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Adaptive Discretization of Liftings for Curvature Regularization

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Outline

Curvature regularization

Adaptive discretization

Proof of concept

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Gestalt theory

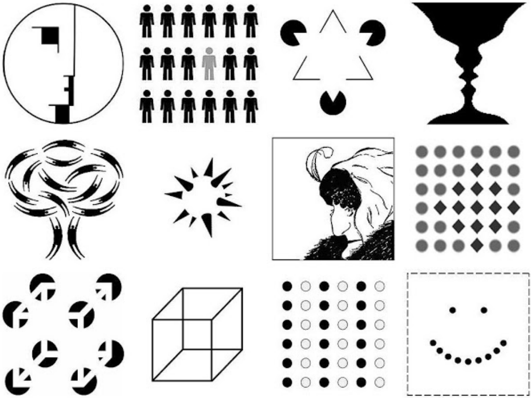
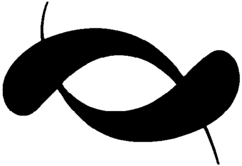


Image: https://en.wikipedia.org/wiki/File:Gestalt_Principles_Composition.jpg

Curvature

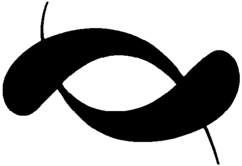


One powerful principle: the law of continuous lines.

- Thus, usage of curvature information great tool in image processing.
- There are models using curvature, e.g. elastica functional.
- But: Strongly non-convex and global minimizer hard to find.

Image: Citti, Sarti: A cortical based model of perceptual completion in the roto-translation space, 2006.

Curvature



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- Thus, usage of curvature information great tool in image processing.
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- But: Strongly non-convex and global minimizer hard to find.

$\Rightarrow$ **Possible solution:** Functional lifting + convex relaxation.

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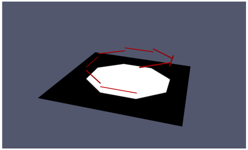
Functional lifting

- Consider problem in a higher dimensional space, e.g. the roto-translations space[1].



- Additional dimension: local orientation $\theta \in S^1$ tangential to the image gradient.

Functional lifting can be used to create a vertex and curvature penalizing functional[2].

[1] Citti, Sarti: A cortical based model of perceptual completion in the roto-translation space, 2006.

[2] Bredies, Pock, Wirth: Convex relaxation of a class of vertex penalizing functionals, 2012.

Quick reminder

Definition (BV)

Let $\Omega \subset \mathbb{R}^d$ be a domain and $u \in L^1(\Omega)$. Then, u is said to be of bounded variation if there exists a finite vector Radon measure $Du \in \mathcal{M}(\Omega, \mathbb{R}^d)$ such that

$$\int_\Omega u \nabla \cdot \phi dx = - \int_\Omega \langle \phi, Du \rangle dx \; \forall \, \phi \in \mathcal{C}_c(\Omega, \mathbb{R}^d).$$

Set of all Radon measures endowed with the total variation measure $|\mu|(\Omega)$ as the norm $\|\mu\|_{\mathcal{M}}$ becomes a Banach space.

Vertex penalizing functional

Let μ be the lifting of the ∇u:

$$T_\rho(\mu) = \sup_{\psi \in M_\rho(\Omega)} \int_{\Omega \times S^1} \nabla_x \psi(x,\theta) \cdot \theta \, \mathrm{d}\mu(x,\theta)$$

with

$$M_\rho(\Omega) = \{\psi \in \mathcal{C}_0(\Omega \times S^1) \colon \nabla_x \psi \in \mathcal{C}_0(\Omega \times S^1, \mathbb{R}^2), \\ \psi(x,\cdot) \in C_\rho \forall x \in \Omega\},$$

$$C_\rho = \{\phi \in \mathcal{C}(S^1) \colon \phi(\theta_1) - \phi(\theta_2) \leq \rho(\theta_1,\theta_2) \,\forall\, (\theta_1,\theta_2) \in S^1 \times S^1\}$$

and $\rho \colon S^1 \times S^1 \to \mathbb{R}$ a lower semi-continuous metric on S^1.

Examples for ρ: the discrete metric ρ_0, the geodesic metric ρ_1.

