Sweeping process and congestion model for crowd motion

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Outline

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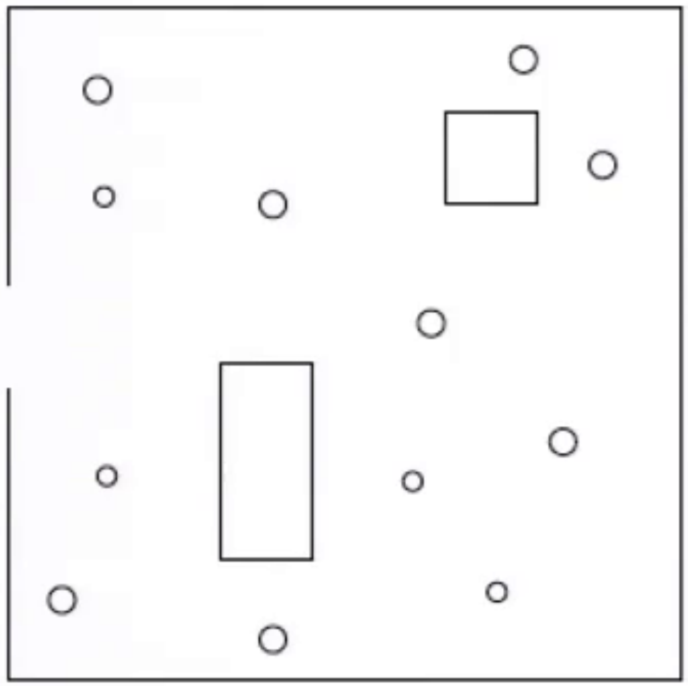
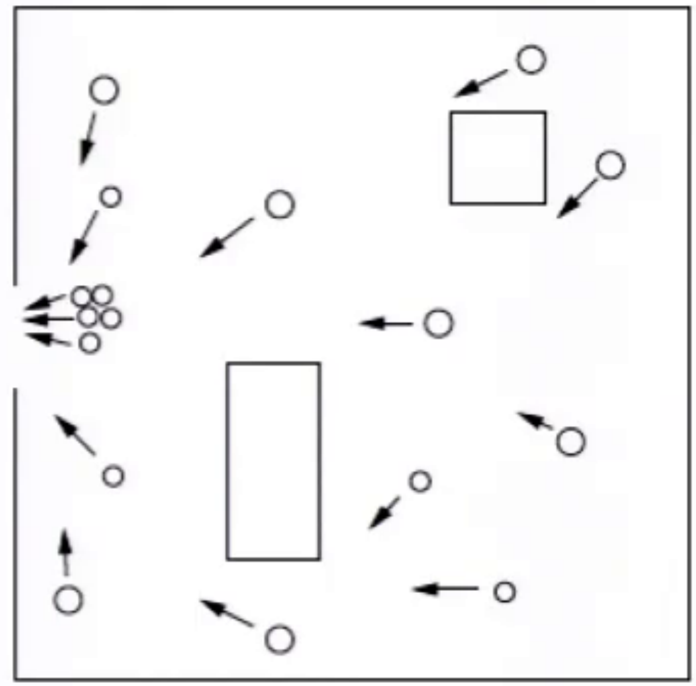
A crowd motion model with several goals

- to deal with emergency evacuation
- to take into account direct contacts between individuals
- to determine the areas where people are crushed

Two principles

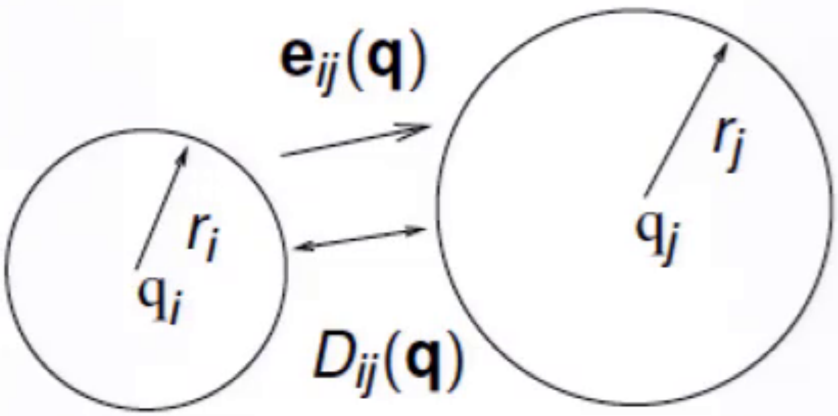




Spontaneous velocity

Actual velocity

Notations



$$\mathbf{q} = (q_1, q_2, .., q_N) \in \mathbb{R}^{2N}$$

$$\mathbf{e}_{ij}(\mathbf{q}) = \frac{q_j - q_i}{|q_j - q_i|}$$

Set of feasible configurations

$$Q_0 = \left\{ \mathbf{q} \in \mathbb{R}^{2N}, \ \forall\, i < j, \quad D_{ij}(\mathbf{q}) = |q_i - q_j| - r_i - r_j \geq 0 \right\}$$

