# Emergent Macroscopic Behavior in Large Systems of Many Coupled Oscillators

Edward Ott

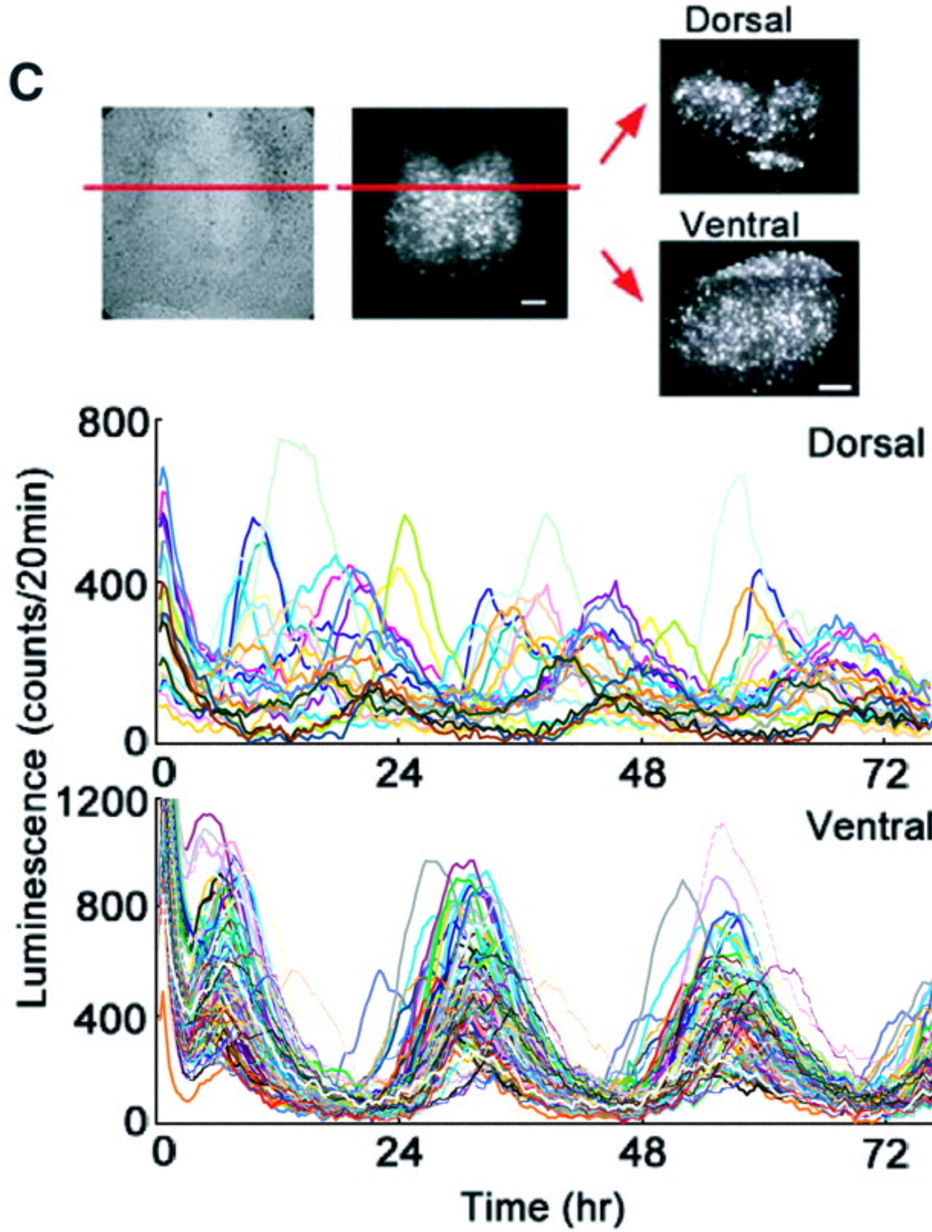
University of Maryland

For the large class of problems considered in this talk, a surprisingly effective method of analysis will be presented.

It is hoped that the results available from this work give valuable insight into the broad topic of emergent dynamical behavior in large complex systems.

# Some Examples of the Types of Problems considered

- **Pacemaker cells in the heart.**
- **Pedestrians on a foot bridge.**
- **Synchronous flashing of fireflies.**
- **Josephson junction circuits.**
- **Laser arrays.**
- **Oscillating chemical reactions.**
- **Brain dynamics.**
- **Etc.**




**Incoherent**

**Coherent**

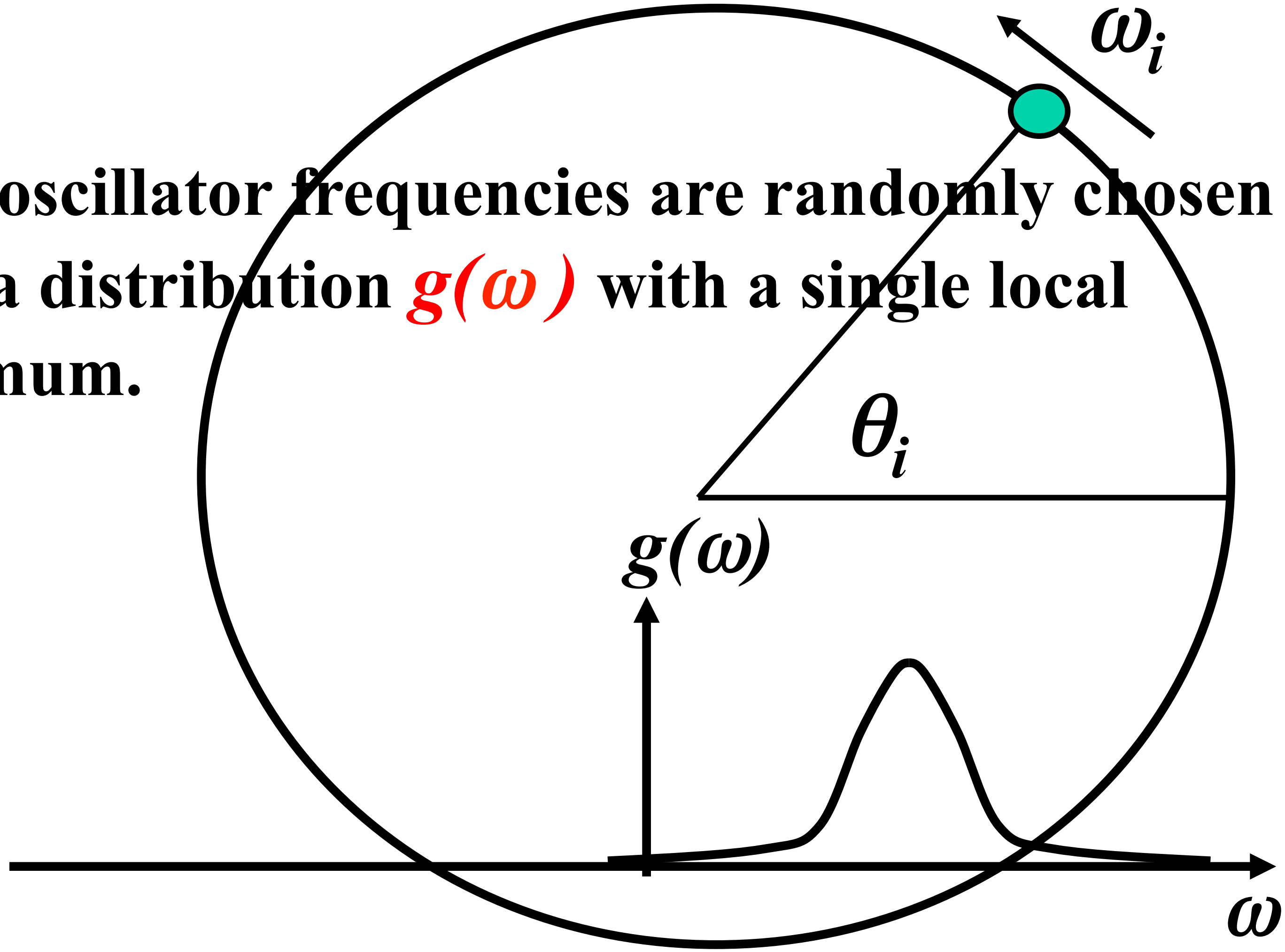
**Cellular clocks in the brain (day-night cycle).**

***Yamaguchi et al., Science <u>302,</u> 1408 (‘03).***

# Framework for the Kuramoto Model

- **$N$ oscillators described only by their phase $\theta$. $N$ is very large.**

- **The oscillator frequencies are randomly chosen from a distribution $g(\omega)$ with a single local maximum.**



# The Kuramoto Model (1975)

$$\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)$$

$i = 1, 2, \ldots., N$ $\quad$ $k$= (global coupling constant)

The $\omega_i$ are randomly chosen from a PDF $g(\omega)$.

