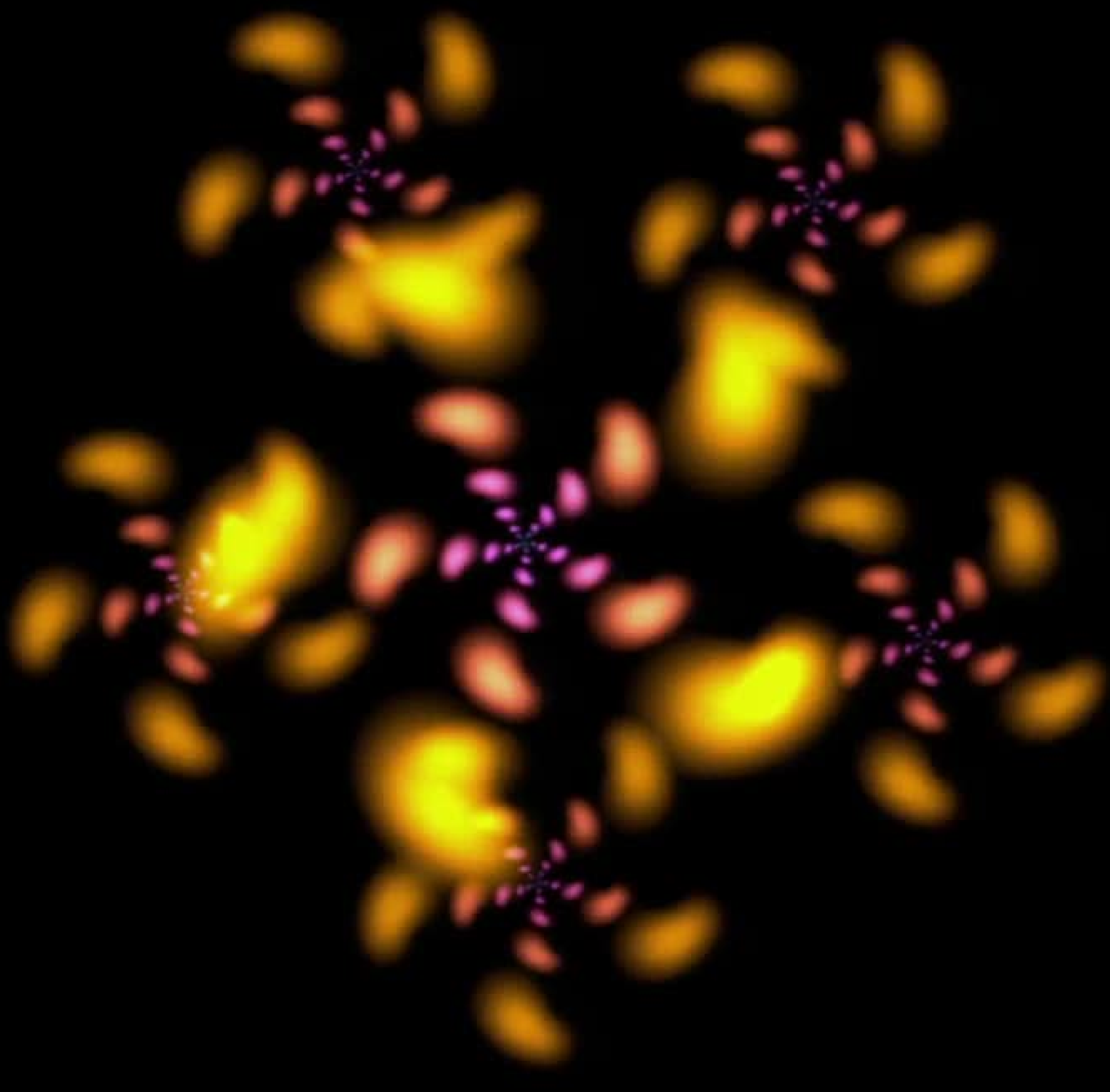
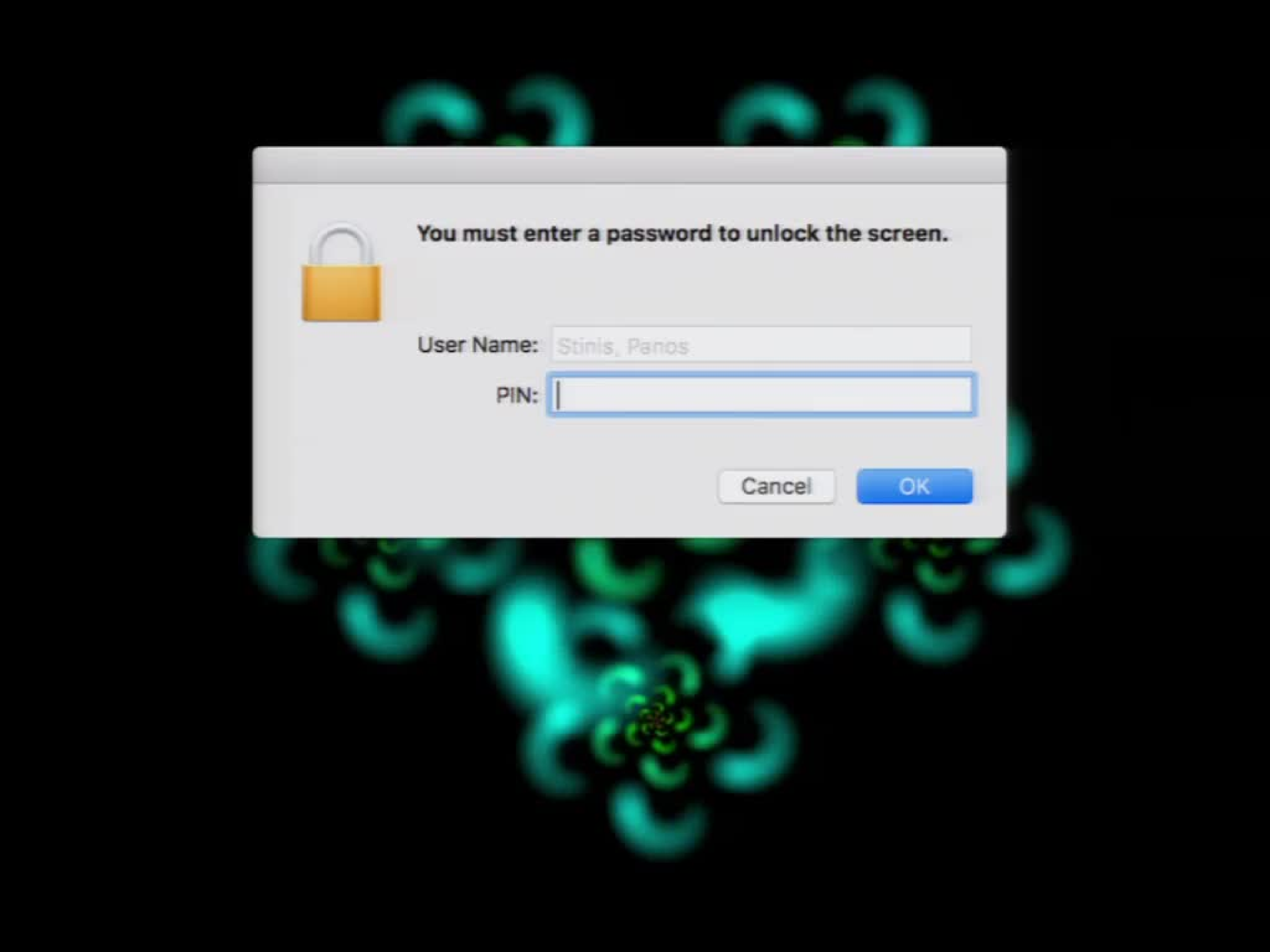
Reduced order models for uncertainty quantification of time-dependent problems