Theorem

Let $P \subset \Omega \subset \mathbb{R}^2$ be with piecewise $\mathcal{C}^2$-boundary $\partial P \subset \Omega$. Assume that ∂P is homeomorphic to S^1. Then, for μ being the lifted gradient of the characteristic function χ_P

$$T_{\rho_0}(\mu) = \begin{cases} \#\{x_i : \partial P \text{ is not } \mathcal{C}^1 \text{ at } x_i\} \text{ if } \kappa = 0 \text{ on } \partial P \setminus \{x_1, \dots, x_l\}, \\ \infty \text{ else}, \end{cases}$$

$$T_{\rho_1}(\mu) = \int_{\partial P \setminus \{x_1,\dots,x_l\}} |\kappa| \, \mathrm{d}\mathcal{H}^1 + \sum_{1 \le i \le l} \gamma(x_i)$$

with the curvature κ and the unsigned external angle $\gamma(x_i)$ at x_i.

More dimensions...

$$T_\rho(\mu) = \sup_{\psi \in M_\rho(\Omega)} \int_{\Omega \times S^2} \mathrm{curl}_x \psi(x,n) \cdot n \, \mathrm{d}\mu(x,n)$$

with

$$M_\rho(\Omega) = \{\psi \in \mathcal{C}_0(\Omega \times S^2, \mathbb{R}^3) \colon \mathrm{curl}_x \psi \in \mathcal{C}_0(\Omega \times S^2, \mathbb{R}^3), \\ \psi(x,\cdot) \in C_\rho \forall x \in \Omega\},$$

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and $\rho \colon S^2 \times S^2 \to \mathbb{R}$ a lower semi-continuous metric on S^2.

Theorem

Let $P \subset \Omega \subset \mathbb{R}^3$ be with piecewise $\mathcal{C}^2$-boundary $\partial P \subset \Omega$. Then, for μ being the lifted gradient of the characteristic function χ_P

$$T_{\rho_0}(\mu) = \begin{cases} \sum_{e_i} \rho_0(n_{F_k}, n_{F_l}) \cdot \mathcal{H}^1(e_i) \text{ if } \kappa_1 = \kappa_2 = 0 \text{ on } \partial P \setminus \bigcup_i e_i, \\ \infty \quad \text{else}, \end{cases}$$

$$T_{\rho_1}(\mu) = \sum_{e_j} \int_{e_j} \rho_1(n_i(x), n_j(x)) \, \mathrm{d}\mathcal{H}^1 + \sum_i \int_{F_i} |\kappa_1(x)| + |\kappa_2(x)| \, \mathrm{d}\mathcal{H}^2$$

with principle curvatures κ_1 and κ_2.

Functional shall act on sublevel-sets of $u \in BV$. Denote by μ_t the lifting of $\nabla \chi_{u<t}$:

$$\int_{\mathbb{R}} \alpha \|\mu_t\|_{\mathcal{M}} + \beta T_\rho(\mu_t)\,\mathrm{d}t$$

Problem: Non-convex due to two operations:

- Extraction of sublevel-sets,
- functional lifting operation.

$\Rightarrow$ Convex relaxation!

Convex relaxation

- Plugging in the functional lifting of ∇u for general $u \in BV$ instead of sublevel-sets gives a relaxation.
- The set of u and μ that contains all μ which are the functional lifting of ∇u can be relaxed by

$$M_\nabla = \{(u,\mu) \in L^1(\Omega) \times \mathcal{M}(\Omega \times S^1) \mid \mu \geq 0,$$
$$\int_\Omega u \nabla \cdot \phi \, \mathrm{d}x + \int_{\Omega \times S^1} \phi(x) \cdot \theta^\perp \, \mathrm{d}\mu(x,\theta) = 0 \; \forall \phi \in C_c^\infty(\Omega, \mathbb{R}^2)\}$$

Relaxed functional $\alpha \|\mu\|_{\mathcal{M}} + \beta T_\rho(\mu)$ is convex, lower semi-continuous and positively one-homogeneous.