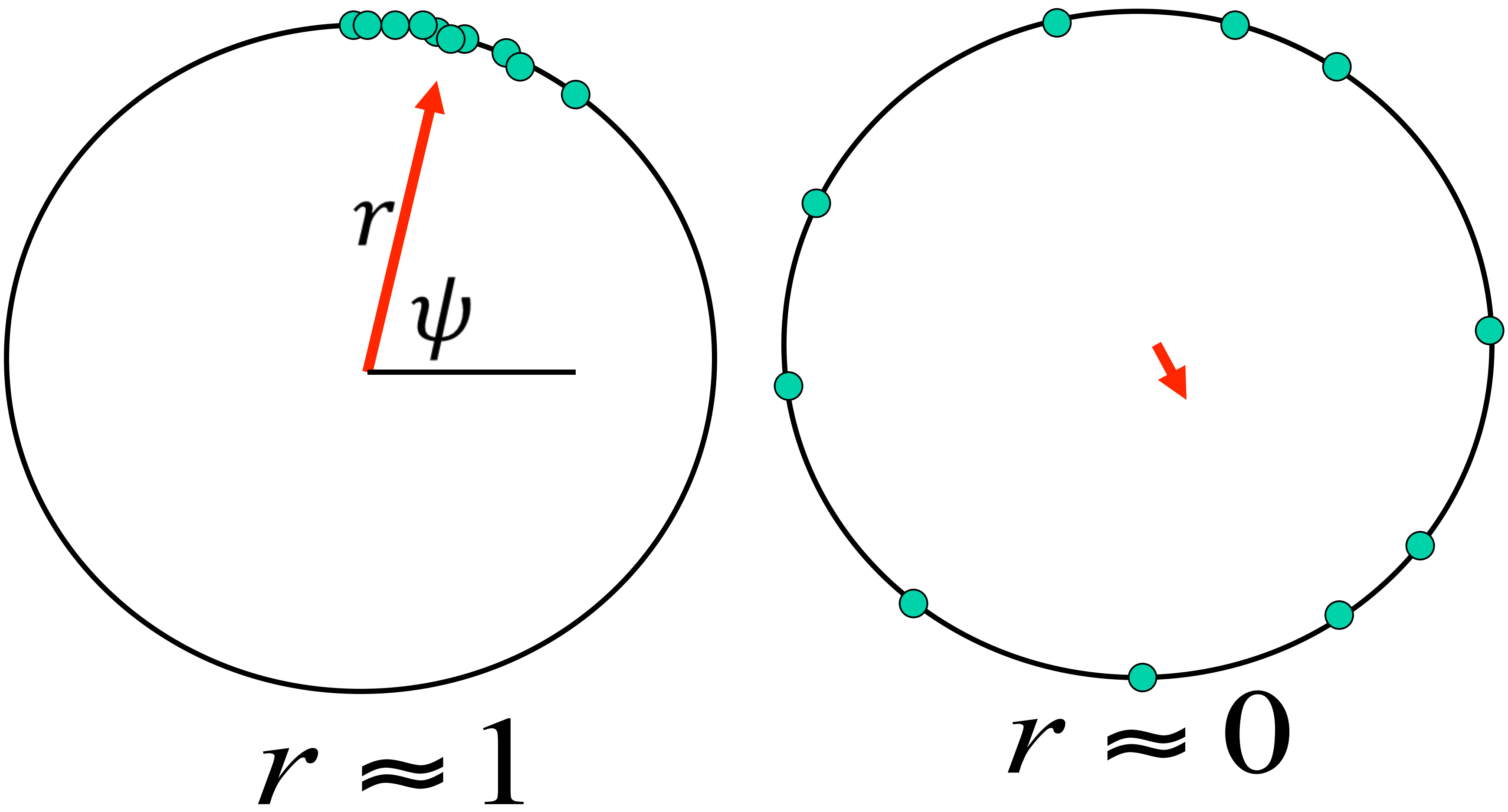
- **Macroscopic coherence of the system is characterized by**

$$R = r\exp(i\psi) = \frac{1}{N}\sum_{i=1}^{N}\exp(i\theta_i) =$$ **(order parameter)**



*Analogy: F=ma for the particles in a fluid coupled by collisions*

# Order parameter measures the coherenc



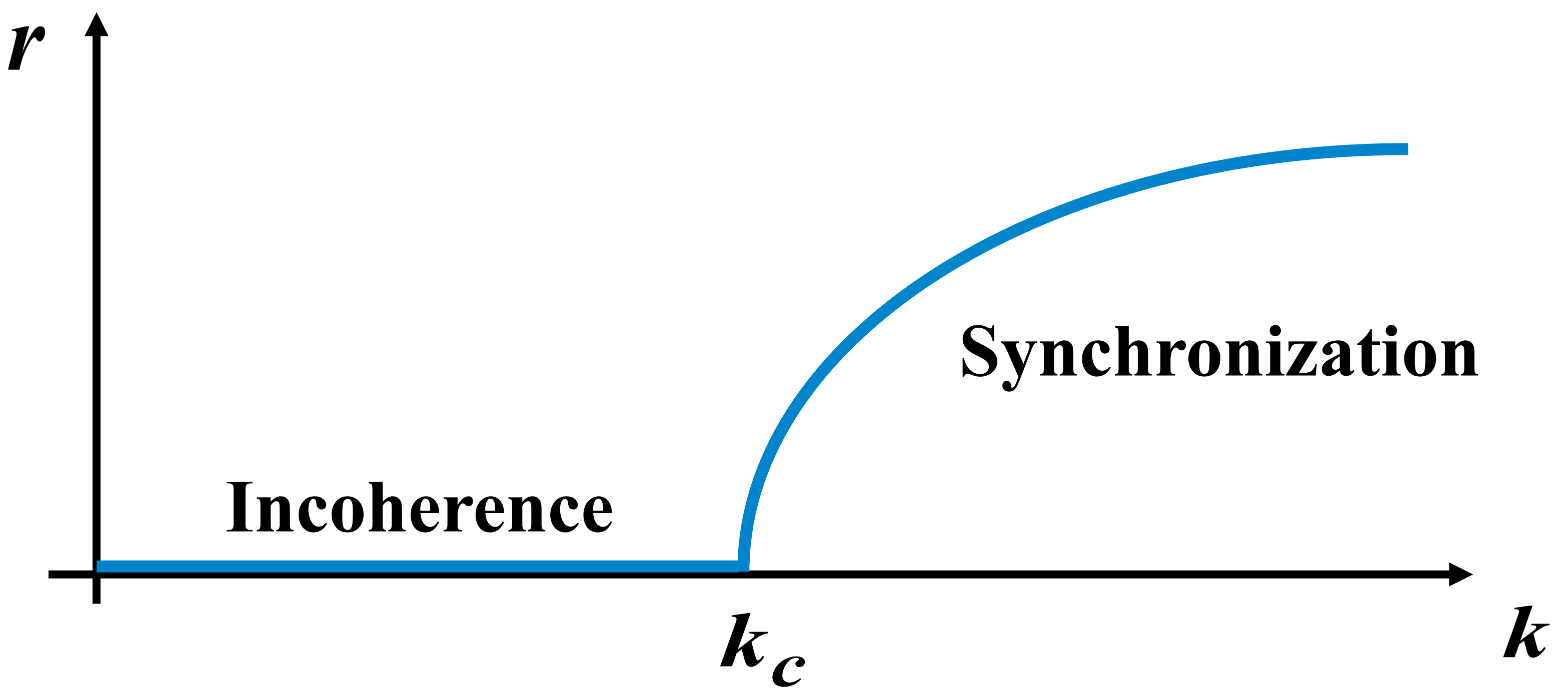



*Analogy: macroscopic description of a fluid*

# Result for the Kuramoto model

**There is a transition of the macroscopic steady state attractor to synchrony that occurs at a critical value of the coupling constant.**

