Error analysis for coupled time-dependent Navier-Stokes and Darcy flows

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SIAM GS

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Outline

- Coupled Darcy and Navier-Stokes model
- Discontinuous Galerkin scheme
- Analysis and numerical results

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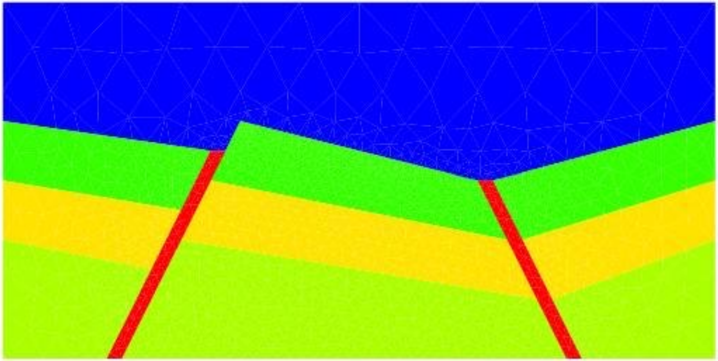
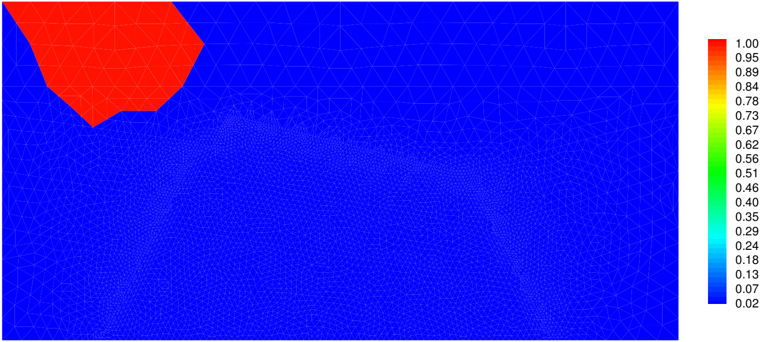
A MultiPhysics Problem

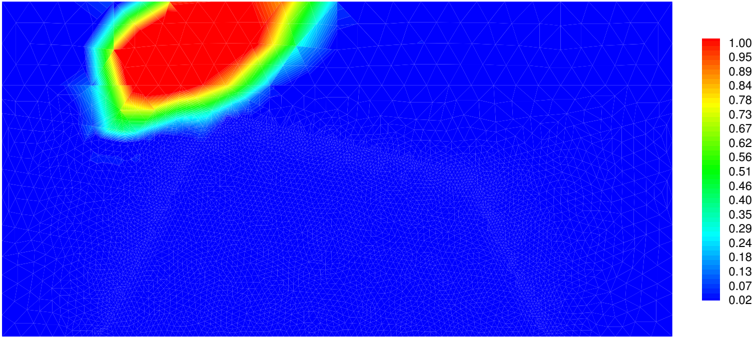
Applications in environment, energy, biomedicine, manufacturing...



Initial Plume



Contamination Simulation



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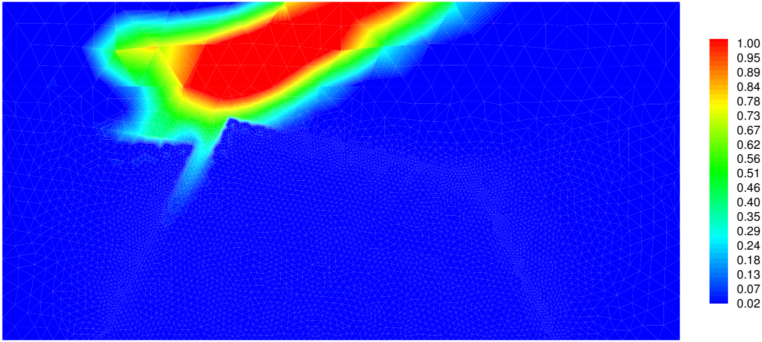

