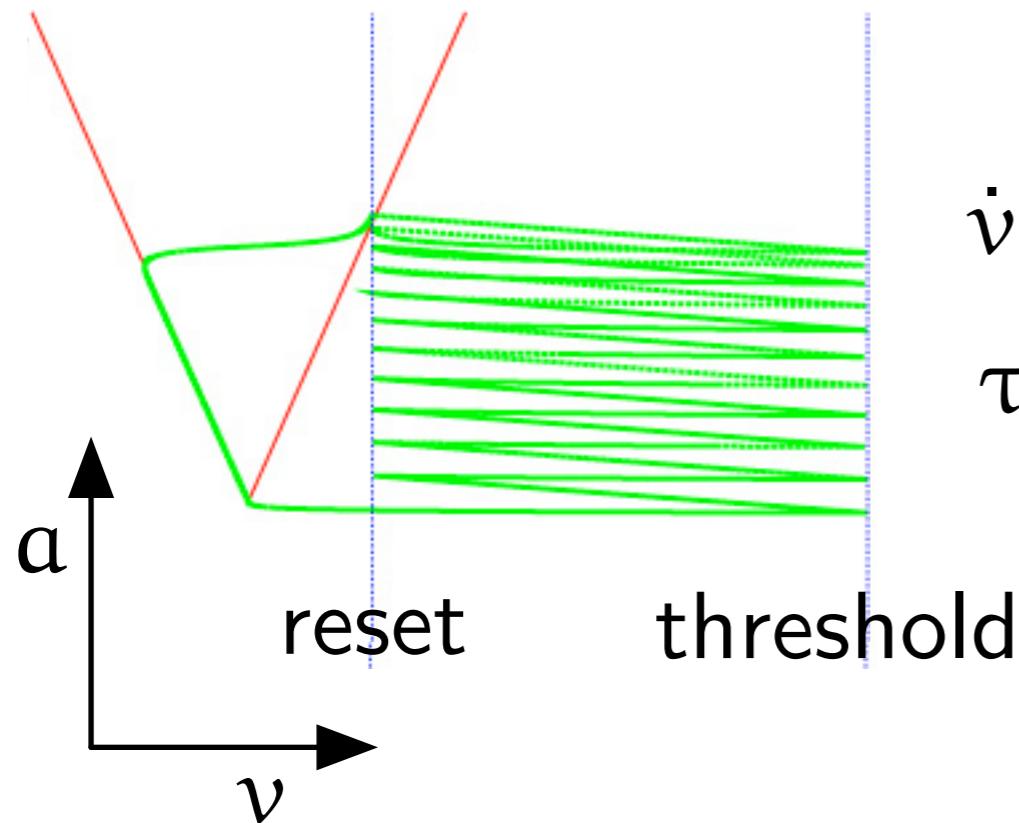
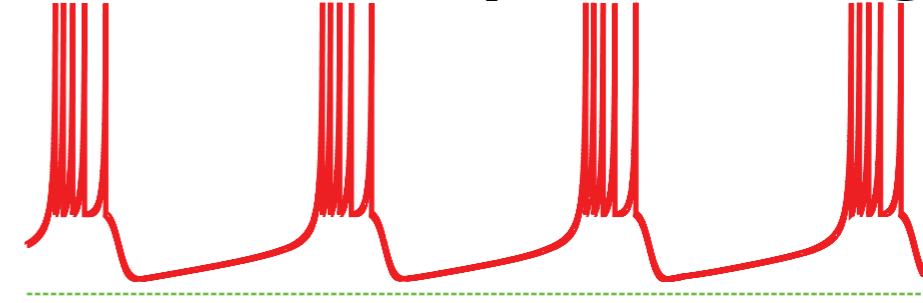
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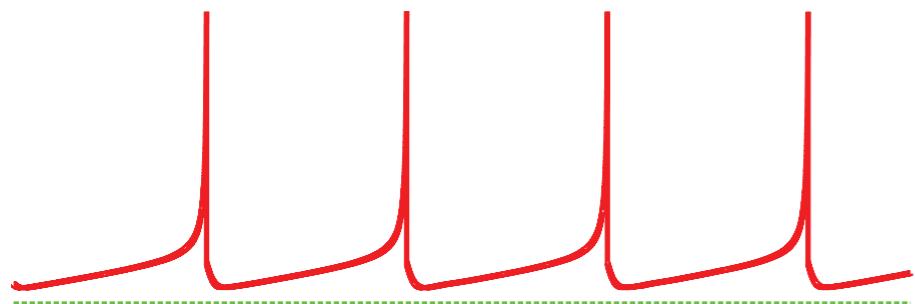
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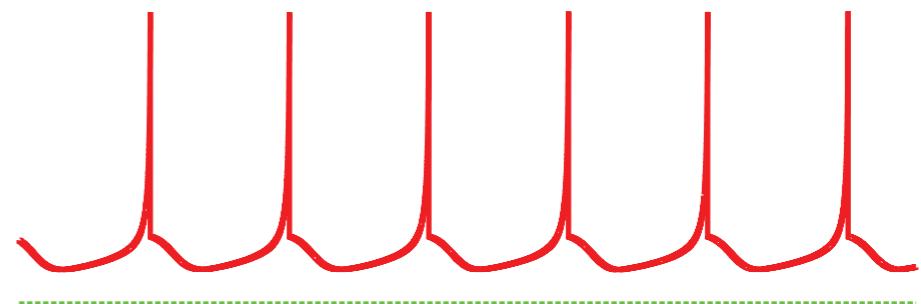
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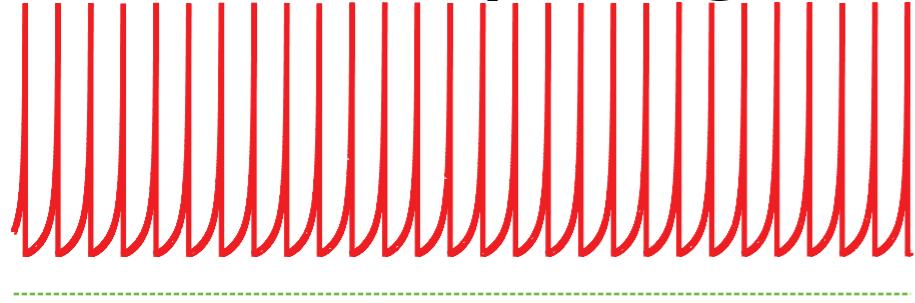
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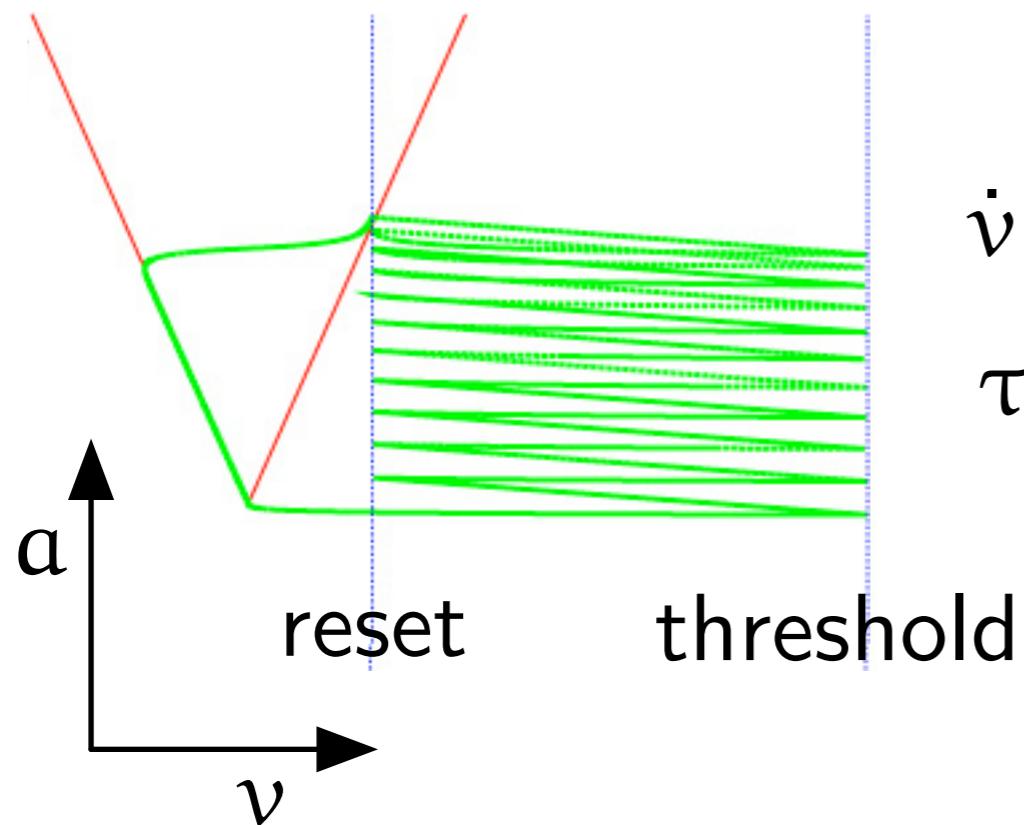
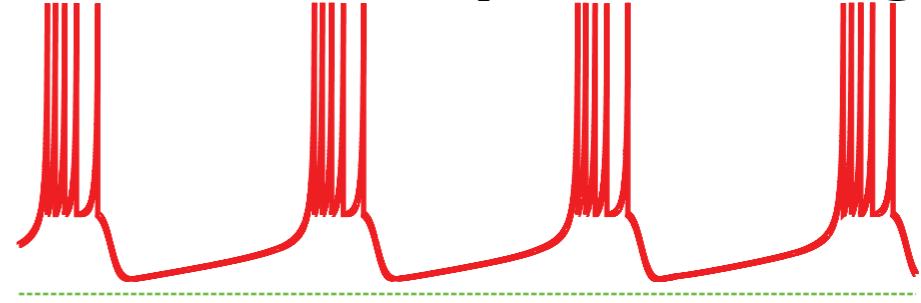
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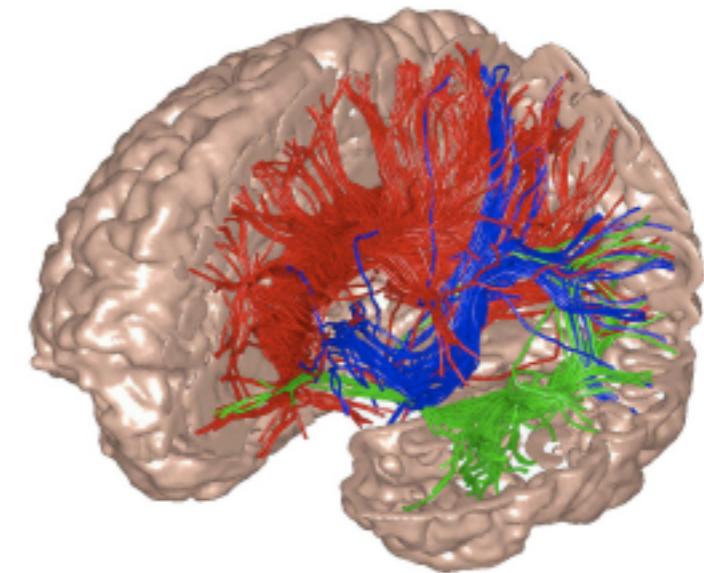