**$g(\omega)$ = PDF of natural oscillator frequencies $\omega$.**




[Explanation of suprachiasmatic nucleus result]

# Generalizations of the Kuramoto model

**Networks of excitable and firing neurons: T.Luke, P.So & E.Barreto, *Neural. Comp.* *25 (2013);* D.Pazo & E.Montbrio, *Phys. Rev. X* *4 (2014)*, C.Laing, *Phys. Rev. E* *90 (2014)*.**

**Modeling circadian rhythm and jet-lag in humans: Z.Lu, T.Antonsen, & M.Girvan, et al., *Chaos (2016)*.**

**Josephson junction circuits: S. A. Marvel & S. H. Strogatz, *Chaos 19 , 013132 (2009)*.**

**Birdsong model compared with experiments: L.M.Alonso, J. A. Alliende, & G. B. Mindlin, *Euro. Phys. J. 60, 361 (2010)*.**

**Effect of network topology: P. S. Skardal, J .G. Restrepo & E. Ott, *Phys. Rev. E 91 060902 (2015)*.**

**Oscillators distributed in space with local coupling: C. Laing, *Chaos 19, 013113 (2009);* and W. S. Lee, J. G. Restrepo, E.Ott, & T. M. Antonsen, *Chaos 21 023122 (2011)*.**

**Main message of this talk: There is an analysis technique for obtaining the macroscopic dynamics of all these problems (see above papers), as well as many others of this type. We now illustrate this using the Kuramoto model as an example.**

# The ‘Order Parameter’ Description

The Kuramoto model as an example:

$$\boxed{\frac{d\theta_i}{dt} = \omega_i + \frac{k}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i)}$$

$$\frac{1}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i) = \frac{1}{N}\,\mathrm{Im}\left\{\sum_{j=1}^{N} e^{i(\theta_j - \theta_i)}\right\}$$

$$= \mathrm{Im}\left\{e^{-i\theta_i}\left(\underbrace{\frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}}_{r\,e^{i\psi}}\right)\right\} = \mathrm{Im}\left[r\,e^{i(\psi-\theta)}\right] = r\sin(\psi - \theta_i)$$

*“The order parameter”*

$$\boxed{\begin{aligned} &d\theta_i / dt = \omega_i + k\,r\sin(\psi - \theta_i) \\ &R = r\,e^{i\psi} \equiv \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j} \end{aligned}}$$

# $N \to \infty$

## Introduce the distribution function *f(θ,ω,t)*

$$f(\theta,\omega,t)\,d\omega d\theta =$$ **[the fraction of oscillators with phases in the range *(θ, θ + dθ)* and frequencies in the range *(ω, ω + dω)* ]**

$$\int_0^{2\pi} f d\theta \equiv g(\omega)$$

## Conservation of number of oscillators:

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left[\frac{d\theta}{dt} f\right] + \frac{\partial}{\partial \omega}\left[\cancelto{0}{\frac{d\omega}{dt}} f\right] = 0$$

$$\boxed{\begin{gathered}\frac{\partial f}{\partial t} + \frac{\partial}{\partial \theta}\left\{\left[\omega + k\, r \sin(\psi - \theta)\right] f\right\} = 0 \\ R = re^{i\psi} = \int\int_0^{2\pi} f\, e^{i\theta}\, d\theta d\omega\end{gathered}}$$

*Analogy: Boltzmann's equation*

# Can We Solve for the Evolution of $f$ ?

Ansatz*:

$$f(\omega,\theta,t)=\frac{g(\omega)}{2\pi}\left\{1+\left[\frac{\alpha\exp(i\theta)}{1-\alpha\exp(i\theta)}+\frac{\alpha^*\exp(-i\theta)}{1-\alpha^*\exp(-i\theta)}\right]\right\}$$

$$\alpha=\alpha\,(\omega,t),\ \left|\alpha\,(\omega,t)\right|\le 1.$$

**This form specifies *M*.**

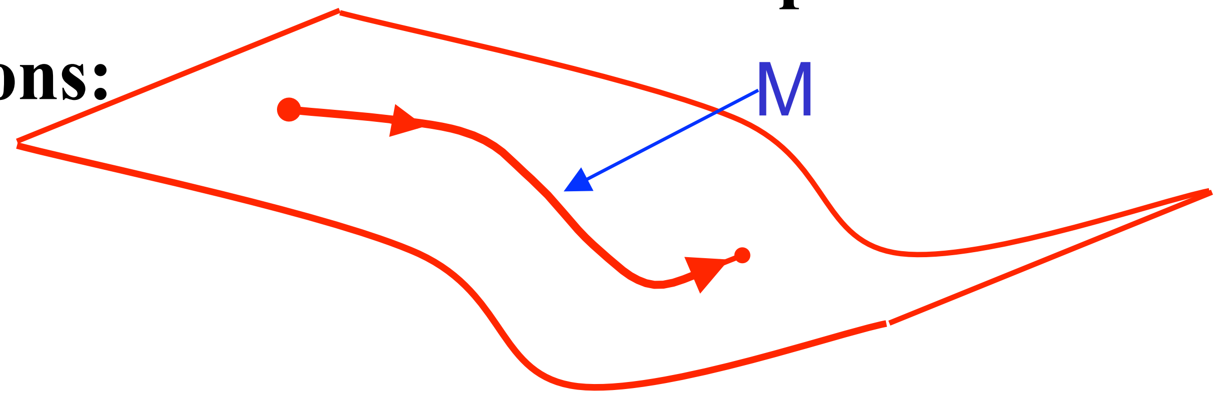
*Analogy: the local Maxwellian for a gas*

$$\Rightarrow\quad \boxed{\begin{aligned}&\frac{\partial\alpha}{\partial t}+\frac{k}{2}\left[R\alpha^2-R^*\right]+i\omega\alpha=0\\ &R^*=\int_{-\infty}^{+\infty}\alpha\, g\, d\omega\end{aligned}}$$

* ***Ott & Antonsen, Chaos 18, 037113 ('08); and Chaos 19, 023119 ('09). Also Ott, Hunt & Antonsen, Chaos 21 025112 ('11).***

# Comments

- *f* lies on an *invariant* manifold M in the space of all possible distributions:




- For *f* on **M** and appropriate *g(ω)* one can obtain a low dimensional <u>macroscopic</u> description of the evolution.
- Is it useful? Yes, very.

•THEOREM (Ott & Antonsen; *Chaos* <u>*19*</u> *‘09*): For a large class of *g(ω) all solutions are attracted to* **M**.