Jing Li & Panos Stinis

Pacific Northwest National Laboratory




You must enter a password to unlock the screen.
User Name:
Stinis, Panos
PIN:
Cancel
OK

Reduced order models for uncertainty quantification of time-dependent problems

Jing Li & Panos Stinis

Pacific Northwest National Laboratory

Uncertainty in time-dependent problems

In many time-dependent problems of practical interest, parameters and/or initial/boundary conditions can be uncertain.

One way to address the problem of how this uncertainty impacts the solution is to expand the solution using polynomial chaos expansions and obtain a system of differential equations for the evolution of the expansion coefficients.

Main idea: Construct reduced models for a *subset* of the polynomial chaos expansion coefficients that are needed for a full description of the uncertainty.

We will use the Mori-Zwanzig formalism to construct such reduced models.

Remark: *Accurate* reduced models require memory *even* for simple systems.

Remark: The construction of *accurate* reduced models can be very costly *even* for simple systems.

The Mori-Zwanzig formalism

Zwanzig(1961), Mori(1965), Chorin, Hald, Kupferman (2000)

Suppose we are given an M-dimensional system of ordinary differential equations

$$\frac{du(t)}{dt} = R(u(t)) \tag{1}$$

with initial condition $u(0) = u_0$.

Transform into a system of linear partial differential equations

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = L e^{tL} u_{0k}, \; k = 1, \ldots, M$$

where the Liouvillian operator $L = \sum_{i=1}^{M} R_i(u_0) \frac{\partial}{\partial u_{0i}}$. Note that $L u_{0j} = R_j(u_0)$.

Let $u_0 = (\hat{u}_0, \tilde{u}_0)$ where $\hat{u}_0$ is N-dimensional and $\tilde{u}_0$ is $M - N$ -dimensional. Define a projection operator $P : \mathcal{F}(u_0) \to \hat{\mathcal{F}}(\hat{u}_0)$. Also, define the operator $Q = I - P$.

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = e^{tL} PLu_{0k} + e^{tQL} QLu_{0k} + \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} \, ds \quad (2)$$

for $k = 1, \ldots, N$.

We have used Dyson's formula (Duhamel's principle)

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} PLe^{sQL} ds. \quad (3)$$

The Mori-Zwanzig formalism

Zwanzig(1961), Mori(1965), Chorin, Hald, Kupferman (2000)

Suppose we are given an M-dimensional system of ordinary differential equations

$$\frac{du(t)}{dt} = R(u(t)) \tag{1}$$

with initial condition $u(0) = u_0$.

Transform into a system of linear partial differential equations

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = L e^{tL} u_{0k}, \ k = 1, \ldots, M$$

where the Liouvillian operator $L = \sum_{i=1}^{M} R_i(u_0) \frac{\partial}{\partial u_{0i}}$. Note that $L u_{0j} = R_j(u_0)$.

Let $u_0 = (\hat{u}_0, \tilde{u}_0)$ where $\hat{u}_0$ is N-dimensional and $\tilde{u}_0$ is $M - N$ -dimensional. Define a projection operator $P : \mathcal{F}(u_0) \to \hat{\mathcal{F}}(\hat{u}_0)$. Also, define the operator $Q = I - P$.

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = e^{tL} P L u_{0k} + e^{tQL} Q L u_{0k} + \int_0^t e^{(t-s)L} P L e^{sQL} Q L u_{0k} \, ds \tag{2}$$

for $k = 1, \ldots, N$.

We have used Dyson's formula (Duhamel's principle)

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} P L e^{sQL} ds. \tag{3}$$

Since the solutions of the orthogonal dynamics equation remain orthogonal to the range of P, we can project the Mori-Zwanzig equation (2) and find

$$\frac{\partial}{\partial t} Pe^{tL} u_{0k} = Pe^{tL} PLu_{0k} + P \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds. \quad (4)$$

Use (4) as the starting point of approximations for the evolution of the quantity $Pe^{tL} u_{0k}$ for $k = 1, \ldots, N$ (note that the equation (4) involves the orthogonal dynamics operator e^{tQL}).

Construct reduced models based on mathematical, physical and numerical observations.

These models come directly from the original equations and the terms appearing in them are not introduced by hand.

Let $u_0 = (\hat{u}_0, \tilde{u}_0)$ where $\hat{u}_0$ is N-dimensional and $\tilde{u}_0$ is $M - N$ -dimensional. Define a projection operator $P : \mathcal{F}(u_0) \to \hat{\mathcal{F}}(\hat{u}_0)$. Also, define the operator $Q = I - P$.

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = e^{tL} PLu_{0k} + e^{tQL} QLu_{0k} + \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds \tag{2}$$

for $k = 1, \dots, N$.

We have used Dyson's formula (Duhamel's principle)

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} PLe^{sQL} ds. \tag{3}$$

Scalar linear differential equation

Consider

$$\frac{du}{dt} = -\kappa u, \tag{5}$$

where $\kappa \sim U[0,1]$. Let $u \approx \sum_{r=0}^{M} u_r L_r(w)$, where $w \sim U[-1,1]$ and $\{L_r\}$ are the Legendre polynomials. We obtain

$$\frac{du_r}{dt} = -\sum_{i=0}^{1}\sum_{j=0}^{M} k_i u_j e_{ijr}, \qquad r = 0,\ldots,M, \tag{6}$$

where $e_{ijr} = \int_{-1}^{1} L_i(w)L_j(w)L_r(w)\frac{1}{2}dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u},\tilde{0})$, with $\hat{u} = (u_0, u_1, \ldots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \ldots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use

$$(P'\varphi_j)(\hat{u},t) \approx \sum_{\nu} (\varphi_j(u,t), h^{\nu}(\hat{u}))h^{\nu}(\hat{u}),$$

where h^{ν} are Hermite polynomials.