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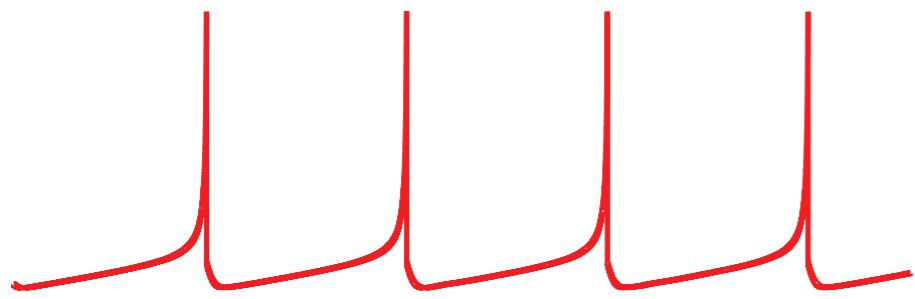
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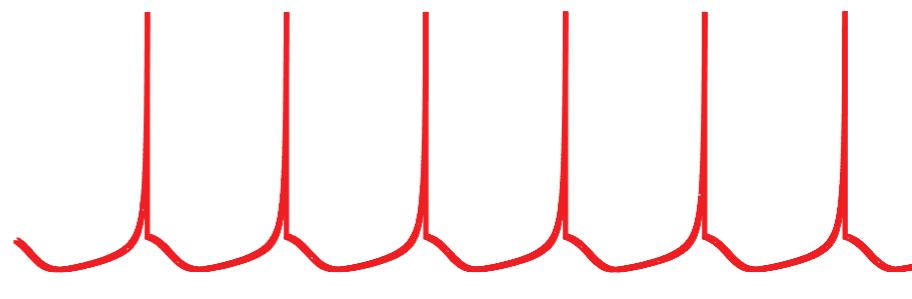