Saddlepoint formulation

Model can be written in a saddlepoint formulation:

$$\inf_{u,\mu} \sup_{\psi,\phi} \lambda G(u) + \beta \int_{\Omega \times S^1} \nabla_x \psi \cdot \theta \, \mathrm{d}\mu + \mathbf{1}_{\mu \geq 0}(\mu) - \mathbf{1}_{C_\rho}(\psi) + \alpha \|\mu\|_{\mathcal{M}} + \int_\Omega u \nabla \cdot \phi \, \mathrm{d}x + \int_{\Omega \times S^1} \phi \cdot \theta^\perp \, \mathrm{d}\mu,$$

with a data fidelity term G $\rightarrow$ optimization with e.g. preconditioned first order primal-dual algorithm[3].

Analogously for the $\Omega \times S^2$-case.

[3] Pock, Chambolle: Diagonal preconditioning for first order primal-dual algorithms, 2011.

Curvature regularization

Adaptive discretization

Proof of concept

Finite Differences

Pros:

- Very easy way to discretize on uniform grids.

Cons:

- High computational cost due to the additional dimension(s) in the lifted variable.
- Uniform grid does not utilize the 1D/2D-structure of μ in "3D"- and "5D"-space respectively.

Finite Differences

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$\Rightarrow$ **Adaptive grid for the lifted variable!**

Finite Elements

Implementation of the $\Omega \times S^1$-case using a Galerkin method with an adaptive grid using the quocmesh library[4]:

- Grid with square and cubic elements respectively,
- bilinear and trilinear basis functions for the 2D- and 3D-functions,
- grid refinement using a 2:1-rule, i.e. neighbouring elements can differ in refinement level only by one level $\rightarrow$ only one hanging node per edge or face possible.

[4] numod.ins.uni-bonn.de/software/quocmesh/

Pros:

- Discrete operators easily computable with implemented routines of the library,
- adding extra directions easy via grid refinement.

Cons:

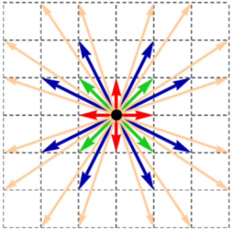
- Poor discretization of measure μ because of trilinear basis functions leading to non-1D-structure.

Line measures

Approximation of the Radon measure μ by line measures $\hat{\mu}$ connecting two nodes of the grid in direction k starting at node j with a weight a:

$$\mu = \sum_j \sum_k a_j^k \hat{\mu}_j^k$$

- Good way to discretize the 1D-structure of the lifted variable.
- Directions are independent from each other in the grid $\rightarrow$ possibility to switch certain directions off.



Master's thesis: Discretization via line measures for a curvature regularization framework, Daniel Tinius, 2015

Implementation using the quocmesh library:

- Usage of the adaptive grid and its refinement-routine,
- 2D-functions with cell-centred degrees of freedom and 3D-functions with node-centred degrees of freedom,
- line measures that are longer than one element are divided into smaller measures to allow local refinement,
- discrete operators have to be assembled by hand.

Operators on adaptive grids

$\int_{\Omega\times S^1} \nabla_x\psi \cdot \theta \,\mathrm{d}\mu$

- Simple differences between nodes connected by the line measure.

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$$\int_{\Omega\times S^1} \nabla_x\psi \cdot \theta \,\mathrm{d}\mu$$

$$\int_\Omega u\nabla \cdot \phi \,\mathrm{d}x$$

- Simple differences between nodes connected by the line measure.
- Test with hat functions with support of two neighbouring elements $\rightarrow$ Finite differences but different cases have to be distinguished dependent on the grid.

Operators on adaptive grids

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- Simple differences between nodes connected by the line measure.

$\int_{\Omega} u\nabla \cdot \phi \,\mathrm{d}x$

- Test with hat functions with support of two neighbouring elements $\rightarrow$ Finite differences but different cases have to be distinguished dependent on the grid.

$\int_{\Omega\times S^1} \phi \cdot \theta^{\perp} \,\mathrm{d}\mu$

- Most complicated operator: Line integrals derived from the line measures have to be evaluated with respect to the support of the hat functions.

Refinement criterion

Needed: Criterion where the grid has to be refined after completion of the algorithm.