Scalar linear differential equation

Consider

$$\frac{du}{dt} = -\kappa u, \tag{5}$$

where $\kappa \sim U[0,1]$. Let $u \approx \sum_{r=0}^{M} u_r L_r(w)$, where $w \sim U[-1,1]$ and $\{L_r\}$ are the Legendre polynomials. We obtain

$$\frac{du_r}{dt} = -\sum_{i=0}^{1}\sum_{j=0}^{M} k_i u_j e_{ijr}, \qquad r = 0, \ldots, M, \tag{6}$$

where $e_{ijr} = \int_{-1}^{1} L_i(w) L_j(w) L_r(w) \frac{1}{2} dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \ldots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \ldots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use

$$(P'\varphi_j)(\hat{u}, t) \approx \sum_{\nu} (\varphi_j(u,t), h^{\nu}(\hat{u})) h^{\nu}(\hat{u}),$$

where h^{ν} are Hermite polynomials.

Since the solutions of the orthogonal dynamics equation remain orthogonal to the range of P, we can project the Mori-Zwanzig equation (2) and find

$$\frac{\partial}{\partial t} Pe^{tL} u_{0k} = Pe^{tL} PLu_{0k} + P \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds. \quad (4)$$

Use (4) as the starting point of approximations for the evolution of the quantity $Pe^{tL} u_{0k}$ for $k = 1, \dots, N$ (note that the equation (4) involves the orthogonal dynamics operator e^{tQL}).

Construct reduced models based on mathematical, physical and numerical observations.

These models come directly from the original equations and the terms appearing in them are not introduced by hand.

Scalar linear differential equation

Consider

$$\frac{du}{dt} = -\kappa u, \tag{5}$$

where $\kappa \sim U[0,1]$. Let $u \approx \sum_{r=0}^{M} u_r L_r(w)$, where $w \sim U[-1,1]$ and $\{L_r\}$ are the Legendre polynomials. We obtain

$$\frac{du_r}{dt} = -\sum_{i=0}^{1}\sum_{j=0}^{M} k_i u_j e_{ijr}, \qquad r = 0, \ldots, M, \tag{6}$$

where $e_{ijr} = \int_{-1}^{1} L_i(w)L_j(w)L_r(w)\frac{1}{2}dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \ldots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \ldots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use

$$(P'\varphi_j)(\hat{u}, t) \approx \sum_{\nu}(\varphi_j(u,t), h^{\nu}(\hat{u}))h^{\nu}(\hat{u}),$$

where h^{ν} are Hermite polynomials.

Let $u_0 = (\hat{u}_0, \tilde{u}_0)$ where $\hat{u}_0$ is N-dimensional and $\tilde{u}_0$ is $M - N$ -dimensional. Define a projection operator $P : \mathcal{F}(u_0) \to \hat{\mathcal{F}}(\hat{u}_0)$. Also, define the operator $Q = I - P$.

$$\frac{\partial}{\partial t} e^{tL} u_{0k} = e^{tL} P L u_{0k} + e^{tQL} Q L u_{0k} + \int_0^t e^{(t-s)L} P L e^{sQL} Q L u_{0k} \, ds \tag{2}$$

for $k = 1, \ldots, N$.

We have used Dyson's formula (Duhamel's principle)

$$e^{tL} = e^{tQL} + \int_0^t e^{(t-s)L} P L e^{sQL} ds. \tag{3}$$

Since the solutions of the orthogonal dynamics equation remain orthogonal to the range of P, we can project the Mori-Zwanzig equation (2) and find

$$\frac{\partial}{\partial t} Pe^{tL} u_{0k} = Pe^{tL} PLu_{0k} + P \int_0^t e^{(t-s)L} PLe^{sQL} QLu_{0k} ds. \quad (4)$$

Use (4) as the starting point of approximations for the evolution of the quantity $Pe^{tL} u_{0k}$ for $k = 1, \ldots, N$ (note that the equation (4) involves the orthogonal dynamics operator e^{tQL}).

Construct reduced models based on mathematical, physical and numerical observations.

These models come directly from the original equations and the terms appearing in them are not introduced by hand.

Scalar linear differential equation

Consider

$$\frac{du}{dt} = -\kappa u, \tag{5}$$

where $\kappa \sim U[0,1]$. Let $u \approx \sum_{r=0}^{M} u_r L_r(w)$, where $w \sim U[-1,1]$ and $\{L_r\}$ are the Legendre polynomials. We obtain

$$\frac{du_r}{dt} = -\sum_{i=0}^{1}\sum_{j=0}^{M} k_i u_j e_{ijr}, \qquad r = 0, \dots, M, \tag{6}$$

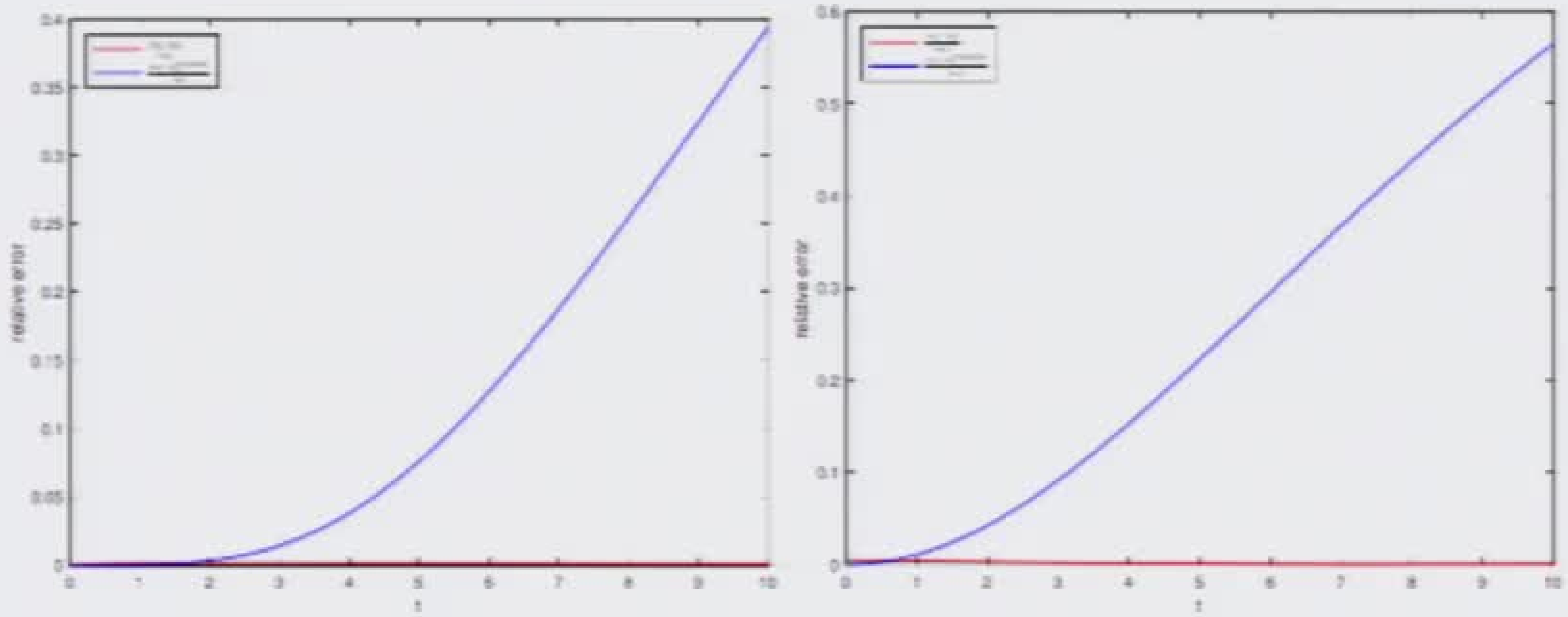
where $e_{ijr} = \int_{-1}^{1} L_i(w) L_j(w) L_r(w) \frac{1}{2} dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \dots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \dots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use

$$(P'\varphi_j)(\hat{u}, t) \approx \sum_{\nu} (\varphi_j(u,t), h^{\nu}(\hat{u})) h^{\nu}(\hat{u}),$$

where h^{ν} are Hermite polynomials.