$$\mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q}) = (0 \ldots 0, \ \underset{i}{-\mathbf{e}_{ij}(\mathbf{q})}, 0 \ldots 0, \ \underset{j}{\mathbf{e}_{ij}(\mathbf{q})}, 0 \ldots 0)$$

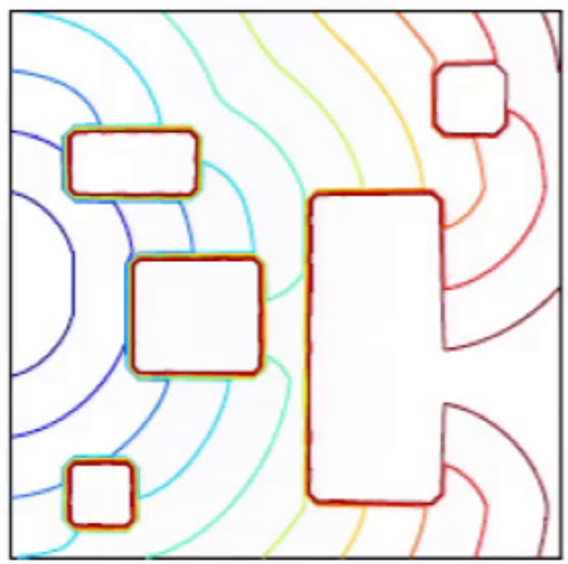
Spontaneous velocity

Notation : $\mathbf{U}(\mathbf{q}) = (\mathrm{U}_1(\mathbf{q}),\ \mathrm{U}_2(\mathbf{q}), \ldots,\ \mathrm{U}_N(\mathbf{q}))$

Example :

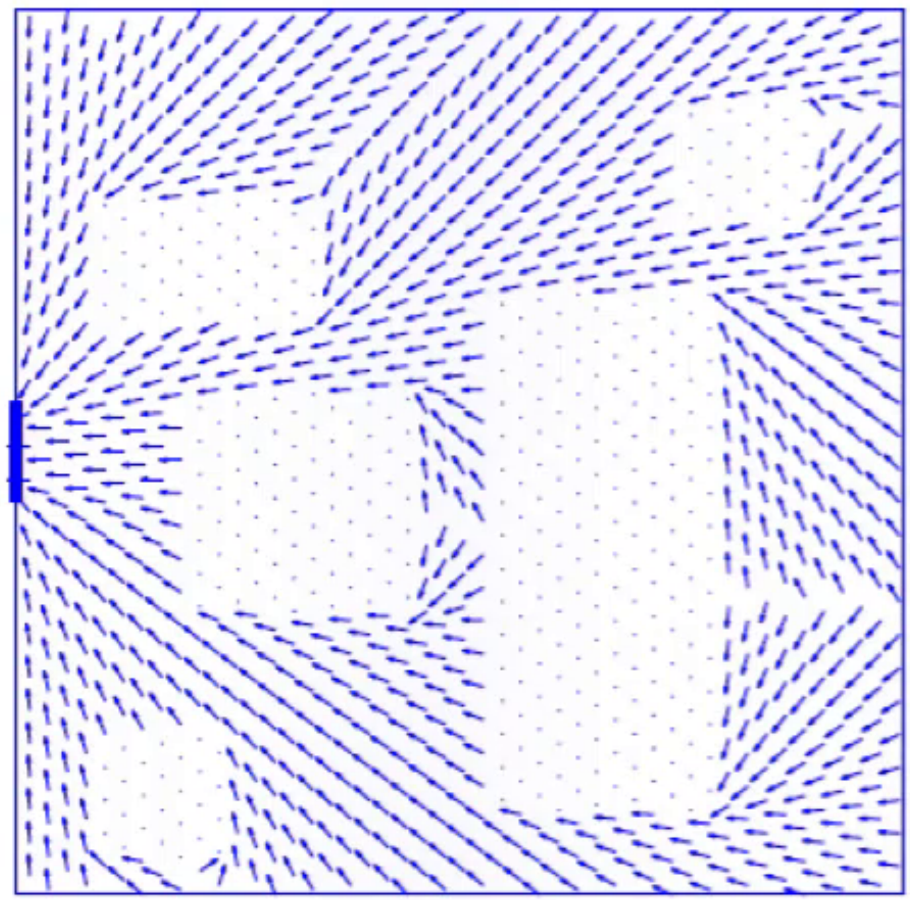
$$\mathrm{U}_i(\mathbf{q}) = -\mathrm{s}_i \nabla \mathcal{D}(\mathrm{q}_i),$$

where $\mathcal{D}(\mathbf{x})$ represents the geodesic distance between $\mathbf{x}$ and the exit.