Theorem

p^* optimal primal value, d^* optimal dual value $\rightarrow$ primal-dual gap
$\Delta := p^* - d^* \geq 0$

- Use local version of the primal-dual gap for local refinement:
  Primal problem - saddlepoint + saddlepoint - dual problem

Example: Primal-dual gap for ρ_0 with binary segmentation

Local primal-dual-gap for binary segmentation:

$$
\begin{aligned}
\Delta(u,\mu,\psi,\phi) = &\int_\Omega (\lambda f + \nabla\cdot\phi)(u - H(-f-\nabla\cdot\phi))\,\mathrm{d}x \\
&+ \int_\Omega \mathbf{1}_{\{\nabla u = \int_{S^1}\theta^\perp\,\mathrm{d}\mu\}}(\mu,u)\,\mathrm{d}x + \mathbf{1}_C(u) \\
&+ \beta\int_{\Omega\times S^1}\left(\frac{1}{2}\operatorname{sgn}(\nabla_\theta\mu)+\psi\right)\,\mathrm{d}\nabla_\theta\mu \\
&+ \int_{\Omega\times S^1}\alpha + \beta\nabla_x\psi\cdot\theta + \phi\cdot\theta^\perp\,\mathrm{d}\mu \\
&+ \int_{\Omega\times S^1}\mathbf{1}_{\{\alpha+\beta\nabla_x\psi\cdot\theta+\phi\cdot\theta^\perp\geq 0\}}\,\mathrm{d}\mu + \mathbf{1}_{C_{\rho_0}}(\psi) + \mathbf{1}_{\mu\geq 0}(\mu)
\end{aligned}
$$

with Heaviside function $H(x)$.

- Gap can be computed for every single 2D- and 3D-element in the grid.
- Gap not a convergence indicator!

Problems:

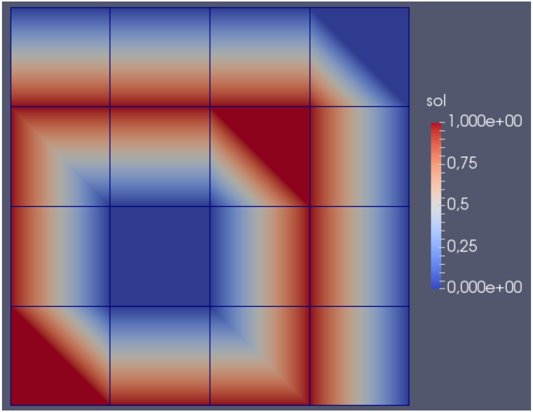
- How to handle the characteristic functions? Set gap value to $+\infty$ or allow a certain range of values without setting gap to $+\infty$? Can a discrete solution fulfil the characteristic functions?
- 2D and 3D grid have to be compatible: 3D grid projected onto the 2D-plane has to look like the 2D grid: What happens if a 2D element has to be refined but no 3D element above?