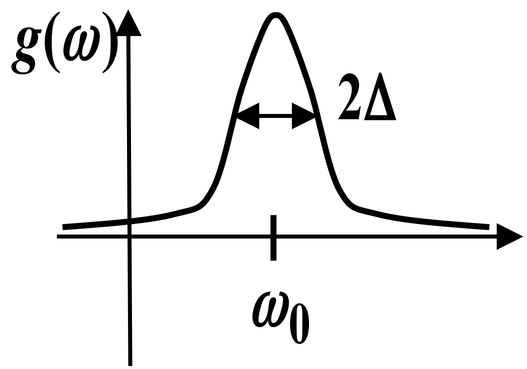
•Relaxation of *f* to ***M*** is due to the spread in *ω*. *(Watanabe & Strogatz PRL)*

•Thus our result can be used to discover and analyze <u>all</u> the long term behaviors of these systems (including all of their bifurcations and attractors) !

# Ex.: Exact Solution of Kuramoto for Lorentzian *g(ω)*

$$g(\omega) = \frac{1}{\pi}\frac{\Delta}{(\omega-\omega_0)^2+\Delta^2} = \frac{1}{2\pi i}\left\{\frac{1}{\omega-\omega_0-i\Delta}-\frac{1}{\omega-\omega_0+i\Delta}\right\}$$

**For this *g(ω)* (and others) the integral for *R(t)* can be done by contour integration in the complex *ω*-plane:**




$$R^*(t) = \int_{-\infty}^{+\infty} \alpha(\omega,t)g(\omega)d\omega = \alpha(\omega_0 - i\Delta, t)$$

**The eq. for $\alpha(\omega,t)$ then yields the macroscopic system dynamics as given by the order parameter *R(t)*:**

$$\Rightarrow \boxed{\frac{dR}{dt}+\frac{k}{2}(|R|^2-1)R+(-i\omega_0+\Delta)R=0}$$

*Analogy: Navier-Stokes equations for macroscopic fluid state*

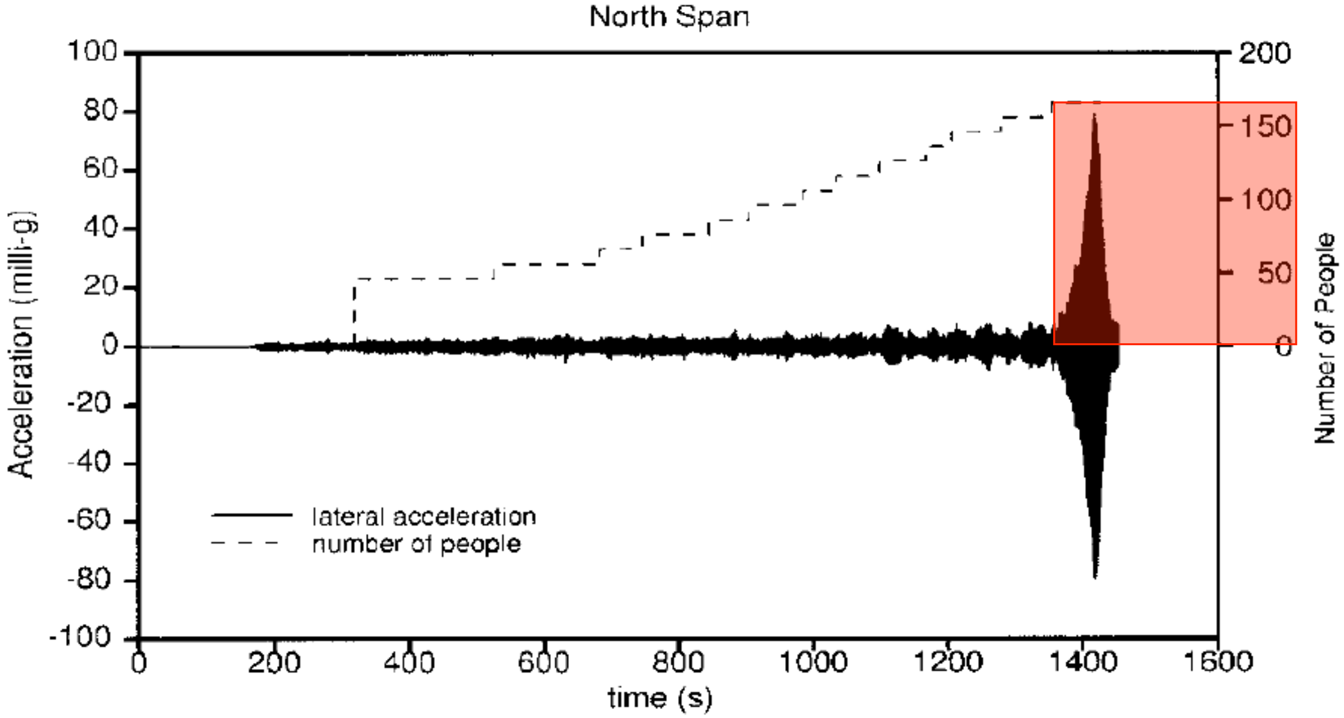
# Another Example: Crowd Synchronization on the Millennium Bridge

**Bridge opened in June 2000, London.**

# The Phenomenon:

**London,**
**Millennium bridge:**
**Opening day**
**June 10, 2000**

# Studies by Arup:



# MODEL

**Model expansion for bridge + phase oscillators for walkers**

$$\frac{d^2y}{dt^2}+\nu\frac{dy}{dt}+\Omega^2 y=\frac{1}{M}\sum_i f_i(t) \quad \textit{(Bridge)}$$

$$f_i(t)=f_{io}\cos\theta_i(t) \quad \textit{(Walker force on bridge)}$$

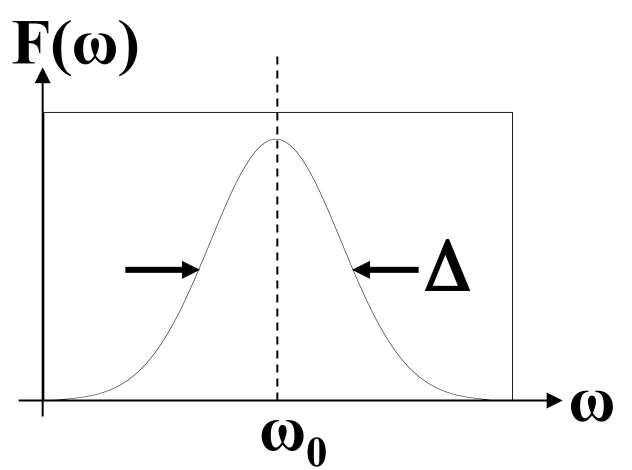
$$\frac{d\theta_i(t)}{dt}=\omega_i-b\frac{d^2y}{dt^2}\cos\theta_i(t) \quad \textit{(Walker phase)}$$