(a) (b)

Figure: Comparison of the relative error for the models with and without memory for the linear equation with uncertain coefficient.

Remark: Memory is very important for accurate prediction even for moderate times.

Periodically forced nonlinearly damped particle

Consider

$$\frac{du}{dt} = u - u^3 + f(t, w), \qquad (7)$$

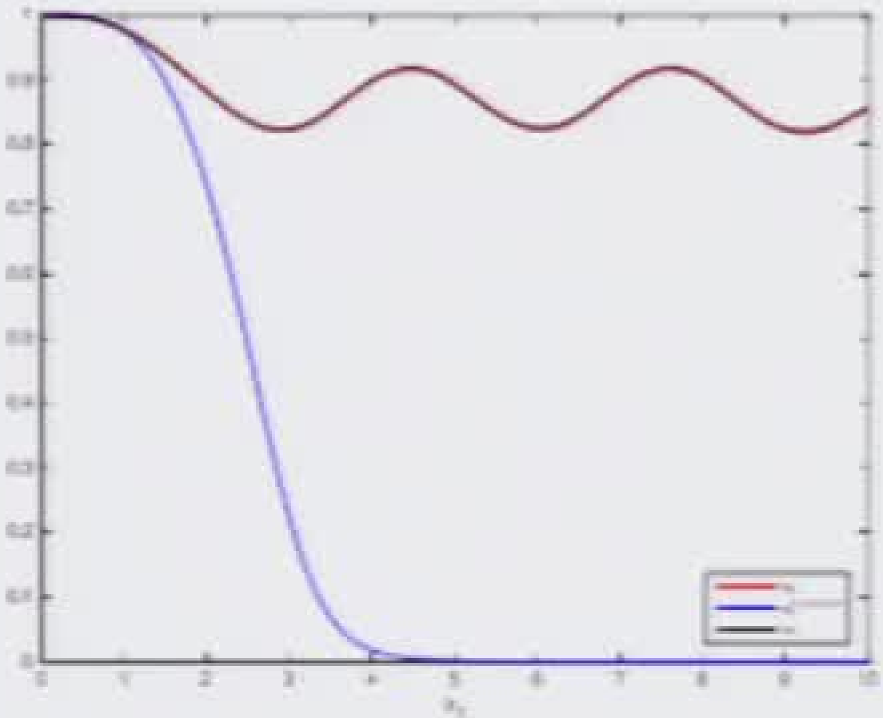
where $f = w \sin t$ and $w \sim U(-1, 1)$. Let $u \approx \sum_{i=0}^{M} u_i L_i(w)$, where $w \sim U[-1, 1]$ and $\{L_i\}$ are the Legendre polynomials. We obtain

$$\frac{du_i}{dt} = u_i - \sum_{j,k,m=0}^{p_f} u_j u_k u_m e_{jkmi} + f_i, \qquad i = 0, \dots, M, \qquad (8)$$

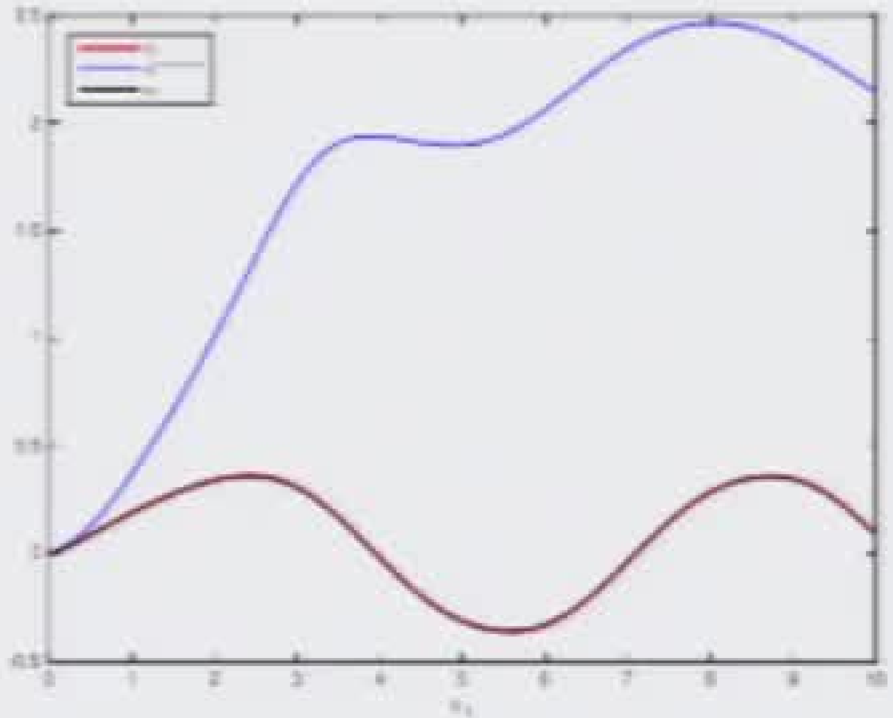
where $e_{jkmi} = \int_{-1}^{1} L_j(w) L_k(w) L_m(w) L_i(w) \frac{1}{2} dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \dots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \dots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use Hermite polynomials as before.