Eugene Izhikevich 2008

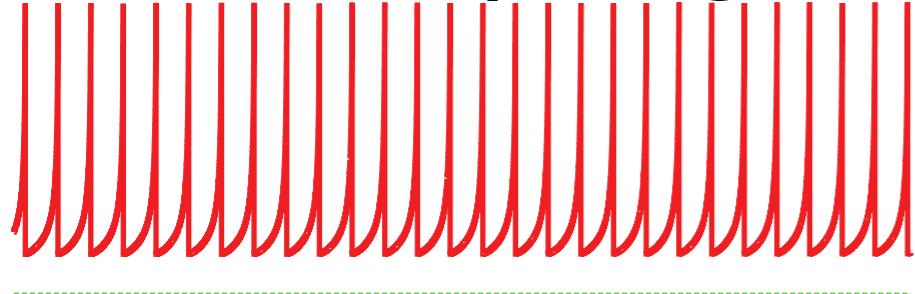
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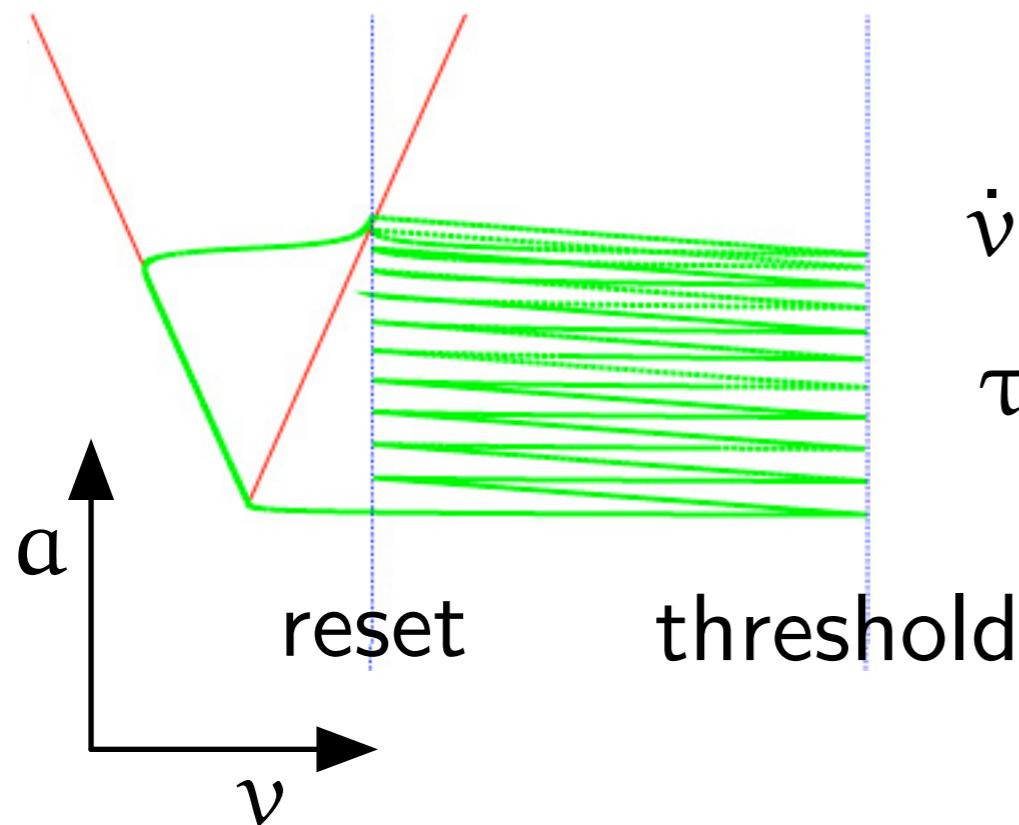
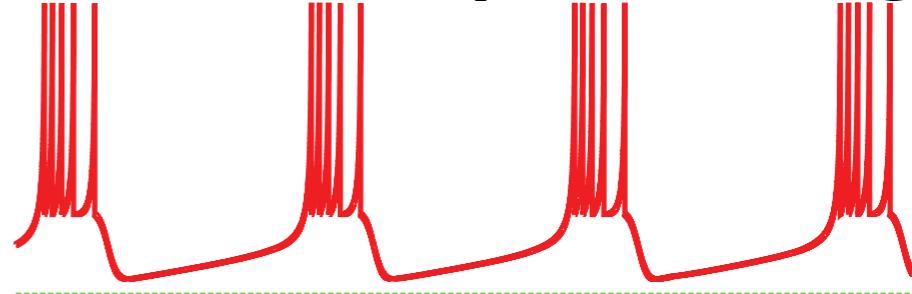
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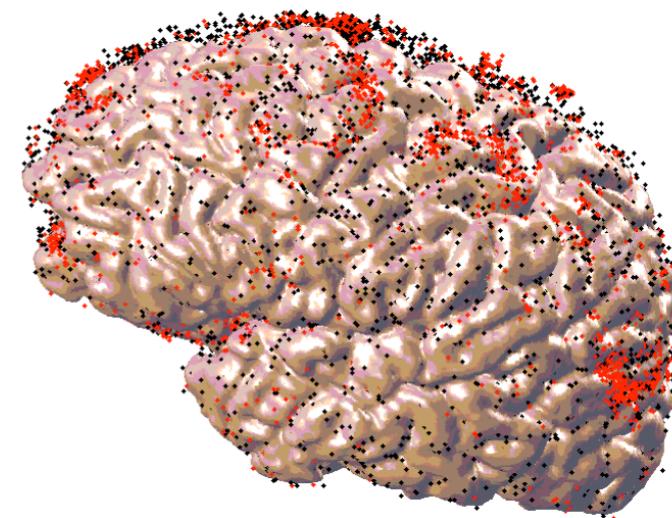


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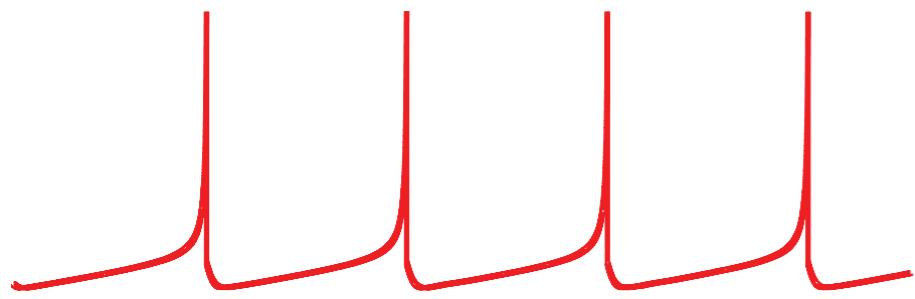