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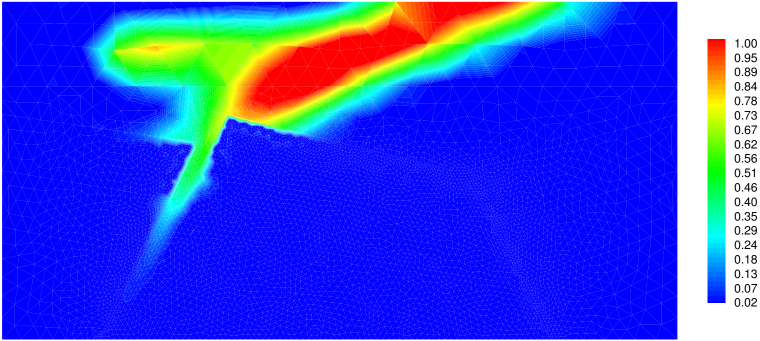

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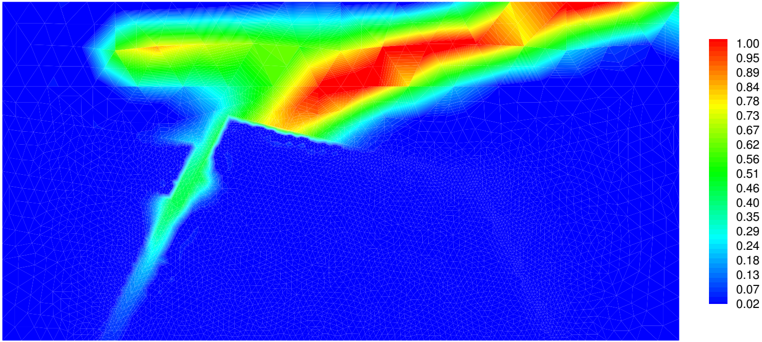

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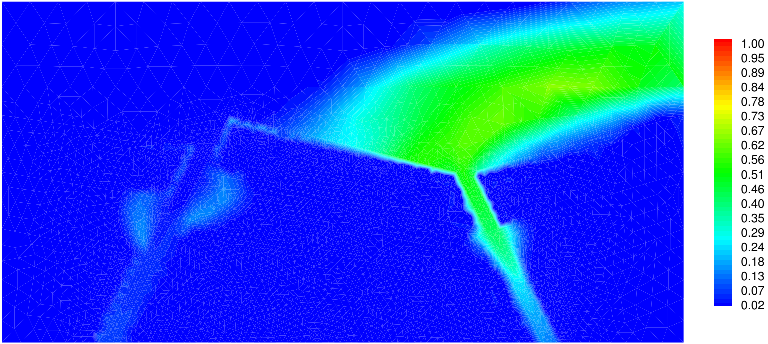

Navier-Stokes/Darcy

Navier-Stokes in Ω_1: (surface flow)

$$
\begin{aligned}
\frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (2\mu \mathbf{D}(\mathbf{u}) - p_1 \mathbf{I}) + \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{f}_1, \quad \text{in} \quad \Omega_1 \times (0, T) \\
\nabla \cdot \mathbf{u} &= 0, \quad \text{in} \quad \Omega_1 \times (0, T)
\end{aligned}
$$

Darcy in Ω_2: (porous media flow)

$$
-\nabla \cdot (\mathbf{K} \nabla p_2) = f_2, \quad \text{in} \quad \Omega_2 \times (0, T)
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Navier-Stokes/Darcy

Navier-Stokes in Ω_1: (surface flow)

$$\begin{aligned} \frac{\partial \mathbf{u}}{\partial t} - \nabla \cdot (2\mu \mathbf{D}(\mathbf{u}) - p_1 \mathbf{I}) + \mathbf{u} \cdot \nabla \mathbf{u} &= \mathbf{f}_1, \quad \text{in} \quad \Omega_1 \times (0, T) \\ \nabla \cdot \mathbf{u} &= 0, \quad \text{in} \quad \Omega_1 \times (0, T) \end{aligned}$$

Darcy in Ω_2: (porous media flow)

$$-\nabla \cdot (\mathbf{K} \nabla p_2) = f_2, \quad \text{in} \quad \Omega_2 \times (0, T)$$