Contour levels of $\mathcal{D}$

Example of spontaneous velocity



Direction opposite to the gradient of the geodesic distance $\mathcal{D}$.

Actual velocity

To handle the contacts, we define the

cone of admissible velocities

$$\mathcal{C}_{\mathbf{q}} = \left\{ \mathbf{v} \in \mathbb{R}^{2N},\ \forall i < j \quad D_{ij}(\mathbf{q}) = 0 \ \Rightarrow \ \mathbf{G}_{ij}(\mathbf{q}) \cdot \mathbf{v} \geq 0 \right\},$$

where $\mathbf{G}_{ij}(\mathbf{q}) = \nabla D_{ij}(\mathbf{q})$.

If $\mathbf{u}$ is the actual velocity of the N pedestrians, the model can be expressed as follows :

$$\begin{cases} \mathbf{q} = \mathbf{q}_0 + \int \mathbf{u}, \\ \mathbf{u} = \mathrm{P}_{\mathcal{C}_{\mathbf{q}}} \mathbf{U}. \end{cases}$$

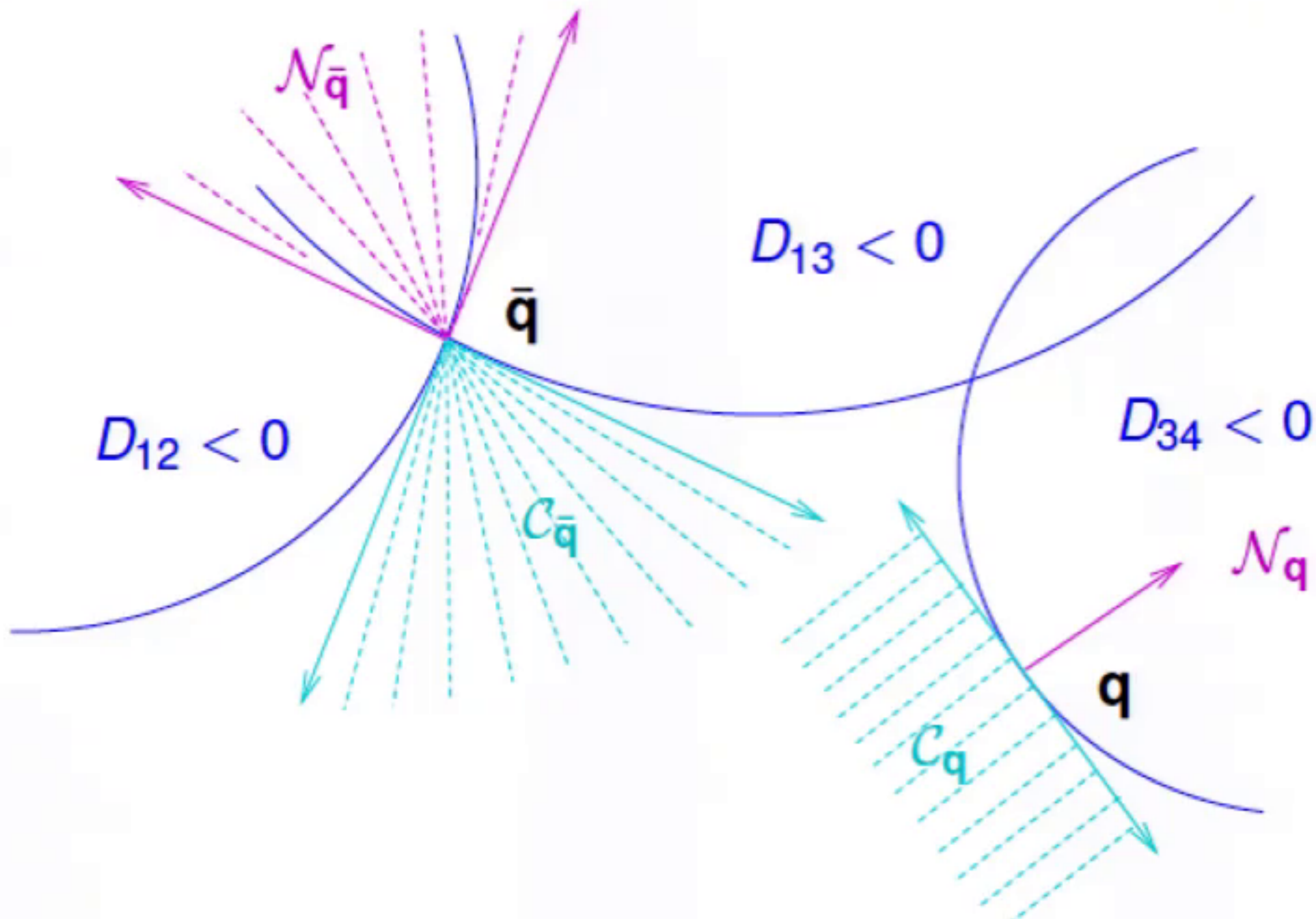
Outline

Cone $\mathcal{N}_{\mathbf{q}}$

Let us define $\mathcal{N}_{\mathbf{q}}$ the polar cone of $\mathcal{C}_{\mathbf{q}}$:

Definition

$$\mathcal{N}_{\mathbf{q}} = \mathcal{C}_{\mathbf{q}}^{\circ} = \{\mathbf{w}\,,\ (\mathbf{w},\mathbf{v}) \leq 0 \quad \forall \mathbf{v} \in \mathcal{C}_{\mathbf{q}}\}\,.$$