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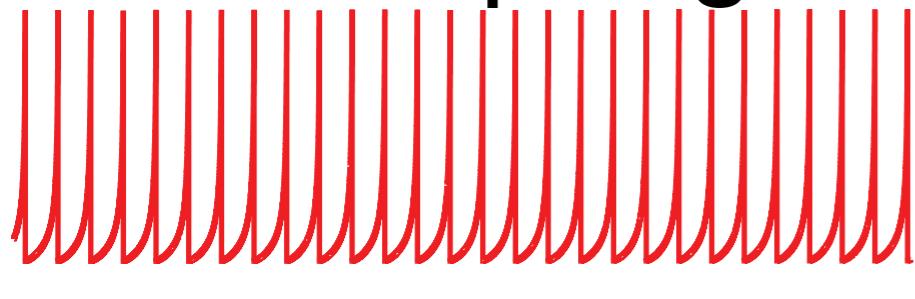
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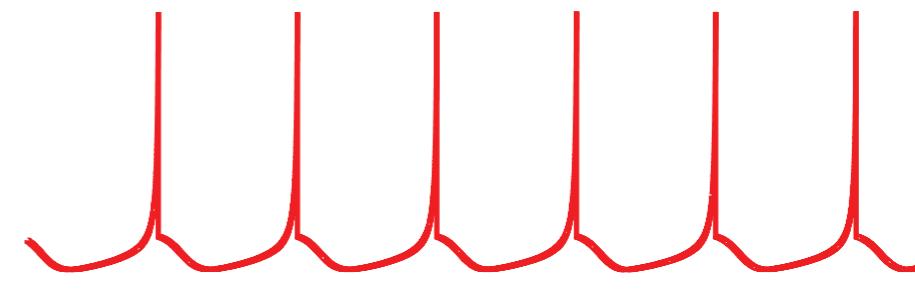
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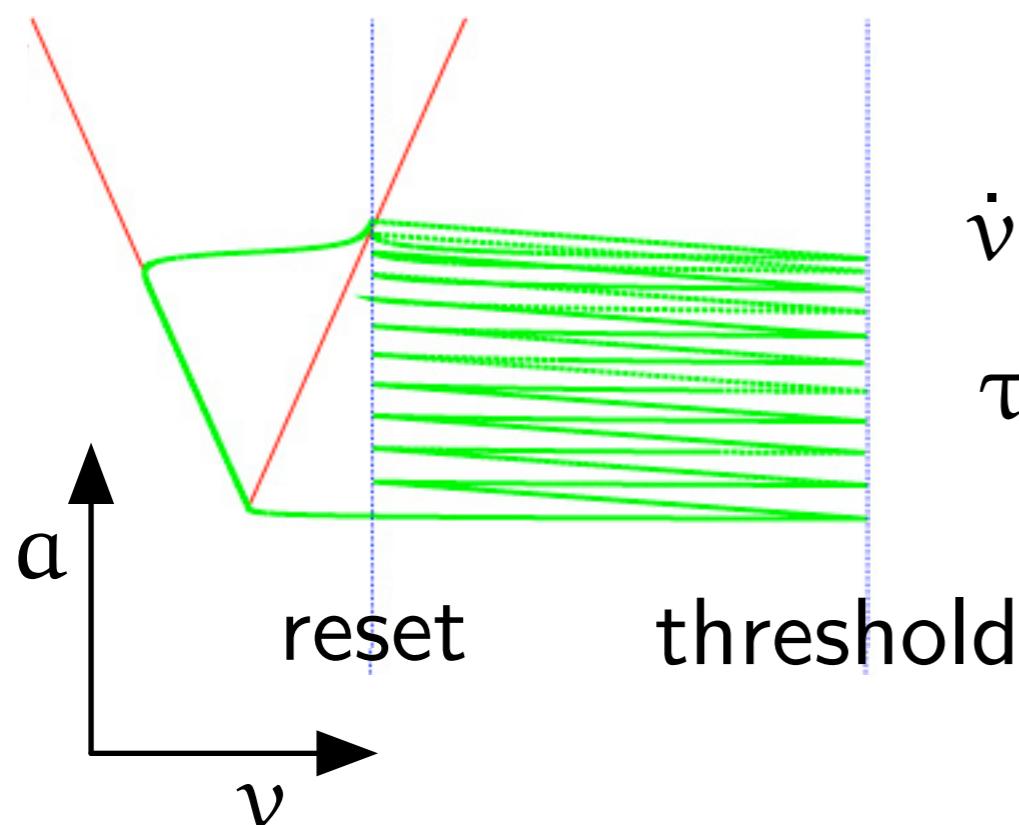
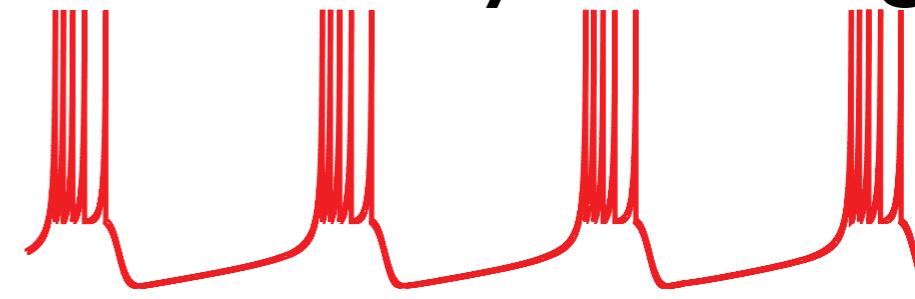
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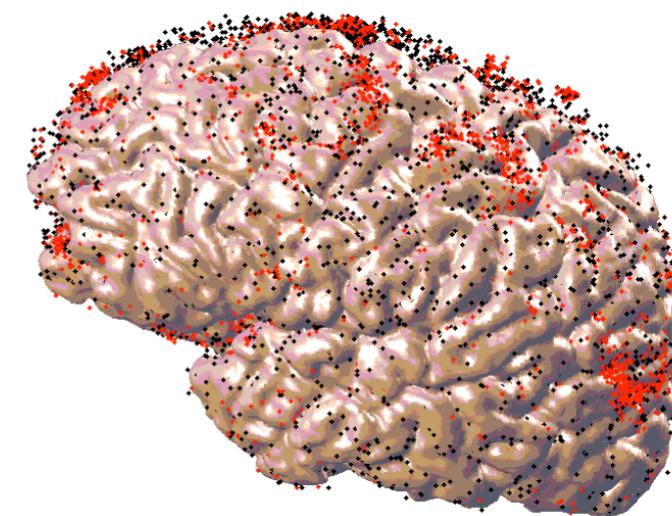