Interface conditions on interface $\Gamma_{12} = \partial\Omega_1 \cap \partial\Omega_2$

$$\mathbf{u} \cdot \mathbf{n}_{12} = -\mathbf{K} \nabla p_2 \cdot \mathbf{n}_{12}, \text{ on } \Gamma_{12} \times (0, T) \tag{1}$$

$$\left((-2\mu \mathbf{D}(\mathbf{u}) + p_1 \mathbf{I}) \mathbf{n}_{12}\right) \cdot \mathbf{n}_{12} = p_2, \text{ on } \Gamma_{12} \times (0, T) \tag{2}$$

$$\mathbf{u} \cdot \boldsymbol{\tau}_{12}^j = -2\mu G^j \left(\mathbf{D}(\mathbf{u}) \mathbf{n}_{12}\right) \cdot \boldsymbol{\tau}_{12}^j, 1 \le j \le d-1, \text{ on } \Gamma_{12} \times (0, T) \tag{3}$$

where

$$G^j = \frac{\mu \alpha}{(\mathbf{K} \boldsymbol{\tau}_{12}^j, \boldsymbol{\tau}_{12}^j)^{1/2}}$$

Coupling via Bilinear Forms

- Γ_{12}: interface between Ω_1 and Ω_2
- Integrate by parts the divergence operators in momentum and Darcy equations (with test functions $\mathbf{v}$ in Ω_1 and q in Ω_2)

$$T_{12} = -\int_{\Gamma_{12}} (2\mu \mathbf{D}(\mathbf{u}) - p_1 \mathbf{I})\mathbf{n}_1 \cdot \mathbf{v} - \int_{\Gamma_{12}} \mathbf{K}\nabla p_2 \cdot \mathbf{n}_2 q$$

- Define $\mathbf{n}_{12} = \mathbf{n}_1$ and use interface conditions to rewrite

$$T_{12} = \int_{\Gamma_{12}} p_2 \mathbf{v} \cdot \mathbf{n}_{12} - \int_{\Gamma_{12}} \mathbf{u} \cdot \mathbf{n}_{12} q + \sum_{j=1}^{d-1} \frac{1}{G^j} \int_{\Gamma_{12}} \mathbf{u} \cdot \boldsymbol{\tau}_{12}^j \mathbf{v} \cdot \boldsymbol{\tau}_{12}^j$$

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Variational methods allow for easy handling of interface conditions via traces of functions

Cesmelioglu, Girault, Riviere, ESAIM M2AN, 2013.

Discrete Scheme

Find $(\mathbf{u}_h^{n+1}, p_{1h}^{n+1}, p_{2h}^{n+1}) \in \mathbf{X}^h \times M_1^h \times M_2^h$, for all $0 \le n \le N_T$ such that

$$
\begin{aligned}
&(\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}, \mathbf{v})_{\Omega_1} + a_S(\mathbf{u}_h^{n+1}, \mathbf{v}) + b_S(\mathbf{v}, p_{1h}^{n+1}) + c_{NS}(\mathbf{u}_h^n, \mathbf{u}_h^n; \mathbf{u}_h^{n+1}, \mathbf{v}) \\
&+ a_D(p_{2h}^{n+1}, q) + (p_{2h}^{n+1}, \mathbf{v} \cdot \mathbf{n}_{12})_{\Gamma_{12}} - (\mathbf{u}_h^{n+1} \cdot \mathbf{n}_{12}, q)_{\Gamma_{12}} + \sum_{j=1}^{d-1} \frac{1}{G^j} (\mathbf{u}_h^{n+1} \cdot \boldsymbol{\tau}_{12}^j, \mathbf{v} \cdot \boldsymbol{\tau}_{12}^j)_{\Gamma_{12}} \\
&= (\mathbf{f}_1^{n+1}, \mathbf{v})_{\Omega_1} + (f_2^{n+1}, q)_{\Omega_2}, \quad \forall (\mathbf{v}, q) \in \mathbf{X}^h \times M_2^h
\end{aligned}
$$

$$
b_S(\mathbf{u}_h^{n+1}, q) = 0. \quad \forall q \in M_1^h.
$$

Interior penalty discontinuous Galerkin discretizations
Discontinuous piecewise polynomials of degree k for NSE velocity and Darcy pressure, and $k-1$ for NSE pressure

Existence and Uniqueness of Solution

- Linear problem: Easy?