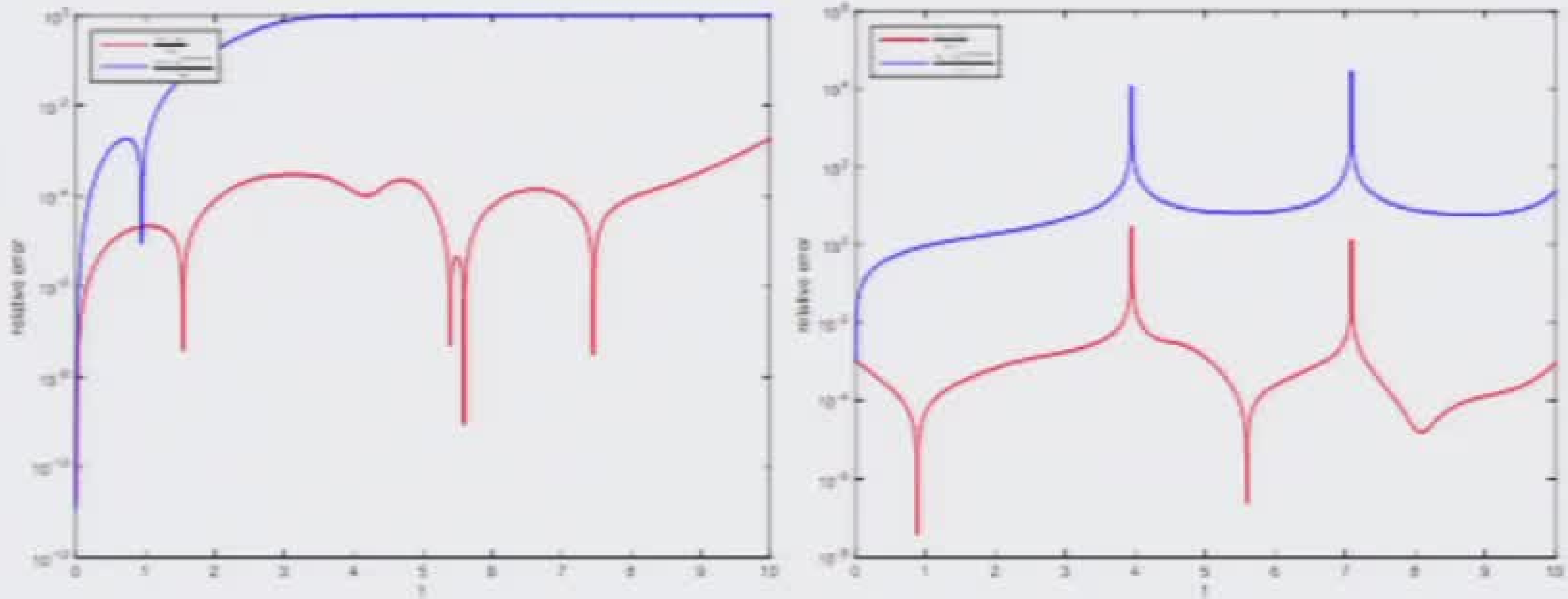
(a)



(b)

Figure: Comparison of the solution without memory, with memory and exact solution for the periodically forced damped particle equation.

Remark: We construct a reduced model with $\Lambda = 1$ and $M = 6$. The model with memory uses 10 basis functions (highest order 3) for the memory term integrand. The integrands for the memory term can be computed by solving Volterra equations.



(a) (b)

Figure: Comparison of the relative error (logarithmic scale) for the models with and without memory for the periodically forced damped particle equation.

Remark: Memory is very important for accurate prediction even for short times.

Application to the 1D Burgers equation

Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}. \tag{9}$$

We assume periodic BCs and $u(0, x) = u_0(x)$.

We can have e.g. uncertain viscosity coefficient and/or uncertain initial conditions.

For a simple case with $\nu = 0.03$ one needs about 100 Fourier modes to resolve the solution. If e.g. $u_0(x, \xi) = (1 + \xi)\sin x$ where $\xi \sim U[-1, 1]$, we need the first 7 Legendre polynomials to fully resolve the uncertainty in the solution.

Remark: If we resolve 2 of the 7 expansion coefficients, then for the memory term term we would need to have a basis in 100×2 dimensions. It is prohibitively expensive to use with high orders.

Remark: If we have some prior information about the system then we may be able to reduce the number of arguments of the basis functions. Otherwise, we need to compute the term $P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$ without relying on a finite-rank projection.

Main idea: Construct a hierarchy of equations for the evolution of the memory term and its derivatives.

Let $w_{0k}(t) = P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$. We find

$$\frac{dw_{0k}}{dt} = Pe^{tL}PLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLu_{0k} + w_{1k}(t)$$

where

$$w_{1k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLu_{0k}ds$$

and we have assumed that the memory extends only t_0 units back.

Also

$$\frac{dw_{1k}}{dt} = Pe^{tL}PLQLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLQLu_{0k} + w_{2k}(t),$$

where

$$w_{2k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLQLu_{0k}ds.$$

This hierarchy continues

$$\frac{dw_{(n-1)k}}{dt} = Pe^{tL}PL(QL)^{n-1}QLu_{0k} -$$
$$Pe^{(t-t_0)L}PLe^{t_0QL}(QL)^{n-1}QLu_{0k} + w_{nk}(t)$$

where

$$w_{nk}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}(QL)^{n}QLu_{0k}ds.$$

Of course, this is the *closure* problem which we can address by assuming that $w_{nk}(t) = 0$.

In addition to the closure, we have to estimate the terms $Pe^{(t-t_0)L}PLe^{t_0QL}QLu_{0k}, \dots Pe^{(t-t_0)L}PLe^{t_0QL}(QL)^{n-1}QLu_{0k}$. This can be achieved by using a discretized version e.g. trapezoidal rule for $w_{0k}(t), \dots, w_{(n-1)k}(t)$ and solving for the unknown terms.

Remark: The expansion above amounts to an expansion of the evolution operator for the *orthogonal dynamics* equation. There are alternative expansions which involve the *full dynamics* operator.

Remark: There is an unknown parameter, the length t_0 of the memory. One can compute estimate t_0 using a construction akin to renormalization. The estimate of t_0 is determined through an optimization problem.

Remark: The whole process of approximating the memory can be recast as a set of ODEs which augments the Galerkin model.

Also

$$\frac{dw_{1k}}{dt} = Pe^{tL}PLQLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLQLu_{0k} + w_{2k}(t),$$

where

$$w_{2k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLQLu_{0k}ds.$$

This hierarchy continues

$$\frac{dw_{(n-1)k}}{dt} = Pe^{tL}PL(QL)^{n-1}QLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}(QL)^{n-1}QLu_{0k} + w_{nk}(t)$$

where

$$w_{nk}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}(QL)^{n}QLu_{0k}ds.$$

Of course, this is the *closure* problem which we can address by assuming that $w_{nk}(t) = 0$.

Remark: If we have some prior information about the system then we may be able to reduce the number of arguments of the basis functions. Otherwise, we need to compute the term $P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$ without relying on a finite-rank projection.

Main idea: Construct a hierarchy of equations for the evolution of the memory term and its derivatives.

Let $w_{0k}(t) = P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$. We find

$$\frac{dw_{0k}}{dt} = Pe^{tL}PLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLu_{0k} + w_{1k}(t)$$

where

$$w_{1k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLu_{0k}ds$$

and we have assumed that the memory extends only t_0 units back.