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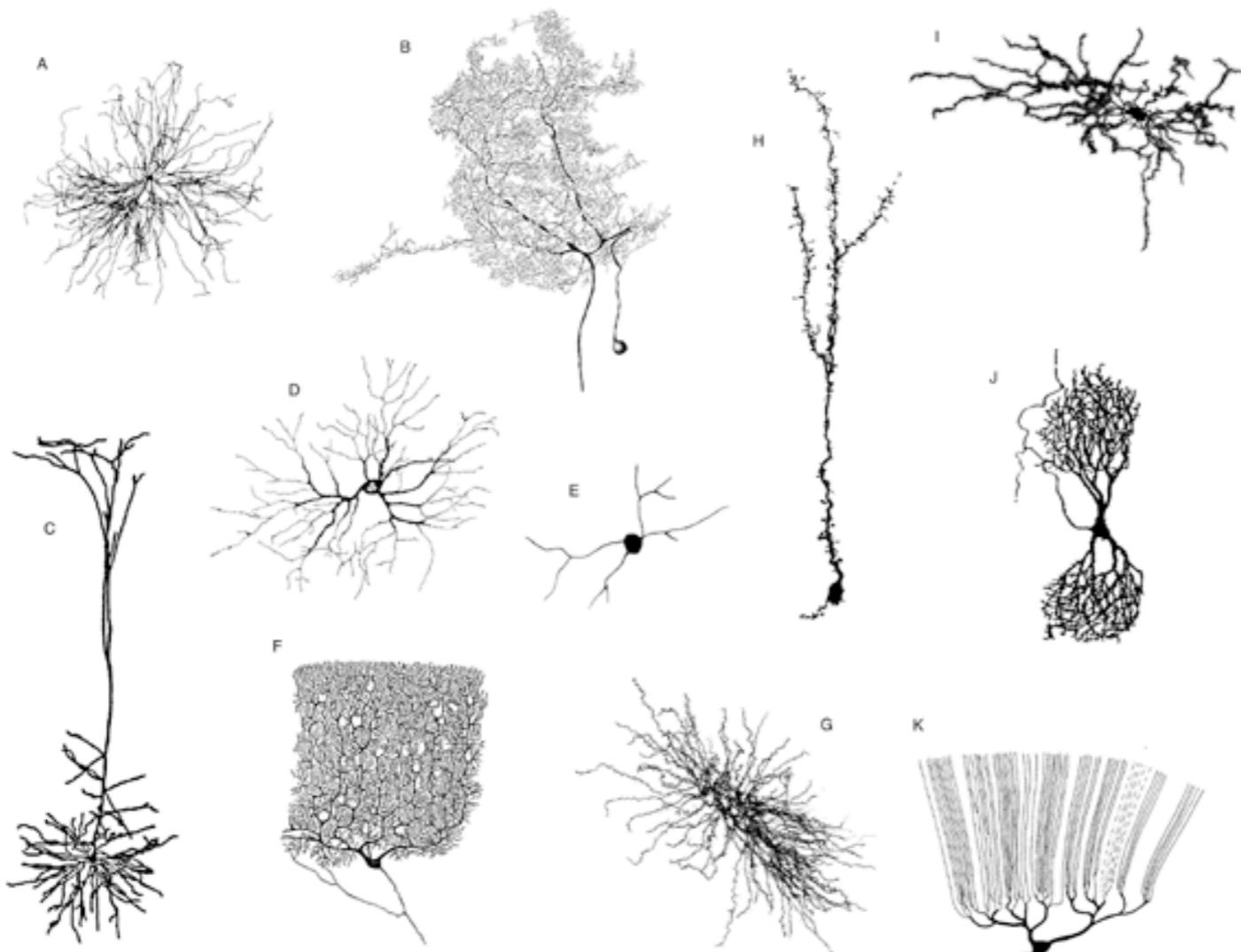
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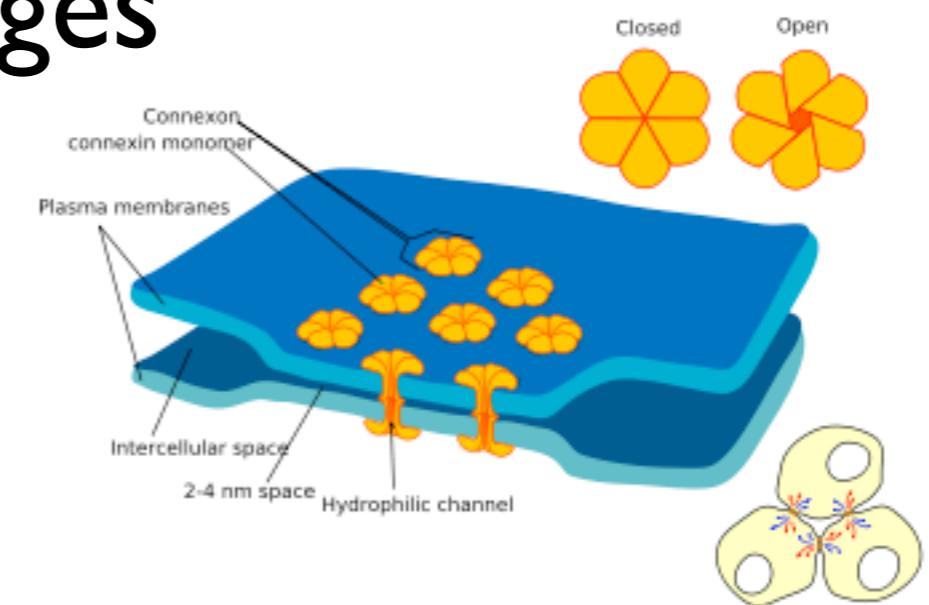
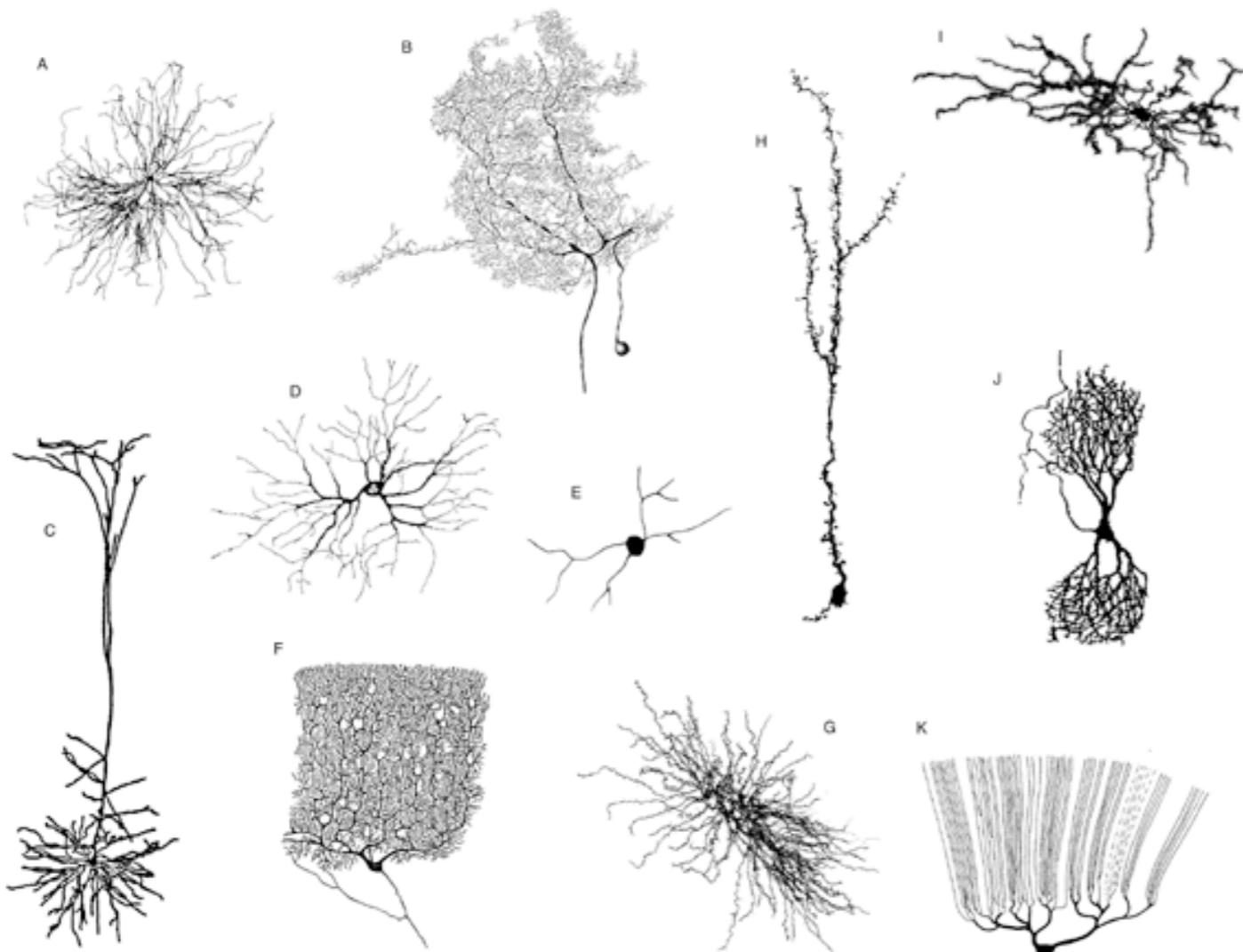


S Coombes and M Zachariou 2009, in  
Coherent Behavior in Neuronal Networks  
(Ed. Rubin, Josic, Matias, Romo), Springer.