**Ref.:Eckhardt, Ott, Strogatz, Abrams and McRobie,**
***Phys.Rev.E 75, 021110 ('07)***

# REDUCED MODEL

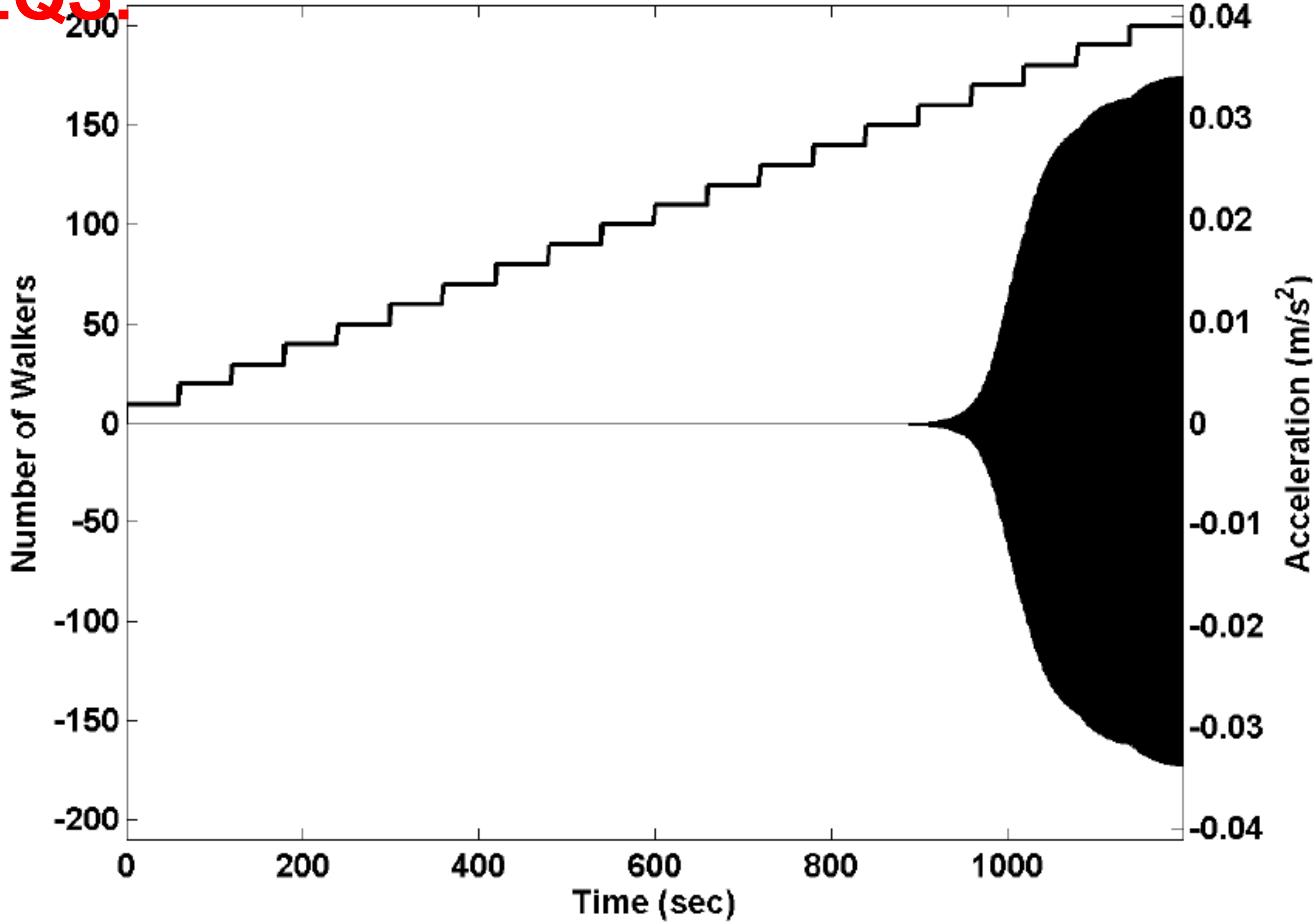
$$M\frac{d^2y}{dt^2}+M\nu\frac{dy}{dt}+M\Omega^2 y=N\overline{F}\,\mathrm{Re}[R(t)]$$

$$\frac{dR(t)}{dt}-i\{(\overline{\omega}+i\Delta)R^*-\beta\frac{d^2y}{dt^2}[R^2(t)+1]\}=0$$




**Ref.: M.M.Abdulrehem and E.Ott, *Chaos 19, 013129('09).***

# NUMERICAL SOLUTIONS OF REDUCED EQS.





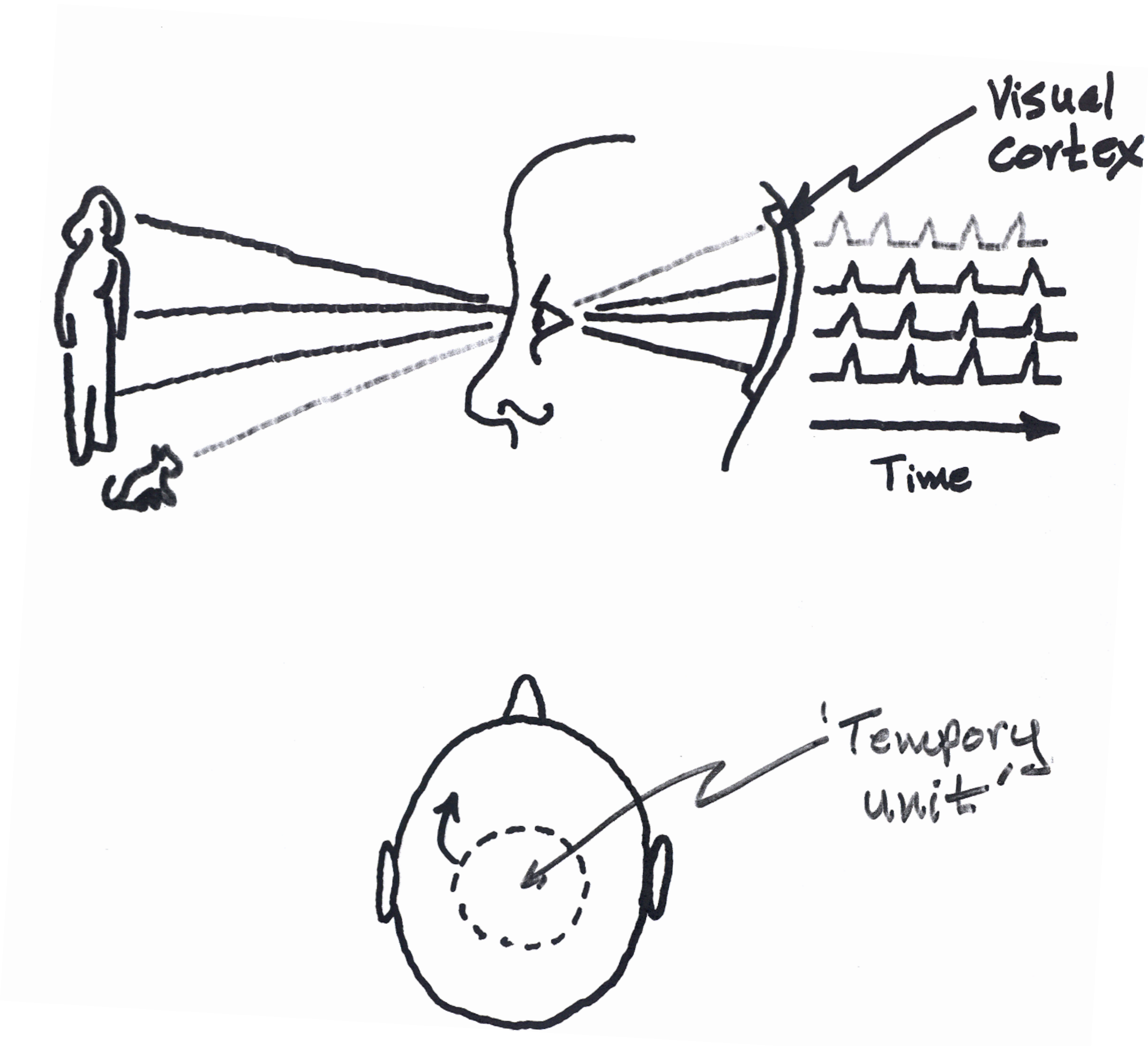

**Arup's fix: They increased the damping *v*.**

# Conclusion

**Explicit mathematical descriptions of the emergent macroscopic behavior of a large class of complex systems of heterogeneous phase oscillators can be obtained and utilized to discover and analyze all the long time behavior (e.g., the attractors and bifurcations) of these systems.**