Also

$$\frac{dw_{1k}}{dt} = Pe^{tL}PLQLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLQLu_{0k} + w_{2k}(t),$$

where

$$w_{2k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLQLu_{0k}ds.$$

This hierarchy continues

$$\frac{dw_{(n-1)k}}{dt} = Pe^{tL}PL(QL)^{n-1}QLu_{0k} -$$
$$Pe^{(t-t_0)L}PLe^{t_0QL}(QL)^{n-1}QLu_{0k} + w_{nk}(t)$$

where

$$w_{nk}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}(QL)^{n}QLu_{0k}ds.$$

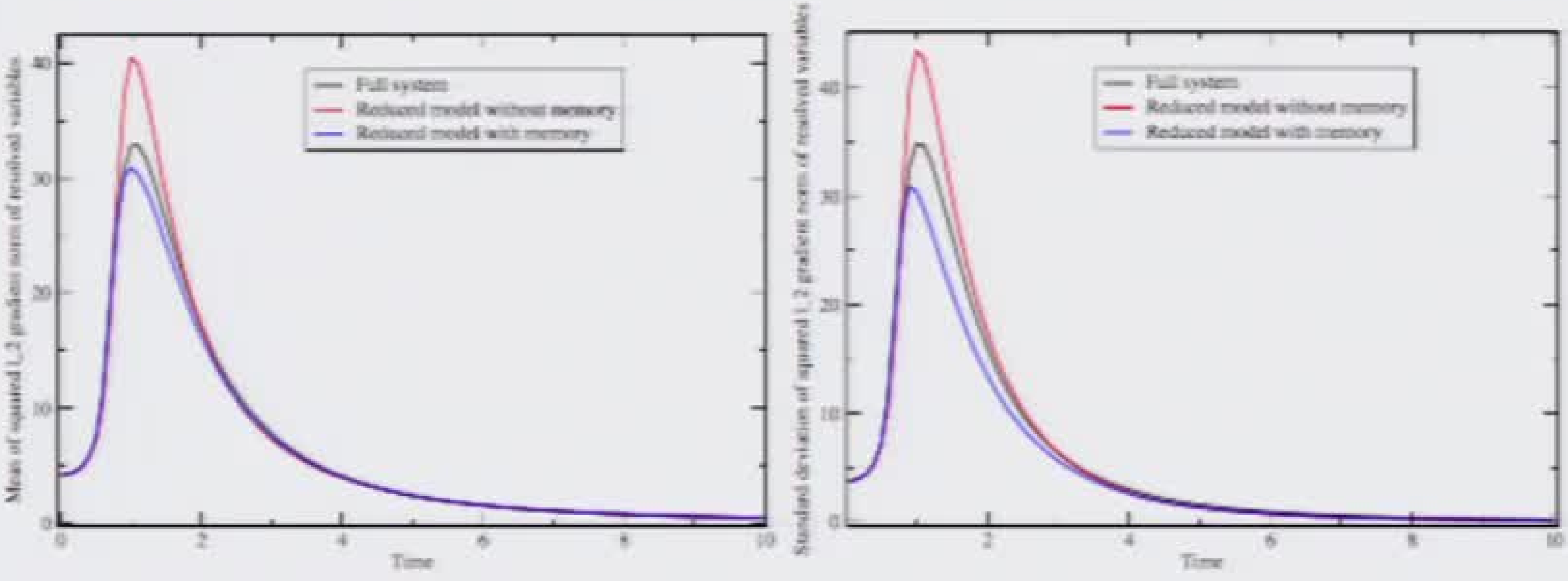
Of course, this is the *closure* problem which we can address by assuming that $w_{nk}(t) = 0$.

In addition to the closure, we have to estimate the terms $Pe^{(t-t_0)L}PLe^{t_0 QL}QLu_{0k},\ldots Pe^{(t-t_0)L}PLe^{t_0 QL}(QL)^{n-1}QLu_{0k}$. This can be achieved by using a discretized version e.g. trapezoidal rule for $w_{0k}(t),\ldots,w_{(n-1)k}(t)$ and solving for the unknown terms.

Remark: The expansion above amounts to an expansion of the evolution operator for the *orthogonal dynamics* equation. There are alternative expansions which involve the *full dynamics* operator.

Remark: There is an unknown parameter, the length t_0 of the memory. One can compute estimate t_0 using a construction akin to renormalization. The estimate of t_0 is determined through an optimization problem.

Remark: The whole process of approximating the memory can be recast as a set of ODEs which augments the Galerkin model.



(a) (b)

Figure: Evolution of the mean and the standard deviation of the gradient of the solution using only the first two Legendre polynomials.

Conclusion: The construction of reduced models for UQ can be necessary and costly. It must account for memory effects.

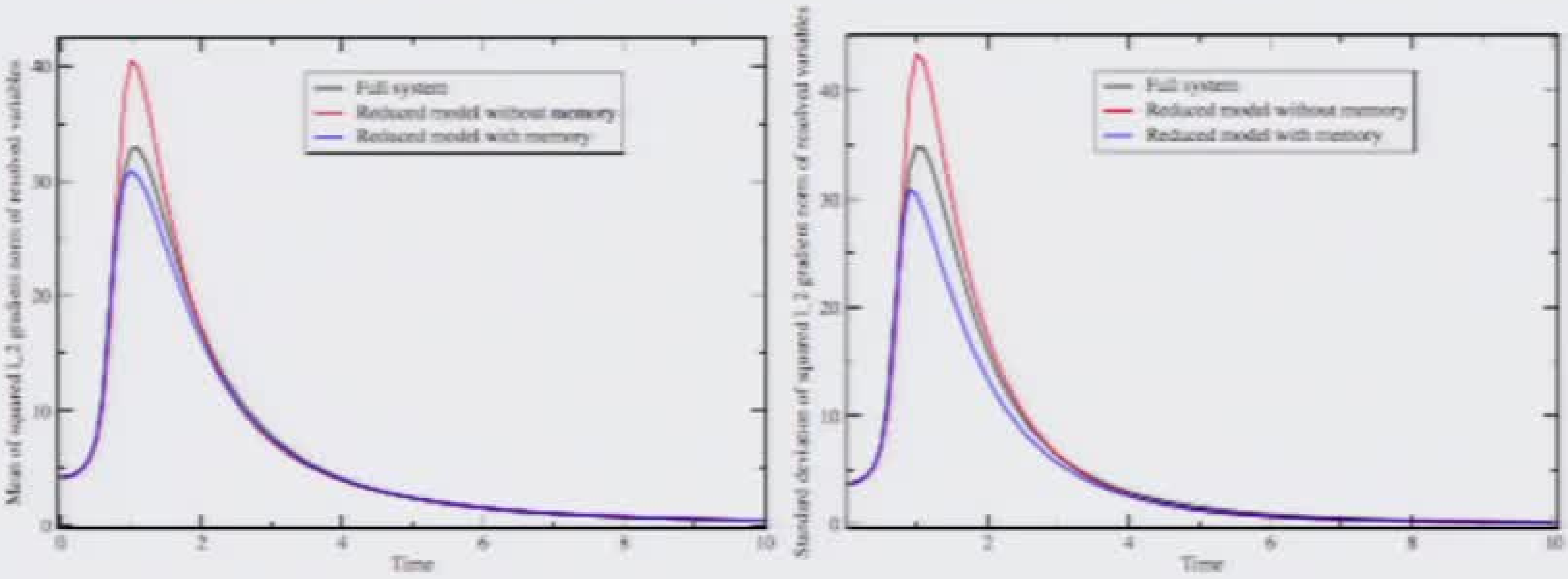
References

1) A. Chorin, O.H. Hald and R. Kupferman, Optimal prediction with memory, Physica D, 166, (2002), pp. 239-257.