Cone $\mathcal{N}_{\mathbf{q}}$

Proposition

$$\mathcal{N}_{\mathbf{q}} = \left\{ -\sum \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}),\ \lambda_{ij} \geq 0,\ D_{ij}(\mathbf{q}) > 0 \Longrightarrow \lambda_{ij} = 0 \right\}.$$

Since $\mathcal{C}_{\mathbf{q}}$ and $\mathcal{N}_{\mathbf{q}}$ are mutually polar cones, the following property holds (J.-J. Moreau 62)

Property

$$\mathrm{P}_{\mathcal{C}_{\mathbf{q}}} + \mathrm{P}_{\mathcal{N}_{\mathbf{q}}} = \mathrm{Id}.$$

Differential inclusion

According to the previous property,

$$\dot{\mathbf{q}} = \mathbf{u} = \mathrm{P}_{\mathcal{C}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})) = \mathbf{U}(\mathbf{q}) - \mathrm{P}_{\mathcal{N}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})),$$

which is equivalent to

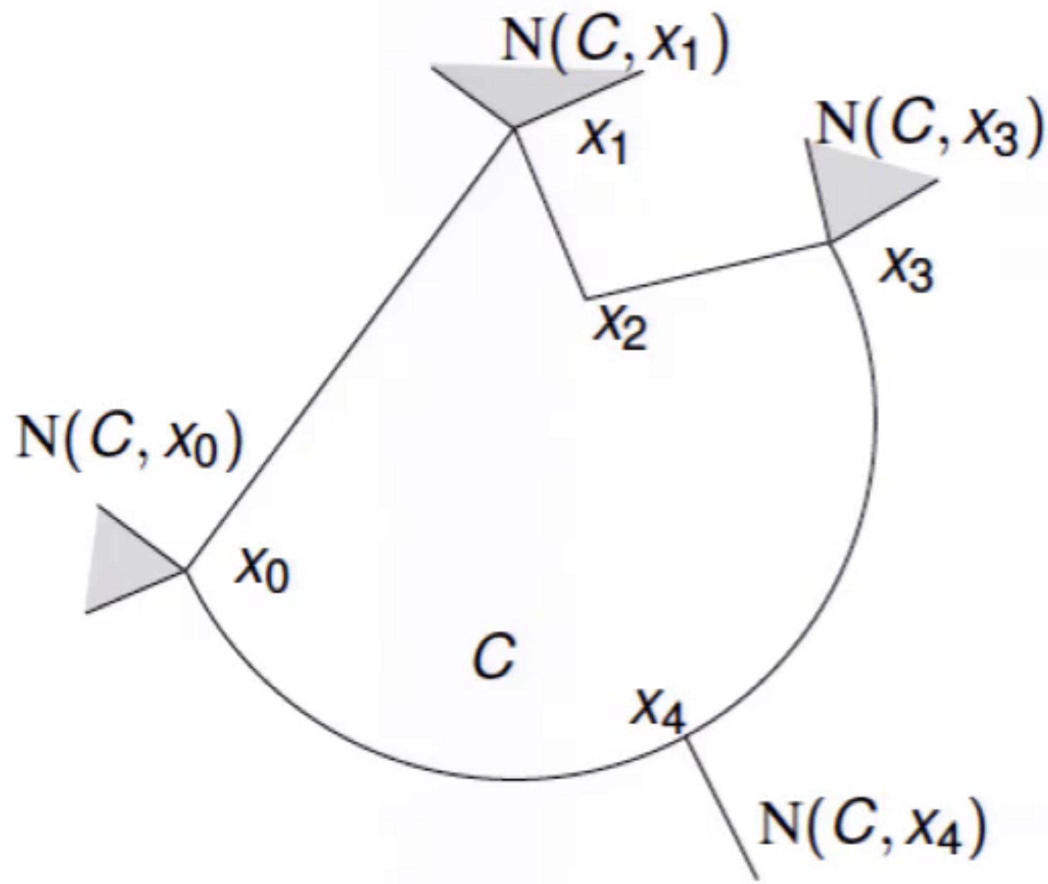
$$\dot{\mathbf{q}} + \mathrm{P}_{\mathcal{N}_{\mathbf{q}}}(\mathbf{U}(\mathbf{q})) = \mathbf{U}(\mathbf{q}).$$