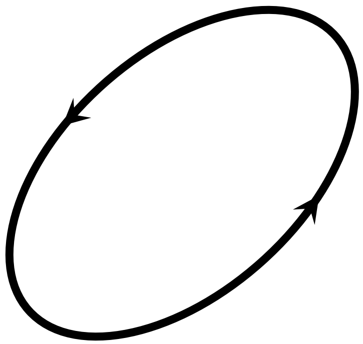
**Refs.: The ansatz for the special form of *f* was given in Ott & Antonsen, *Chaos 18, 037113 (2008)*. For the demonstration of attraction to *M* see: *Chaos 19, 023117 (2009)*, and Ott, Hunt & Antonsen, *Chaos 21, 025112 (2011)*.**

# x Synchrony in the brain



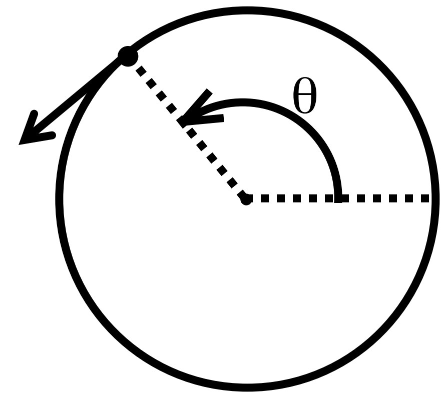
x

# Coupled phase oscillators



Limit cycle in phase space

Change of variables




$$d\theta / dt = \omega$$

**Many such 'phase oscillators':**

$$\frac{d\theta_i}{dt} = \omega_i \quad ; i=1,2,\ldots,N \gg 1$$

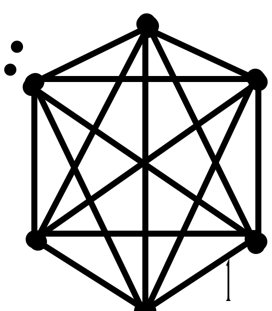
**Couple them:**

$$\frac{d\theta_i}{dt} = \omega_i + \sum_{j=1}^{N} k_{ij}(\theta_j - \theta_i)$$

$$k_{ii}(\phi) = 0, \quad k_{ij}(\phi) = k_{ij}(\phi \pm 2\pi)$$

**Kuramoto:** $k_{ij}(\phi) = \frac{k}{N}\sin(\phi)$

$N = 6$ :



Global coupling

X

[redacted]

# A Key Point of This Lecture*

**Considering the Kuramoto model and its generalizations, for i.c.'s $f(\omega,\theta,0)$ of a specific special form (specified later),**

- **$f(\omega,\theta,t)$ continues to have that specific special form,**

* ***Ott & Antonsen, Chaos 18, 037113 ('08); and Chaos 19, 023119 ('09). Also Ott, Hunt & Antonsen, Chaos 21 025112 ('11).***

# The Kuramoto Model as an Example

$$\partial f / \partial t + \partial / \partial \theta \left\{ \left[ \omega + (k / 2i)\left( R e^{i\theta} - R^{*} e^{-i\theta} \right) \right] f \right\} = 0$$

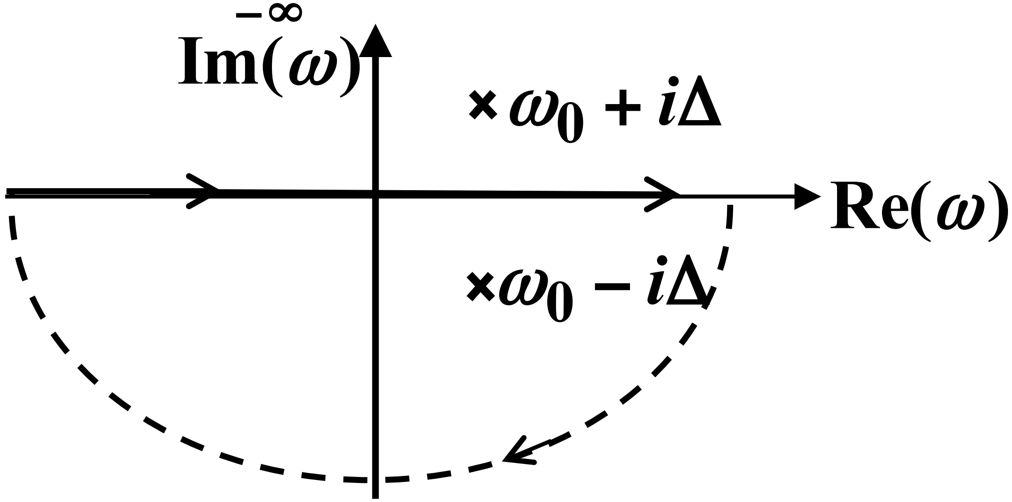
$$R = r e^{i\psi} = \int_{0}^{2\pi} d\theta \int_{-\infty}^{+\infty} d\omega \; f \; e^{i\theta}, \quad g(\omega) = \int_{0}^{2\pi} f(\omega, \theta, t) d\theta$$

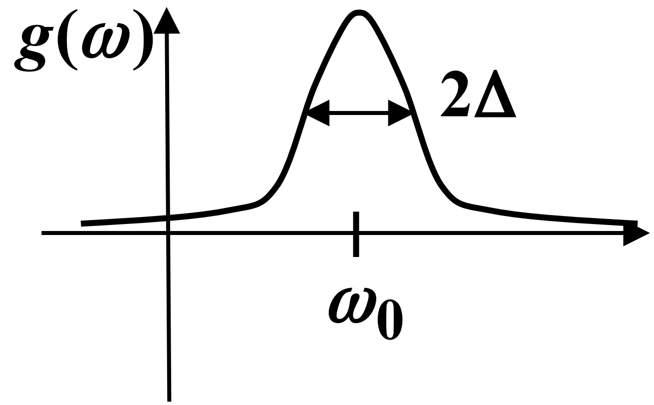
**The manifold M is specified by a constraint on the form of** $f(\omega,\theta, t)$**.**

# Ex.: Exact Solution of Kuramoto for Lorentzian $g(\omega)$

$$g(\omega) = \frac{1}{\pi}\frac{\Delta}{(\omega-\omega_0)^2+\Delta^2} = \frac{1}{2\pi i}\left\{\frac{1}{\omega-\omega_0-i\Delta} - \frac{1}{\omega-\omega_0+i\Delta}\right\}$$

$$R^*(t) = \int_{-\infty}^{+\infty}\alpha(\omega,t)\,g(\omega)\,d\omega = \alpha(\omega_0 - i\Delta, t)$$