Curvature regularization

Adaptive discretization

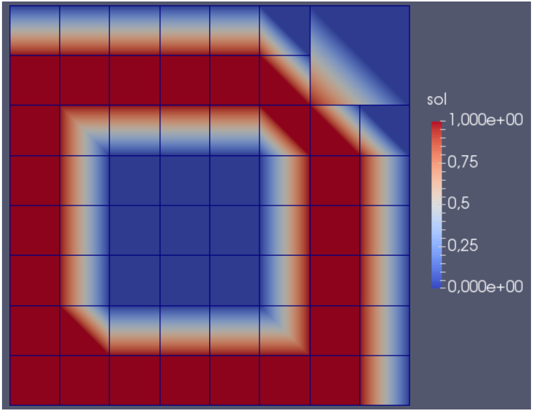
Proof of concept

Square



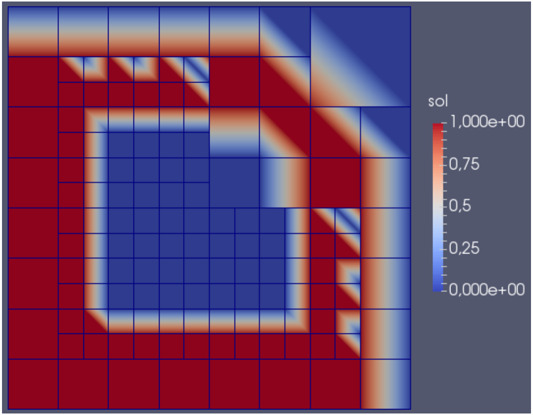
Initial level

Square



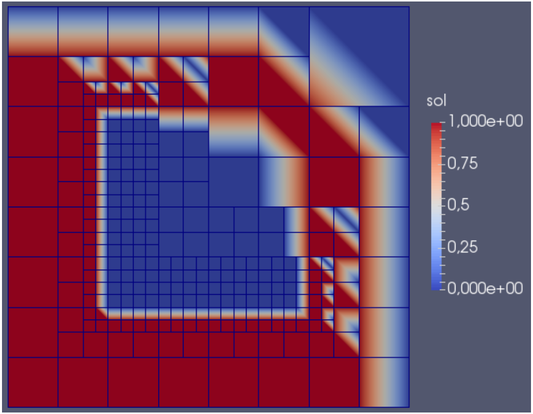
First refinement

Square



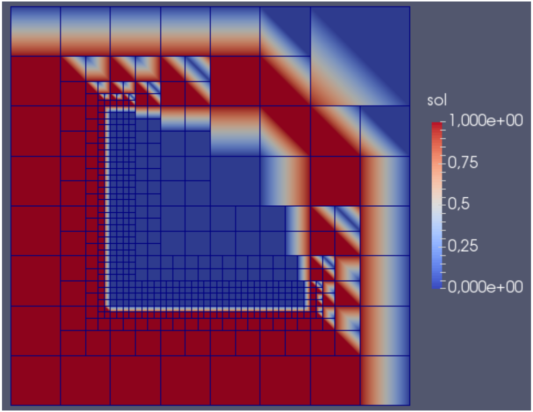
Second refinement

Square



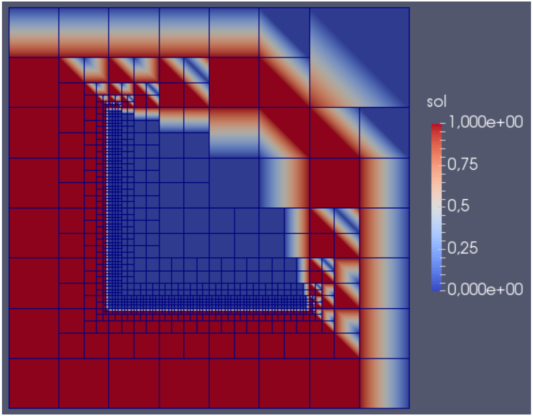
Third refinement

Square



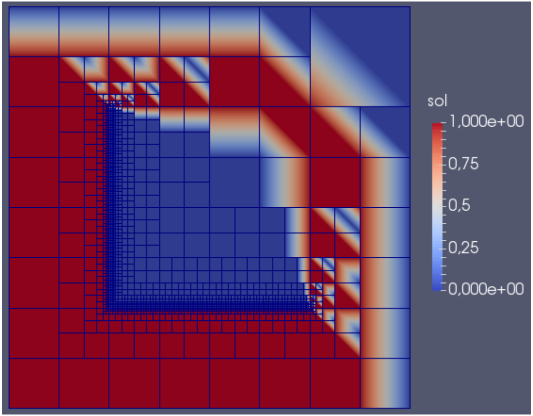
Fourth refinement

Square



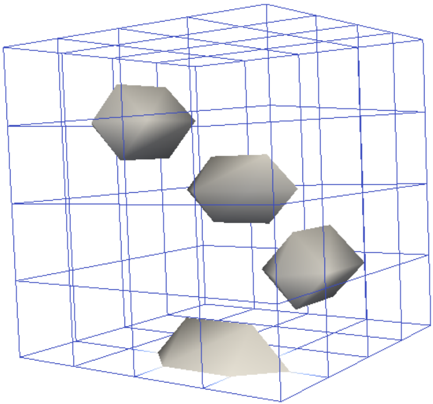
Fifth refinement

Square



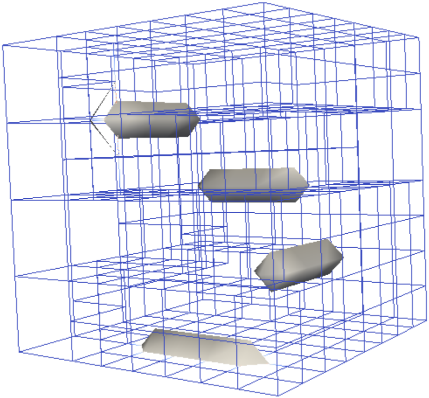
Sixth refinement

Square



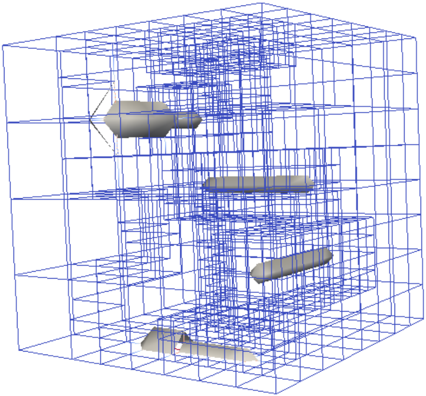
Initial level

Square



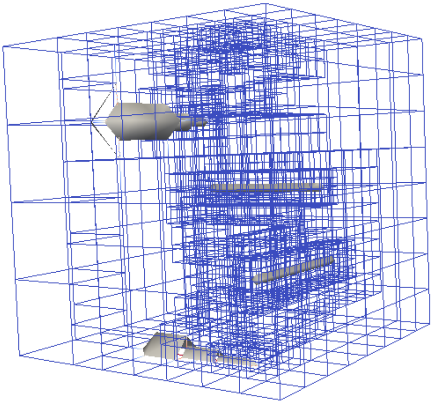
First refinement

Square



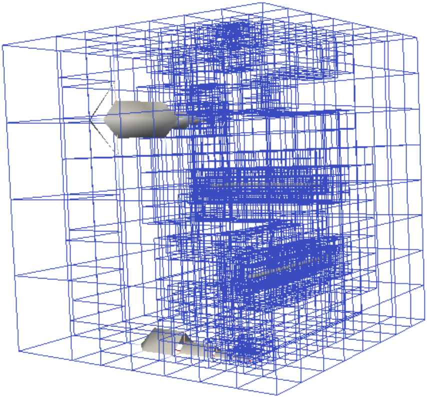
Second refinement

Square



Third refinement

Square



Fourth refinement

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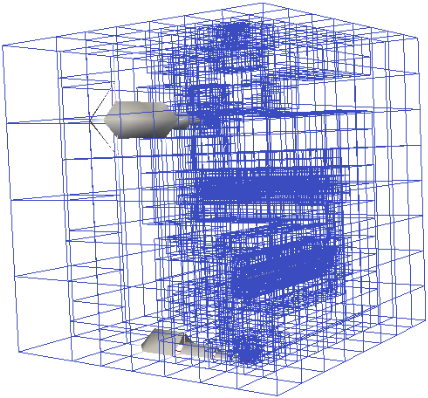

Fifth refinement