and so the problem can be formulated as a first order differential inclusion .

Model

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathcal{N}_{\mathbf{q}} \ni \mathbf{U}(\mathbf{q}), \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

Proximal normal cone



Proximal normal cone of *C* at *x*

$$\mathrm{N}(C,x) = \{v \in H,\ \exists\alpha > 0,\ x \in \mathrm{P}_C(x+\alpha v)\}$$

(F. Clarke, R. Stern, P. Wolenski 95)

Proximal normal cone

Proposition

For every $\mathbf{q}$,

$$\mathcal{N}_{\mathbf{q}} = \mathrm{N}(Q_0, \mathbf{q}).$$

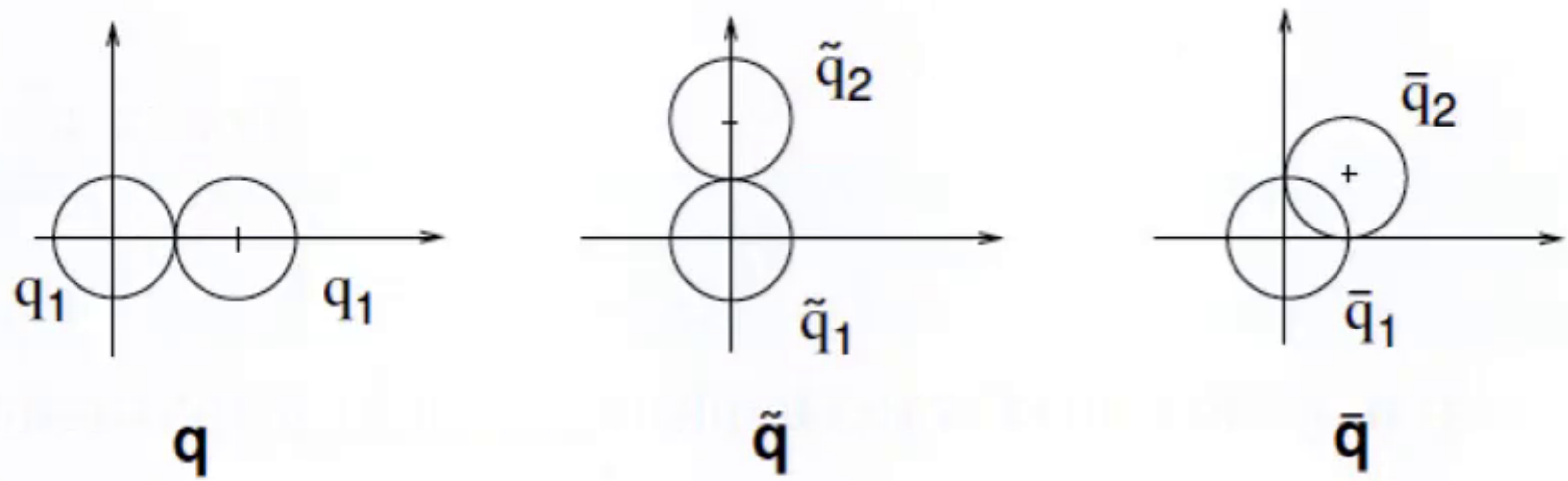
where $\mathrm{N}(Q_0, \mathbf{q})$ is the proximal normal cone of Q_0 at $\mathbf{q}$.