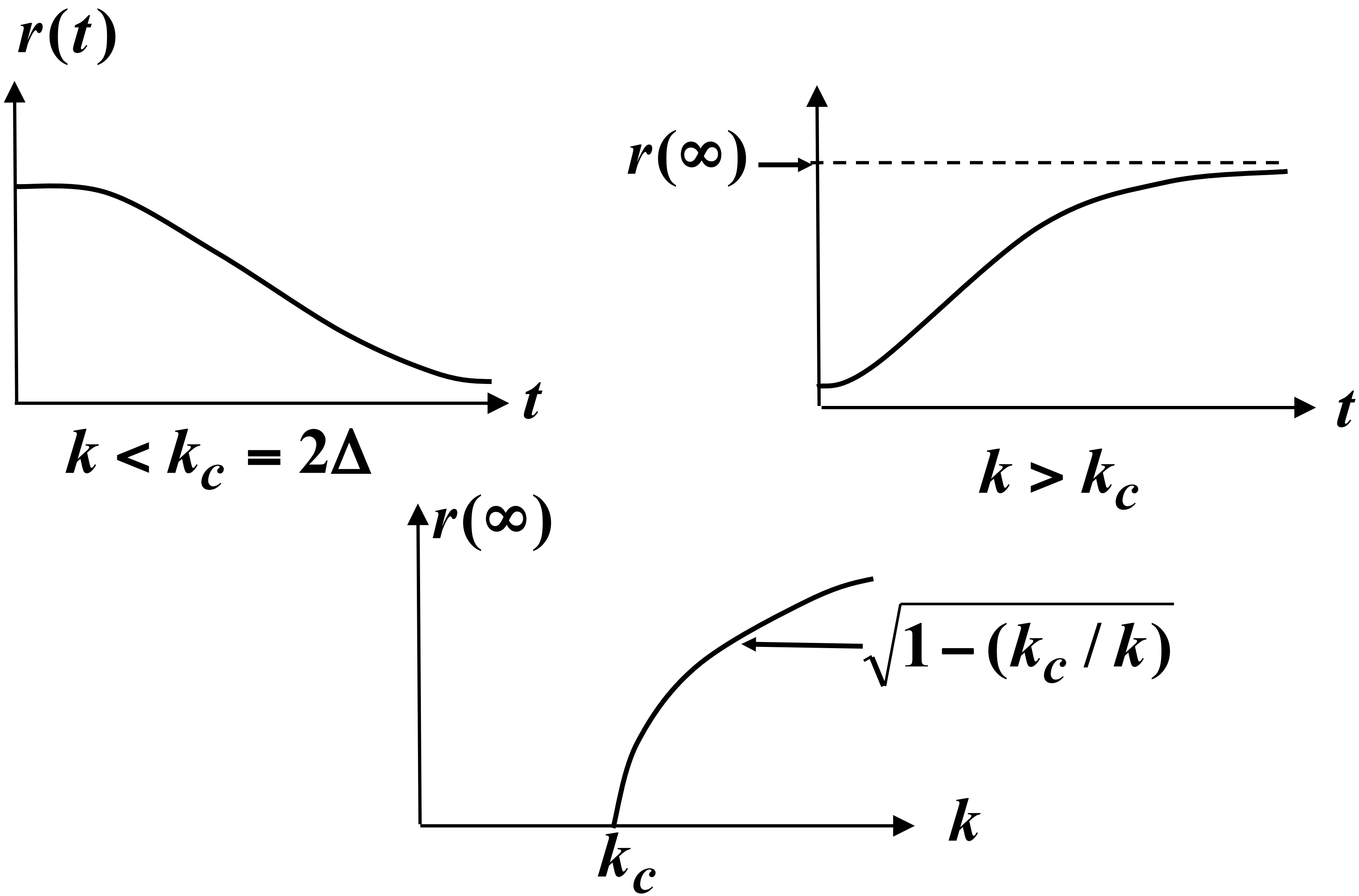




Set $\omega = \omega_0 - i\Delta$ in $\partial\alpha/\partial t + (k/2)(R\alpha^2 - R^*) + i\omega\alpha = 0$

$$\Longrightarrow \boxed{\frac{dR}{dt} + \frac{k}{2}(|R|^2 - 1)R + (-i\omega_0 + \Delta)R = 0}$$

*Analogy: Navier-Stokes equations for macroscopic fluid state*

# Solution for $|R(t)|=r(t)$




Thus the steady solution is *nonlinearly* stable and *globally* attracting.

# x An Analogy (Continued)

**Kinetic Theory** **Relaxation to local Maxwellian**

$$f(\underline{x},\underline{v},t)=\frac{\rho(\underline{x},t)}{(2\pi KT(\underline{x},t)/m)^{3/2}}\exp\left\{-\frac{[\underline{v}-\underline{u}(\underline{x},t)]^2}{2KT(\underline{x},t)/m}\right\}$$

**Phase Oscillators** **Relaxation to M ('Poisson kernel')**

$$f(\theta,\omega,t)=\frac{g(\omega)}{2\pi}\left\{1+\left[\left(\sum_{n=1}^{\infty}\alpha^n(\omega,t)e^{in\theta}\right)+c.c\right]\right\}$$

$$\alpha(\omega,t)=a(\omega,t)\exp[-i\phi(\omega,t)]$$

$$f=\frac{g(\omega)}{2\pi}\bullet\frac{1-a^2}{(1-a)^2+4a\sin^2\left[\frac{1}{2}(\theta-\phi)\right]}$$

**$f$ is a delta function for $a\rightarrow 1$. $f$ is flat for $a=0$. For $0<a<1$, $f$ is peaked at $\theta=\varphi$, and the peak's width decreases with $a$.**

**Mixing**

**Kinetic theory: Mixing is due to chaos caused by collisions.**

**Phase oscillators: Mixing is due to the spread in osc. freqs.**

x

# AN ANALOGY

## PHASE OSCILLATORS

**N eqs. for N>>1 oscillator phases.**

↓

**Relaxation to M (exact!).**

↓

**ODE description for order parameter.**

## KINETIC THEORY FOR A GAS

**Hamilton's eqs. for N>>1 interacting fluid particles.**

↓

**Relaxation to a local Maxwellian (asymptotic expansion)**

↓

**Fluid eqs. for moments (density, velocity, temp., …).**

# x References

- Main Refs.: *E. Ott and T.M. Antonsen,*

**"Low Dimensional Behavior of Large Systems of Globally Coupled Oscillators,"** ***Chaos 18 ,037113 ('08).***

**"Long Time Behavior of Phase Oscillator Systems",** ***Chaos 19, 023117 ('09).***

Our other related work that is referred to in this talk can be found at:

***http://www-chaos.umd.edu/umdsyncnets.htm***

# Generalizations of the Kuramoto Model

## External Drive:

$$d\theta_i / dt = \omega_i + (k/N)\sum_{j=1}^{N} \sin(\theta_j - \theta_i) + \underbrace{M_0 \sin(\Omega_0 t - \theta_i)}_{\text{drive}}$$

**E.g., circadian rhythm.**

**Ref.: Sakaguchi, *ProgTheorPhys('88);* Zhixin Lu et al., *Chaos* *26* *('16);* Childs & Strogatz, *Chaos* *18* *('08)*.**

## Groups of Oscillators:

**$\sigma$ = group ($\sigma = 1,2,.., s$); $N_\sigma$ = # of oscillators in group $\sigma$.**

$$d\theta_i^\sigma / dt = \omega_i^\sigma + \sum_{\sigma'=1}^{s} (k_{\sigma\sigma'} / N_{\sigma'}) \sum_{j=1}^{N_{\sigma'}} \sin(\theta_j^{\sigma'} - \theta_i^\sigma + \beta^{\sigma\sigma'})$$

**E.g., Barreto et al., *PhysRevE ('08);*Martens et al., *PhysRevE('09)*; Abrams,et al., *PhysRevLett('08);* Laing, *Chaos19('09);* and Pikovsky & Rosenblum, *PhysRevLett* *101* *('08)*.**

# X Generalizations (continued)

## Millennium Bridge Problem:

$$d^2y/dt^2 + \nu\, dy/dt + \Omega^2 y = \frac{1}{M}\sum_i f_i \quad \textit{(Bridge mode)}$$

$$f_i(t) = f_{i0}\cos(\theta_i(t)) \quad \textit{(Walker force on bridge)}$$

$$d\theta_i/dt = \omega_i - bd^2y/dt^2\cos(\theta_i + \beta) \quad \textit{(Walker phase)}$$