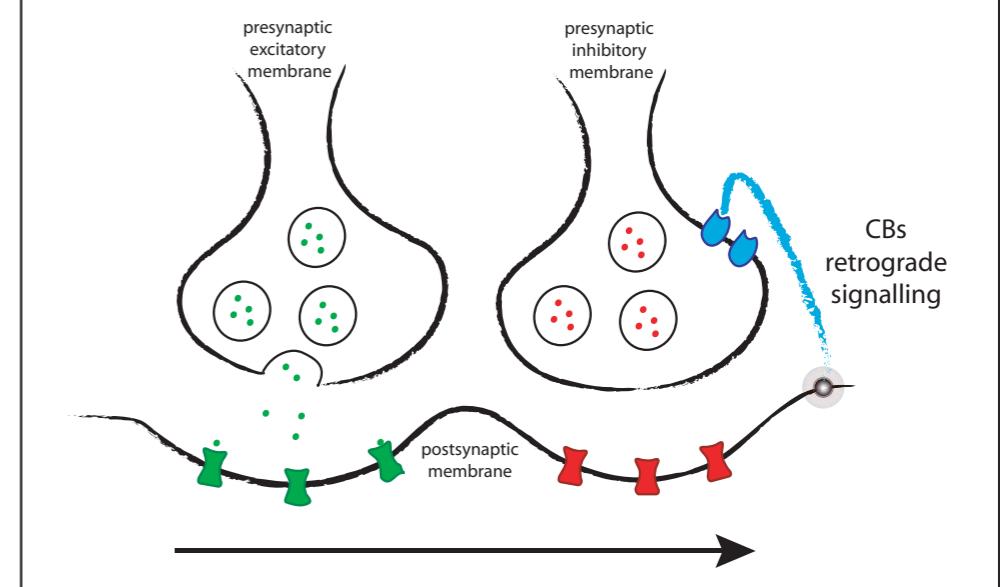
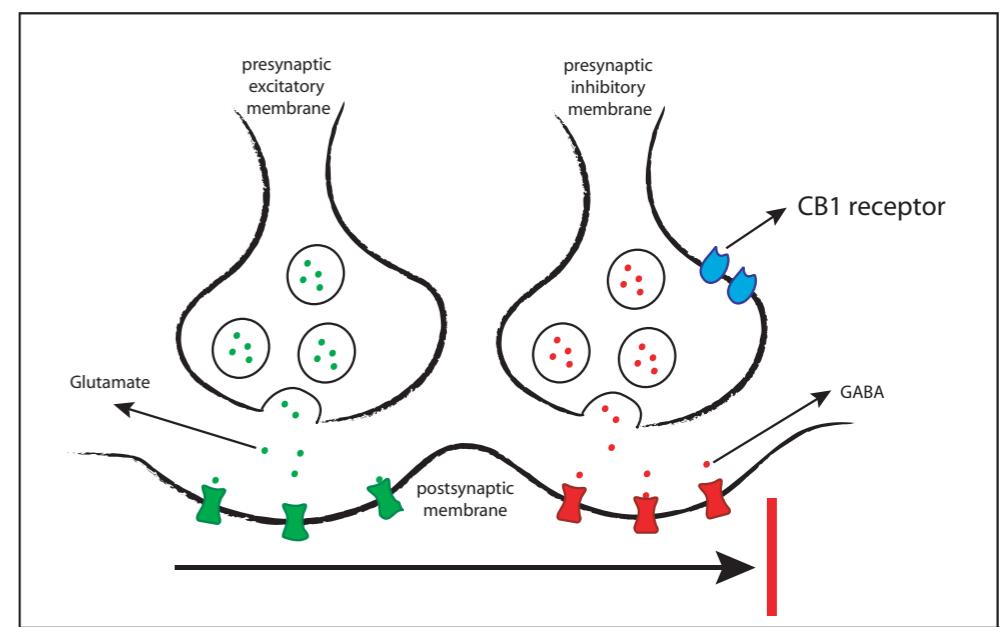
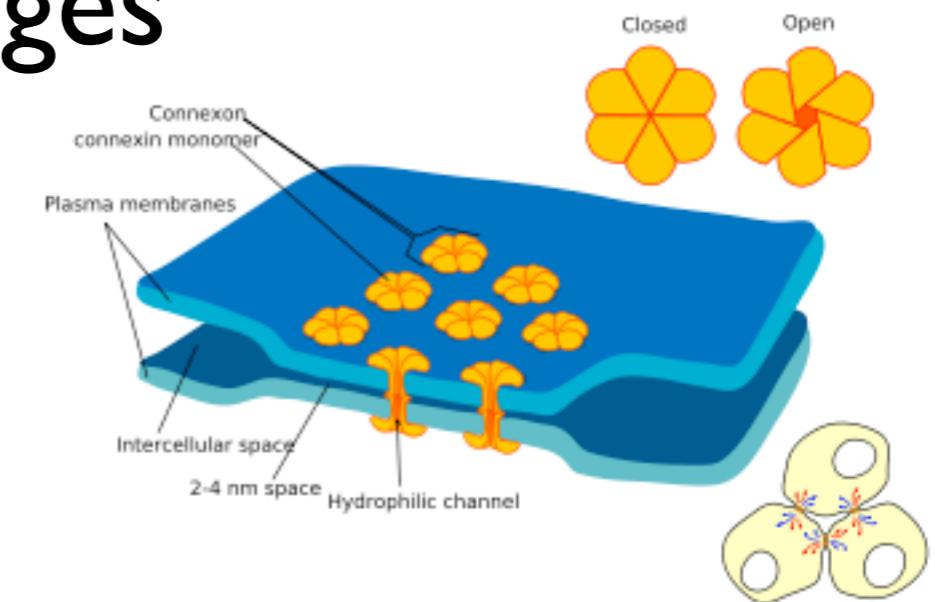
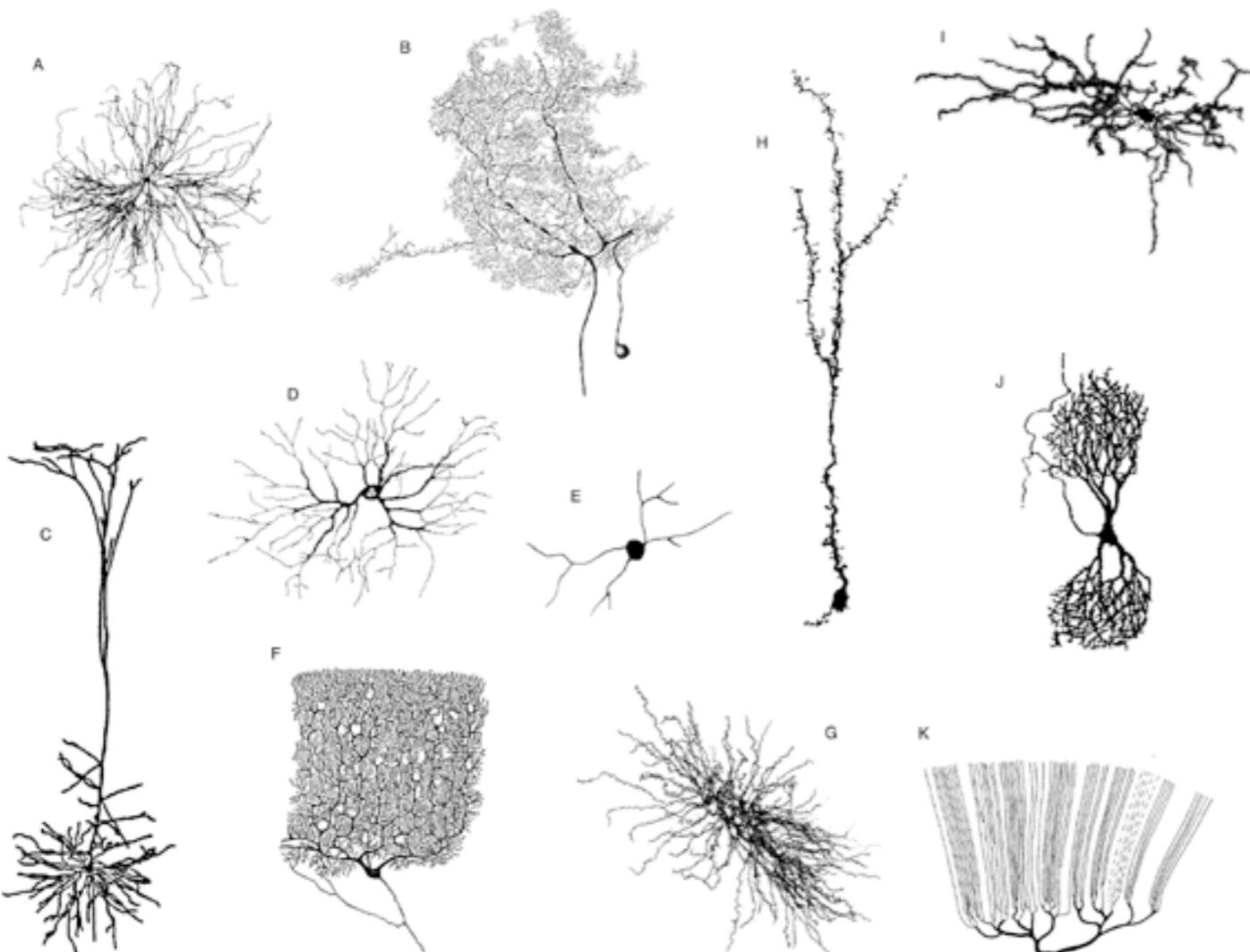
# Further Challenges