Hence,

Model

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}), \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

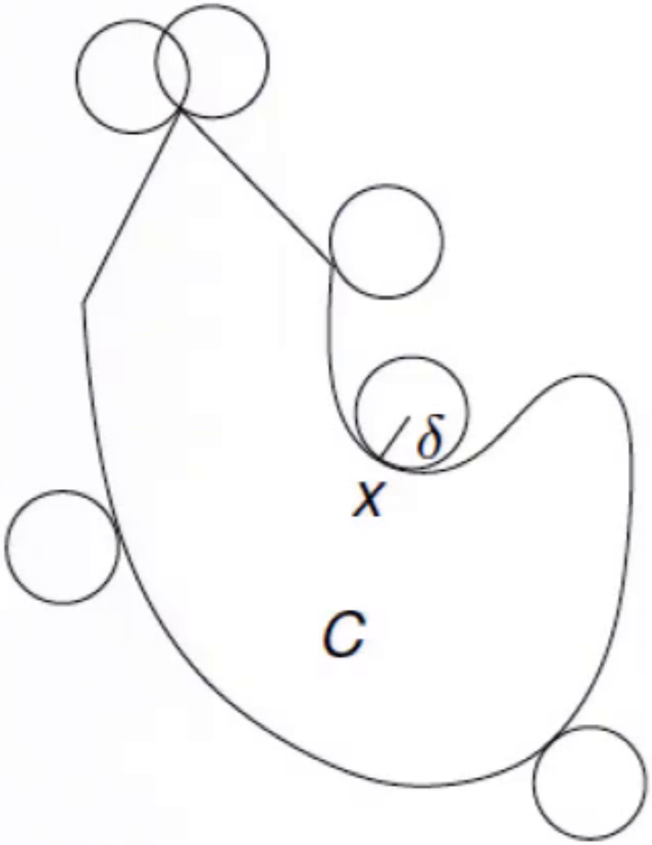
Non-convexity of the feasible set Q_0



where $\bar{\mathbf{q}} = \dfrac{\mathbf{q} + \tilde{\mathbf{q}}}{2}$.

So Q_0 is not a convex set !

Uniform prox-regularity



Uniformly prox-regular set

Let C be a closed subset of a Hilbert space H,
C is η-prox-regular if the projection on C is single-valued and continuous at any point x satisfying $d_C(x) < \eta$.

H. Federer 59, *positively reached sets*
A. Canino 88, *p-convex sets*
F. Clarke, R. Stern, P. Wolenski 95, *proximally smooth sets*
R. Poliquin, R. Rockafellar, L. Thibault 00, *prox-regular sets*

Prox-regularity of Q_0

Proposition

Q_0 is η-prox-regular with $\eta = \eta(N, r_i)$.

Sketch of the proof :

One constraint's case :

$Q_{ij} = \{\mathbf{q} \in \mathbb{R}^{2N},\ D_{ij}(\mathbf{q}) = |q_j - q_i| - (r_j + r_i) \geq 0\}$ is η_{ij}-prox-regular with $\eta_{ij} = \dfrac{r_i + r_j}{\sqrt{2}}$.

Extension to several constraints : $Q_0 = \bigcap_{i<j} Q_{ij}$.

Key point of the proof

A reverse triangle inequality

For every $\mathbf{q} \in Q_0$, for every $\lambda_{ij} \geq 0$, there exists $\gamma > 1$ such that

$$\sum_{(i,j)\in I(\mathbf{q})} \lambda_{ij} |\mathbf{G}_{ij}(\mathbf{q})| \leq \gamma \left| \sum_{(i,j)\in I(\mathbf{q})} \lambda_{ij} \mathbf{G}_{ij}(\mathbf{q}) \right|,$$

where

$$I(\mathbf{q}) = \{(i,j), i < j, D_{ij}(\mathbf{q}) = 0\}.$$

Well-posedness

Theorem

Assume that $\mathbf{U}$ is bounded and Lipschitz continuous.
Then for any $\mathbf{q}_0$ in Q_0, there is a unique absolutely continuous map $\mathbf{q}$ satisfying

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}) & \text{a.e. in } [0, T], \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