**Ref.:** Eckhardt, Ott, Strogatz, Abrams, & McRobie, *PhysRevE* 75, *021110('07);* Abdulrehem and Ott, *Chaos* 19, *013129 ('09).*

x

# Other Frequency Distributions

## $g(\omega)$

**Our method can treat certain other $g(\omega)$'s, e.g.,**

$$g(\omega) \sim [(\omega-\omega_0)^4 + \Delta^4]^{-1}, \text{ or}$$

$$g(\omega)=[polynomial]/[polynomial].$$

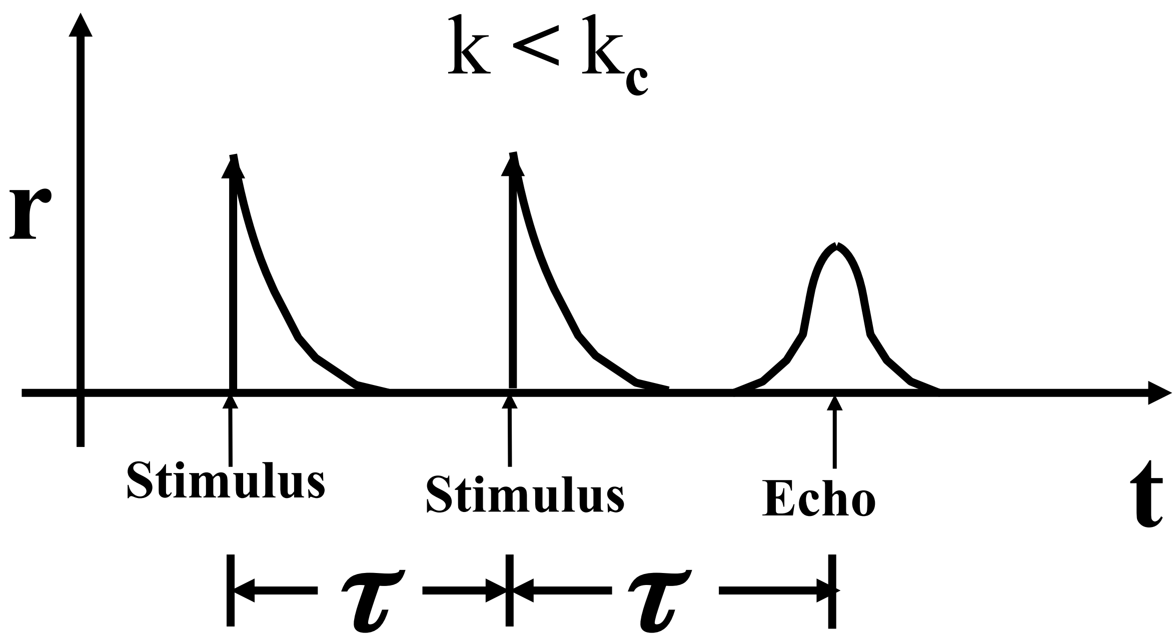
**Then there are $s$ coupled ODE's for $s$ order parameters where $s$ is the number of poles of $g(\omega)$ in $Im(\omega)<0$.**

# X COMMENT ON ATTRACTION TO M

**Due to the analyticity requirement on *g(ω)*, our result of attraction to M does *not* apply if *g(ω)* is a delta function, and, in that case, long time behavior not on M can occur [e.g., Pikovsky & Rosenblum, *PhysRevLett (‘08);* Marvel & Strogatz, *Chaos* <u>*19*</u> *(‘09)*]. Thus the long time behavior is, in a sense, <u>simpler</u> when the oscillator frequencies are heterogeneous.**

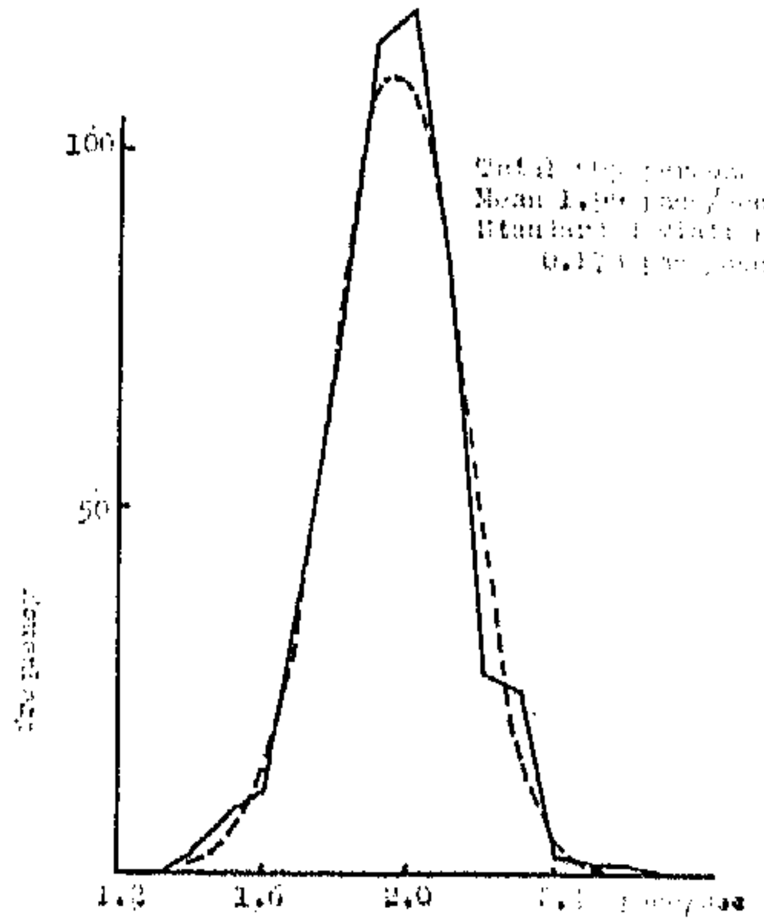
# X TRANSIENT BEHAVIOR: ECHOES

- **The transient behavior that occurs as the orbit relaxes to M can be nontrivial. An example of this is the 'echo' phenomenon studied in Ott, Platig, Antonsen & Girvan, *Chaos* <u>*18,*</u> *037115 ('08)*. [Similar to Landau echoes in plasmas; e.g., T.M.O'Neil & R.W.Gould, *Phys. Fluids (1968)*.]**
- **For the classical Kuramoto model with *k* below its critical value and external stimuli (figure below).**
- **Chemical experiment in progress by Showalter et al.**



# The Frequency of Walking:

**People walk at a rate of about 2 steps per second (one step with each foot).**




**Matsumoto et al., *Trans JSCE* <u>*5,*</u> *50 (1972)***

**If $\alpha(\omega,0) \to 0$ as $Im(\omega) \to -\infty$ ,**

**then so does $\alpha(\omega,t)$**

Since $|\alpha| < 1$, we also have (recall that $R^* = \int_{-\infty}^{+\infty} \alpha\, g\, d\omega$)

$$|R(t)| < 1 \text{ and } \left|\left(1 - |\alpha|^2\right) \mathrm{Re}\{\alpha R\}\right| < 1 .$$

Thus

$$\frac{\partial |\alpha|^2}{\partial t} < K + 2Im(\omega)\, |\alpha|^2$$

$|\alpha| \to 0$ as $Im(\omega) \to -\infty$ for all time $t$. ✓

## If, for *Im(ω)< 0*, *|α(ω,0)|<1*, then *|α(ω,t)|<1* :

$$\partial \alpha / \partial t + \frac{1}{2} k \left[ R\alpha^2 - R^* \right] + i\omega\alpha = 0$$

**Multiply by $\alpha^*$ and take the real part:**

$$\partial |\alpha|^2 / \partial t + k\left(1 - |\alpha|^2\right) \mathrm{Re}\{\alpha R\} - 2\,\mathrm{Im}(\omega)\,|\alpha|^2 = 0$$

**At $|\alpha(\omega,t)|=1$:** $\partial |\alpha|^2 / \partial t = 2\,\mathrm{Im}(\omega)\,|\alpha|^2 \le 0$

⇨ **$|\alpha|$ starting in $|\alpha(\omega,0)| < 1$ cannot cross into $|\alpha(\omega,t)| > 1$.**

**$|\alpha(\omega,t)| < 1$ and the solution exists for all $t$ ( $Im(\omega) < 0$ ) .** ✓