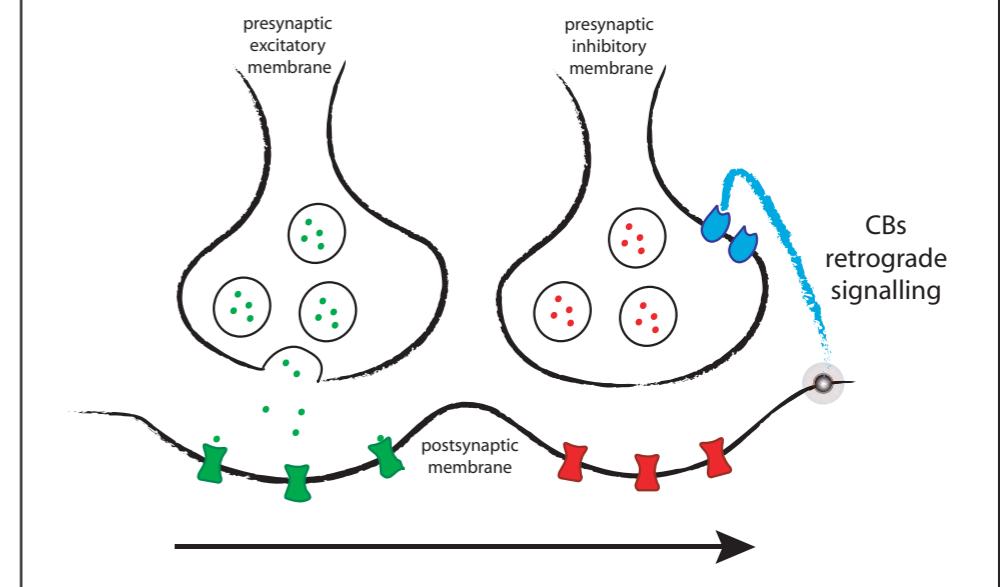
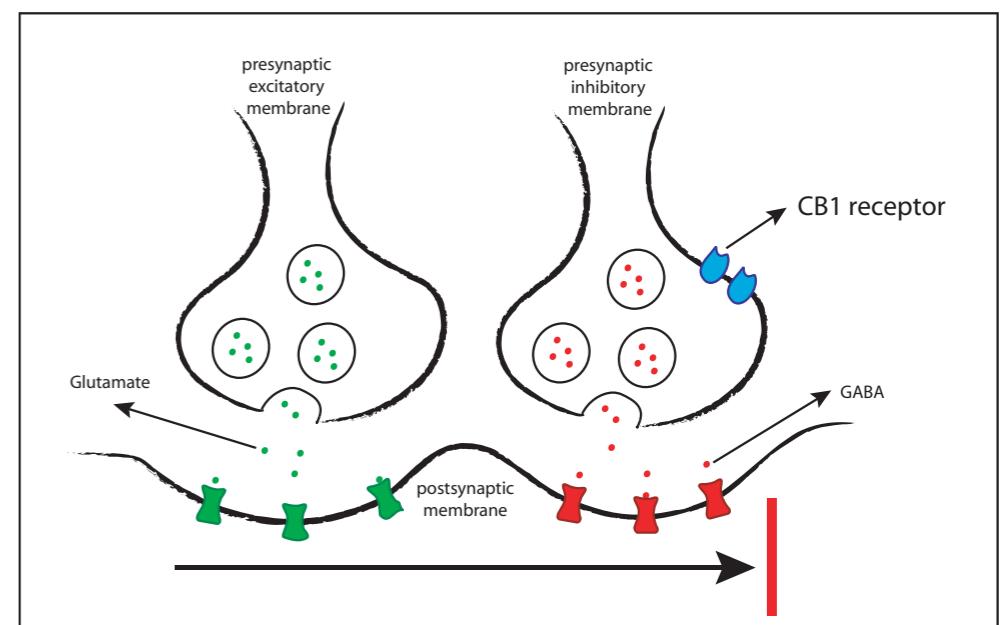
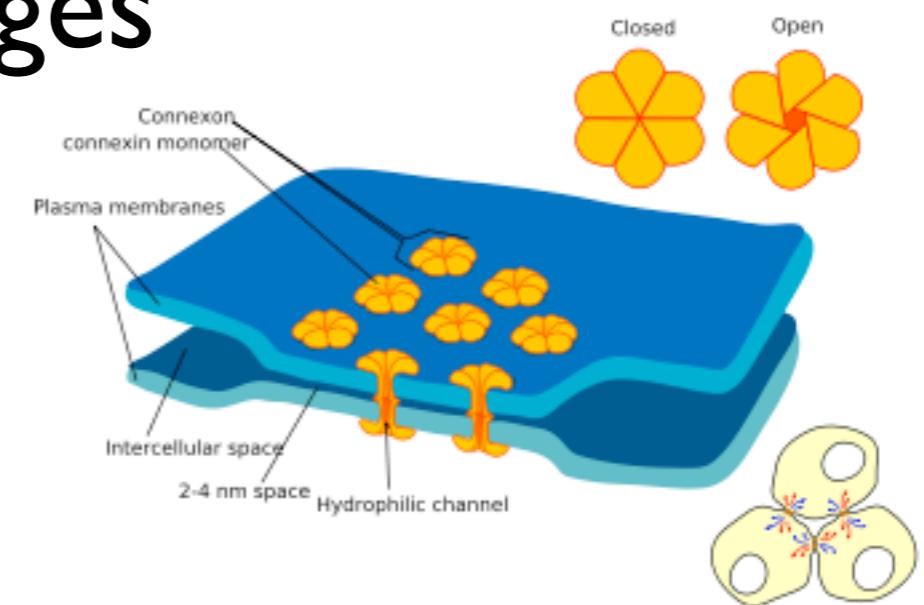
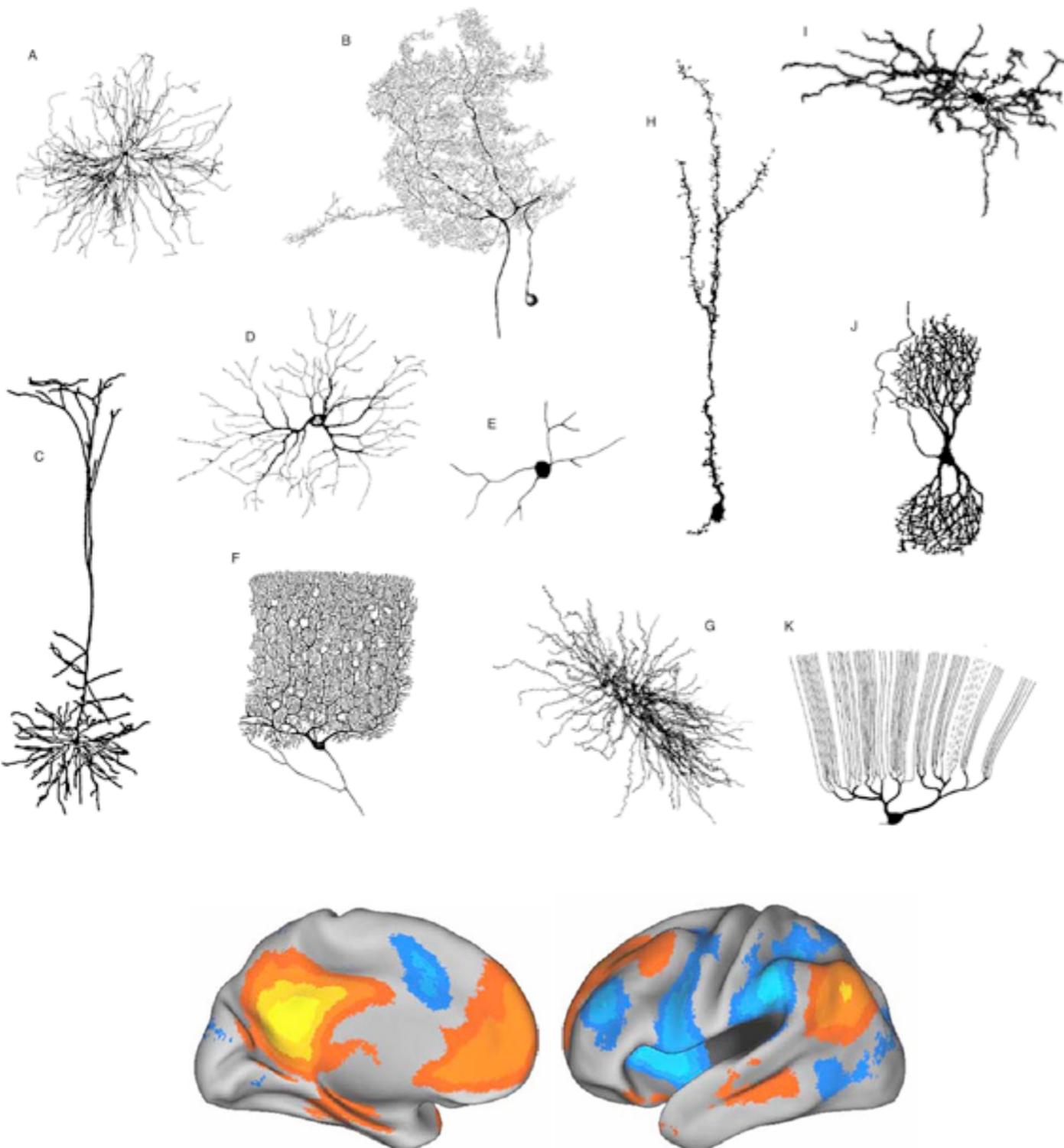
# Further Challenges