Outline

Numerical scheme

Initialization : $\mathbf{q}^0 = \mathbf{q}_0$

Time-loop : $\mathbf{q}^n$ is known

$$\mathbf{u}^n = \mathrm{P}_{C_h(\mathbf{q}^n)}(\mathbf{U}(\mathbf{q}^n))$$

$$\mathbf{q}^{n+1} = \mathbf{q}^n + \mathrm{h}\,\mathbf{u}^n$$

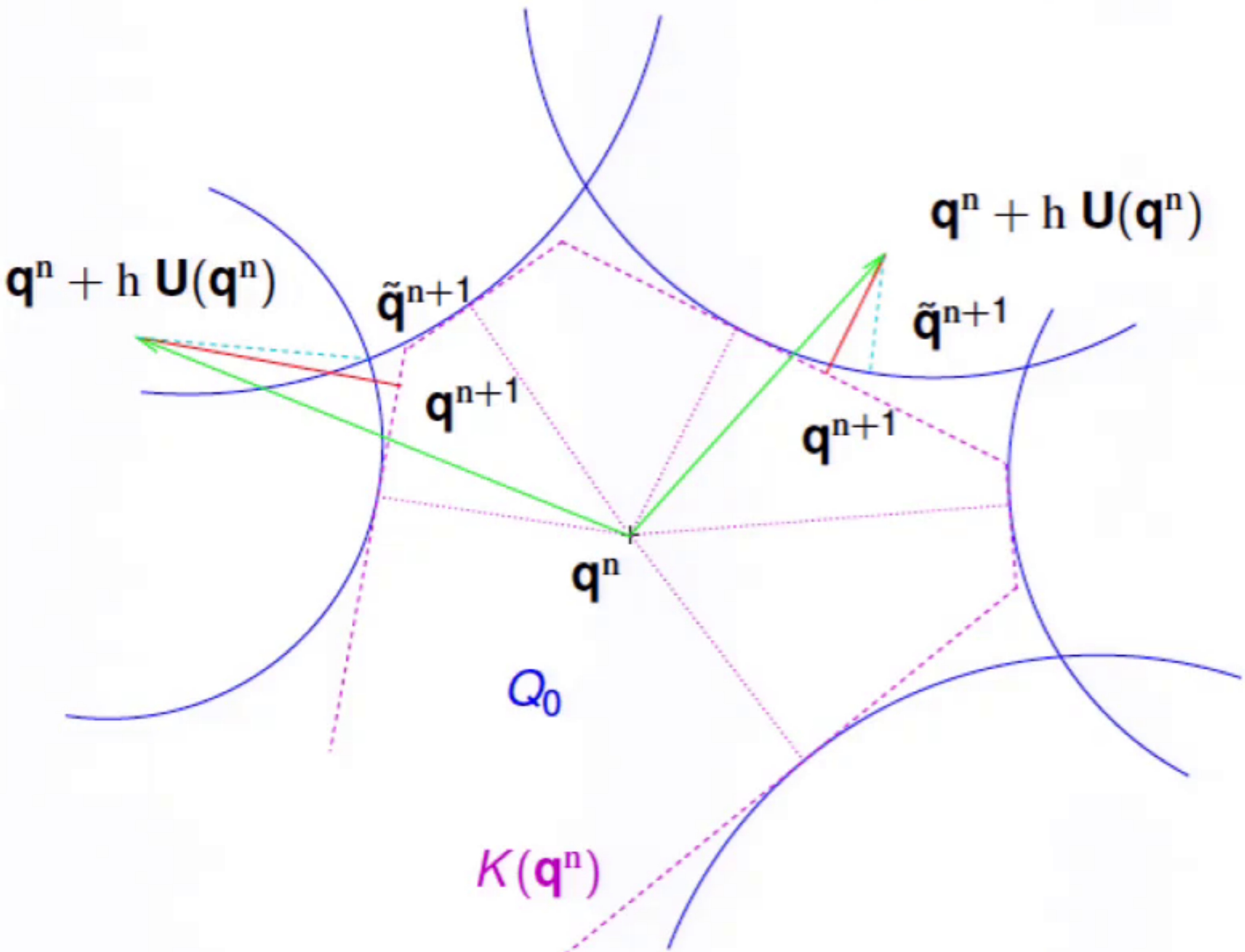
where $C_h(\mathbf{q}^n) = \left\{\mathbf{v} \in \mathbb{R}^{2N}, \forall i < j,\ D_{ij}(\mathbf{q}^n) + \mathrm{h}\,\mathbf{G}_{ij}(\mathbf{q}^n) \cdot \mathbf{v} \geq 0\right\}$.