2) P. Stinis, Mori-Zwanzig reduced models for uncertainty quantification I: Parametric uncertainty (2012), arXiv:1211.4285v1.

3) P. Stinis, Mori-Zwanzig reduced models for uncertainty quantification II: Initial condition uncertainty (2012), arXiv:1212.6360v1.

4) J. Li and P. Stinis, Mori-Zwanzig reduced models for uncertainty quantification (2018), arXiv:1803.02826v1 (*submitted*).



(a) (b)

Figure: Evolution of the mean and the standard deviation of the gradient of the solution using only the first two Legendre polynomials.

Conclusion: The construction of reduced models for UQ can be necessary and costly. It must account for memory effects.

In addition to the closure, we have to estimate the terms $Pe^{(t-t_0)L}PLe^{t_0 QL}QLu_{0k}, \ldots Pe^{(t-t_0)L}PLe^{t_0 QL}(QL)^{n-1}QLu_{0k}$. This can be achieved by using a discretized version e.g. trapezoidal rule for $w_{0k}(t), \ldots, w_{(n-1)k}(t)$ and solving for the unknown terms.

Remark: The expansion above amounts to an expansion of the evolution operator for the *orthogonal dynamics* equation. There are alternative expansions which involve the *full dynamics* operator.

Remark: There is an unknown parameter, the length t_0 of the memory. One can compute estimate t_0 using a construction akin to renormalization. The estimate of t_0 is determined through an optimization problem.

Remark: The whole process of approximating the memory can be recast as a set of ODEs which augments the Galerkin model.

In addition to the closure, we have to estimate the terms $Pe^{(t-t_0)L}PLe^{t_0QL}QLu_{0k}, \dots Pe^{(t-t_0)L}PLe^{t_0QL}(QL)^{n-1}QLu_{0k}$. This can be achieved by using a discretized version e.g. trapezoidal rule for $w_{0k}(t), \dots, w_{(n-1)k}(t)$ and solving for the unknown terms.

Remark: The expansion above amounts to an expansion of the evolution operator for the *orthogonal dynamics* equation. There are alternative expansions which involve the *full dynamics* operator.

Remark: There is an unknown parameter, the length t_0 of the memory. One can compute estimate t_0 using a construction akin to renormalization. The estimate of t_0 is determined through an optimization problem.

Remark: The whole process of approximating the memory can be recast as a set of ODEs which augments the Galerkin model.

Remark: If we have some prior information about the system then we may be able to reduce the number of arguments of the basis functions. Otherwise, we need to compute the term $P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$ without relying on a finite-rank projection.

Main idea: Construct a hierarchy of equations for the evolution of the memory term and its derivatives.

Let $w_{0k}(t) = P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$. We find

$$\frac{dw_{0k}}{dt} = Pe^{tL}PLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0 QL}QLu_{0k} + w_{1k}(t)$$

where

$$w_{1k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLu_{0k}ds$$

and we have assumed that the memory extends only t_0 units back.

Application to the 1D Burgers equation

Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}. \quad (9)$$

We assume periodic BCs and $u(0, x) = u_0(x)$.

We can have e.g. uncertain viscosity coefficient and/or uncertain initial conditions.

For a simple case with $\nu = 0.03$ one needs about 100 Fourier modes to resolve the solution. If e.g. $u_0(x, \xi) = (1 + \xi)\sin x$ where $\xi \sim U[-1, 1]$, we need the first 7 Legendre polynomials to fully resolve the uncertainty in the solution.

Remark: If we resolve 2 of the 7 expansion coefficients, then for the memory term term we would need to have a basis in 100×2 dimensions. It is prohibitively expensive to use with high orders.

Remark: If we have some prior information about the system then we may be able to reduce the number of arguments of the basis functions. Otherwise, we need to compute the term $P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$ without relying on a finite-rank projection.

Main idea: Construct a hierarchy of equations for the evolution of the memory term and its derivatives.

Let $w_{0k}(t) = P\int_0^t e^{(t-s)L}PLe^{sQL}QLu_{0k}ds$. We find

$$\frac{dw_{0k}}{dt} = Pe^{tL}PLQLu_{0k} - Pe^{(t-t_0)L}PLe^{t_0QL}QLu_{0k} + w_{1k}(t)$$

where

$$w_{1k}(t) = P\int_{t-t_0}^{t} e^{sL}PLe^{(t-s)QL}QLQLu_{0k}ds$$

and we have assumed that the memory extends only t_0 units back.

Application to the 1D Burgers equation

Consider the Burgers equation

$$u_t + uu_x = \nu u_{xx}. \tag{9}$$

We assume periodic BCs and $u(0, x) = u_0(x)$.

We can have e.g. uncertain viscosity coefficient and/or uncertain initial conditions.

For a simple case with $\nu = 0.03$ one needs about 100 Fourier modes to resolve the solution. If e.g. $u_0(x, \xi) = (1 + \xi) \sin x$ where $\xi \sim U[-1, 1]$, we need the first 7 Legendre polynomials to fully resolve the uncertainty in the solution.

Remark: If we resolve 2 of the 7 expansion coefficients, then for the memory term term we would need to have a basis in 100×2 dimensions. It is prohibitively expensive to use with high orders.