# Further Challenges



# Further Challenges



Default mode network and ultra slow coherent oscillations

# In collaboration with

Nikola Venkov  
(Notts)



Gabriel Lord  
(Heriot-Watt)



Yulia Timofeeva  
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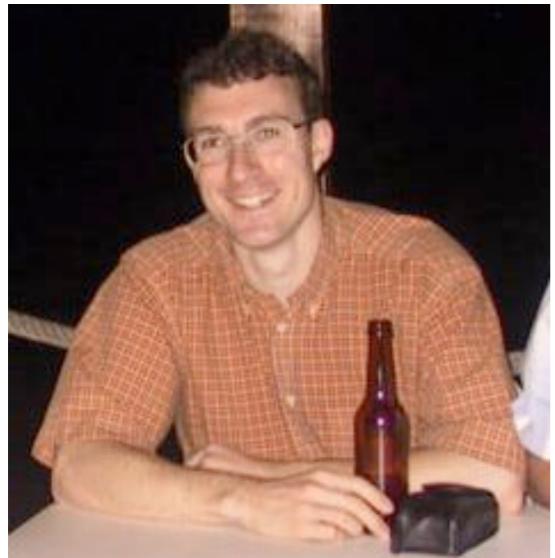


David Liley  
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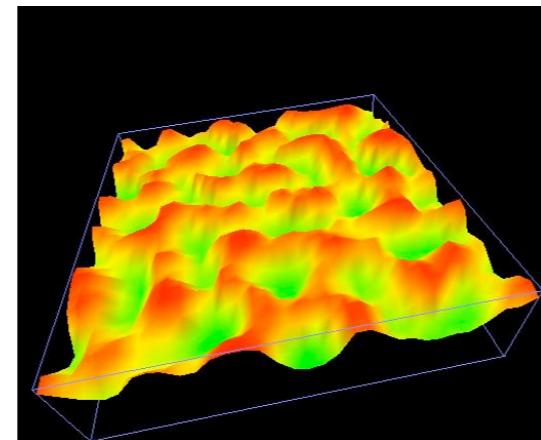


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