In terms of position, this algorithm can be formulated as follows :

$$\mathbf{q}^{n+1} = \mathrm{P}_{K(\mathbf{q}^n)}(\mathbf{q}^n + \mathrm{h}\,\mathbf{U}(\mathbf{q}^n))$$

with $K(\mathbf{q}^n) = \left\{\mathbf{q} \in \mathbb{R}^{2N}, \forall i < j,\ D_{ij}(\mathbf{q}^n) + \mathbf{G}_{ij}(\mathbf{q}^n) \cdot (\mathbf{q} - \mathbf{q}^n) \geq 0\right\}$

Comparison between theoretical and numerical projections



Continuous and discrete problems

Discrete differential inclusion :

$$\mathbf{u}^{\mathrm{n}} + \mathrm{N}(K(\mathbf{q}^{\mathrm{n}}), \mathbf{q}^{\mathrm{n}+1}) \ni \mathbf{U}(\mathbf{q}^{\mathrm{n}}).$$

Continuous differential inclusion :

$$\frac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}).$$

Proposition

$$\mathrm{N}(Q_0, \mathbf{q}) = \mathrm{N}(K(\mathbf{q}), \mathbf{q}).$$

Convergence

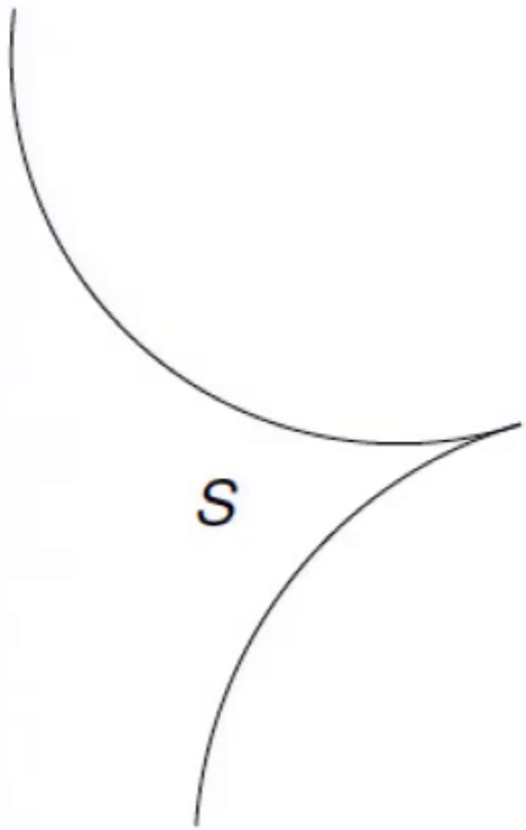
Let $\mathbf{q}_h$ be the continous piecewise linear function associated to the numerical scheme

Theorem

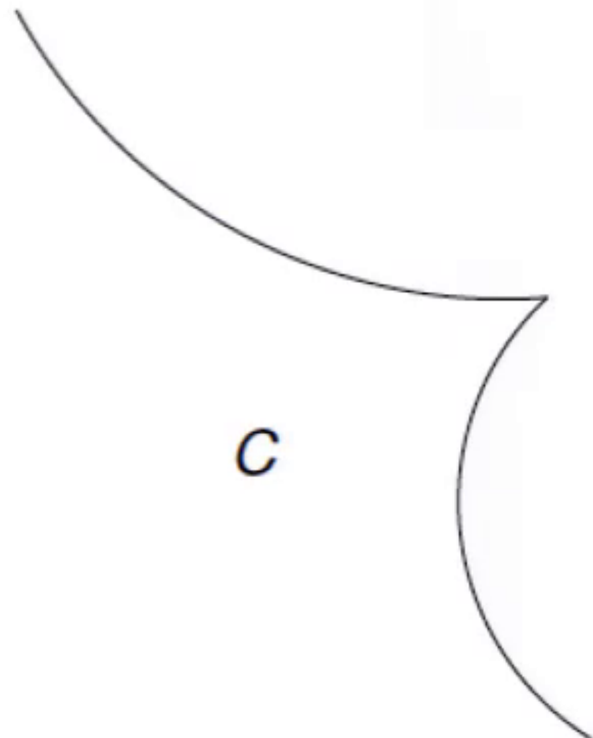
Assume that $\mathbf{U}$ is bounded and Lipschitz continous.
Then $\mathbf{q}_h$ uniformly converges in $[0, T]$ to the map $\mathbf{q}$ satisfying :

$$\begin{cases} \dfrac{d\mathbf{q}}{dt} + \mathrm{N}(Q_0, \mathbf{q}) \ni \mathbf{U}(\mathbf{q}) \quad \text{a.e. in } [0, T], \\ \mathbf{q}(0) = \mathbf{q}_0. \end{cases}$$

A second important geometrical assumption



The set S
is not suitable.



The set C
is suitable. No "thin peaks".

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where $C_h(\mathbf{q}^n) = \left\{\mathbf{v} \in \mathbb{R}^{2N}, \forall i < j,\ D_{ij}(\mathbf{q}^n) + h\,\mathbf{G}_{ij}(\mathbf{q}^n) \cdot \mathbf{v} \geq 0\right\}$.

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