Periodically forced nonlinearly damped particle

Consider

$$\frac{du}{dt} = u - u^3 + f(t, w), \tag{7}$$

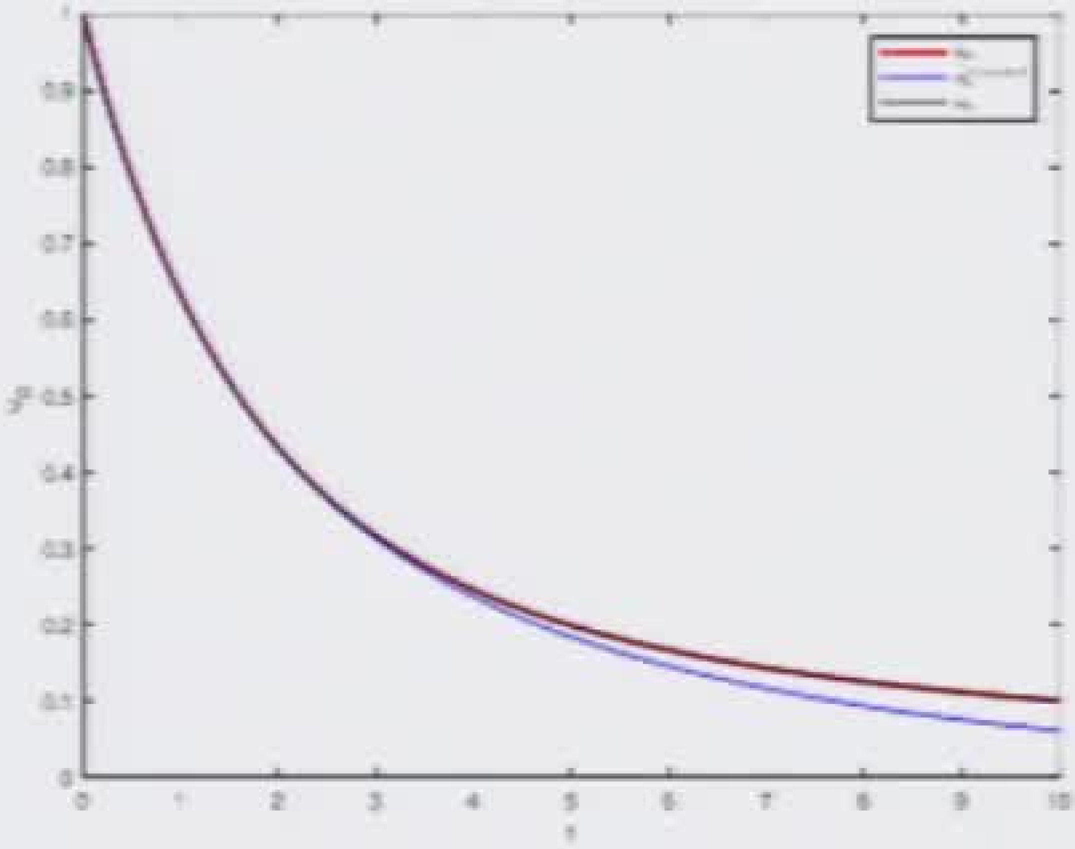
where $f = w \sin t$ and $w \sim U(-1, 1)$. Let $u \approx \sum_{i=0}^{M} u_i L_i(w)$, where $w \sim U[-1, 1]$ and $\{L_i\}$ are the Legendre polynomials. We obtain

$$\frac{du_i}{dt} = u_i - \sum_{j,k,m=0}^{p_f} u_j u_k u_m e_{jkmi} + f_i, \qquad i = 0, \ldots, M, \tag{8}$$

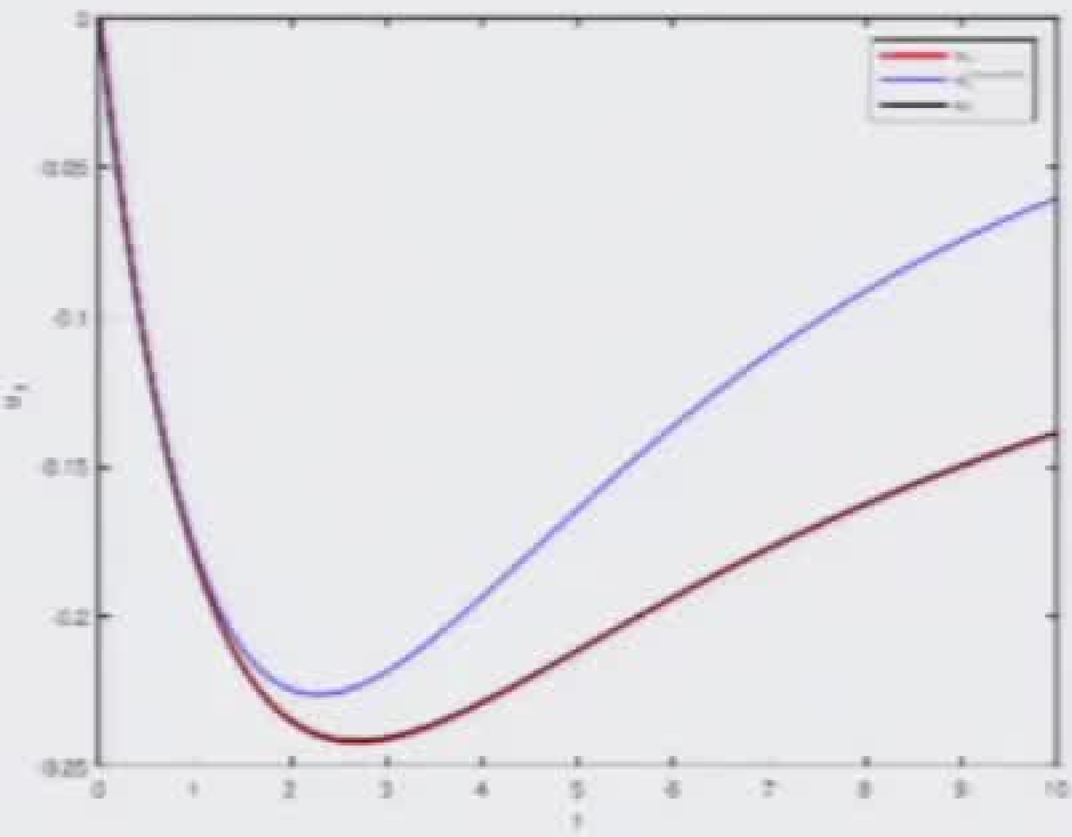
where $e_{jkmi} = \int_{-1}^{1} L_j(w) L_k(w) L_m(w) L_i(w) \frac{1}{2} dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \ldots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \ldots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use Hermite polynomials as before.



(a)



(b)

Figure: Comparison of the solution without memory, with memory and exact solution for the linear equation with uncertain coefficient.

Remark: We construct a reduced model with $\Lambda = 1$ and $M = 6$. The model with memory uses 21 basis functions (highest order 5) for the memory term integrand. The integrands for the memory term can be computed by solving Volterra equations.

Scalar linear differential equation

Consider

$$\frac{du}{dt} = -\kappa u, \tag{5}$$

where $\kappa \sim U[0,1]$. Let $u \approx \sum_{r=0}^{M} u_r L_r(w)$, where $w \sim U[-1,1]$ and $\{L_r\}$ are the Legendre polynomials. We obtain

$$\frac{du_r}{dt} = -\sum_{i=0}^{1}\sum_{j=0}^{M} k_i u_j e_{ijr}, \qquad r = 0, \dots, M, \tag{6}$$

where $e_{ijr} = \int_{-1}^{1} L_i(w) L_j(w) L_r(w) \frac{1}{2} dw$.

The projection P is defined as, $(Pf)(\hat{u}) = f(\hat{u}, \tilde{0})$, with $\hat{u} = (u_0, u_1, \dots, u_\Lambda)$ and $\tilde{u} = (u_{\Lambda+1}, \dots, u_M)$.

Remark: To compute the memory term in *closed form* we need to introduce a finite-rank projection (in addition to P). We use

$$(P'\varphi_j)(\hat{u}, t) \approx \sum_{\nu} (\varphi_j(u,t), h^{\nu}(\hat{u})) h^{\nu}(\hat{u}),$$

where h^{ν} are Hermite polynomials.