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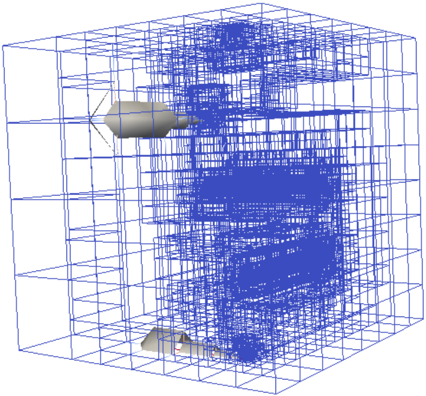

Sixth refinement

Where is the $\Omega \times S^2$-case?

Still ongoing work....

Conclusion

- Curvature regularizing functional can be obtained by functional lifting and made convex by convex relaxation.
- High computational cost due to the lifting require an alternative approach for the discretization.
- Line measures with an adaptive grid are a good solution.
- Local primal-dual gap can be used as a refinement criterion.

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- High computational cost due to the lifting require an alternative approach for the discretization.
- Line measures with an adaptive grid are a good solution.
- Local primal-dual gap can be used as a refinement criterion.

To do:

- Better understanding of the gap and where to refine.
- Get the $\Omega \times S^2$-case implemented!

Thank you for your attention!

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Questions...?