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- Because of coupling, discretization of $\mathbf{u} \cdot \nabla \mathbf{u}$ does not satisfy positivity property

$$\forall \mathbf{u}_h, \mathbf{v}_h \in \mathbf{X}^h, \quad c_{NS}(\mathbf{u}_h, \mathbf{u}_h; \mathbf{v}_h, \mathbf{v}_h) \geq \frac{1}{2}(\mathbf{u}_h \cdot \mathbf{n}_{12}, \mathbf{v}_h \cdot \mathbf{v}_h)_{\Gamma_{12}}.$$

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- To control this term, we need an a priori bound on velocity

$$\|\mathbf{u}_h^n\|_{\mathrm{DG},\Omega_1} \leq \mu C^*, \quad \forall n$$

- Challenge: bound in $L^\infty(0, T; H^1)$ is needed instead of $L^\infty(0, T; L^2)$!

Bounding Velocity in $L^\infty(0, T; H^1(\mathcal{T}_h))$

- We need to bound the time derivative of velocity

$$\|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\|_{L^2(\Omega_1)} \leq \mathcal{C}$$

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$$\|\frac{\mathbf{u}_h^{n+1} - \mathbf{u}_h^n}{\Delta t}\|_{L^2(\Omega_1)} \leq \mathcal{C}$$

- **Theorem:** There is a constant $\tilde{C}$, independent of h such that for all $\mathbf{u}_h \in \mathbf{V}^h$ and for all $q_{2h} \in M_2^h$,

$$|(q_{2h}, \mathbf{u}_h \cdot \mathbf{n}_{12})_{\Gamma_{12}}| \leq \tilde{C}\|q_{2h}\|_{\mathrm{DG},\Omega_2}\|\mathbf{u}_h\|_{L^2(\Omega_1)}$$

Key ingredient: regularization of discrete functions by Scott-Zhang interpolant

Existence and Uniqueness

Theorem:
Assume small data condition that depends on:

$$\mu, \|\mathbf{f}_1\|_{L^\infty(L^2)}, \|\delta_t \mathbf{f}_1\|_{\ell^2(L^2)}, \|f_2\|_{L^\infty(L^2)}, \|\delta_t f_2\|_{\ell^2(L^2)}$$

Then there is a unique solution $(\mathbf{u}_h^{n+1}, p_{1h}^{n+1}, p_{2h}^{n+1})$ to the numerical scheme.

Notation:

$$\delta_t g^i = \frac{g^{i+1} - g^i}{\Delta t}$$

Error Estimates

Under the small data assumption, there is a constant C independent of h and Δt such that for all $1 \leq m \leq N_T$, we have,

$$\|\mathbf{u}^m - \mathbf{u}_h^m\|_{L^2(\Omega_1)}^2 + \mu \Delta t \sum_{n=1}^{m} \|\mathbf{u}^n - \mathbf{u}_h^n\|_{\mathrm{DG},\Omega_1}^2 + \Delta t \sum_{n=1}^{m} \|p_2^n - p_{2h}^n\|_{\mathrm{DG},\Omega_2}^2$$
$$+ \Delta t \sum_{n=1}^{m} \sum_{j=1}^{d-1} \|\frac{1}{\sqrt{G^j}}(\mathbf{u}^n - \mathbf{u}_h^n) \cdot \tau_{12}^j\|_{L^2(\Gamma_{12})}^2 \leq C(h^{2k} + \Delta t^2).$$

This bound is valid if the exact solution satisfies the following regularity assumptions: $\mathbf{u} \in L^\infty(0, T; H^{k_1+1}(\Omega_1)^d)$, $\frac{\partial \mathbf{u}}{\partial t} \in L^2(0, T; L^\infty(\Omega_1)^d) \cap L^2(0, T; H^{k_1}(\Omega_1)^d)$, $\frac{\partial^2 \mathbf{u}}{\partial t^2} \in L^2((0, T) \times \Omega_1)^d$ and $p_2 \in L^\infty(0, T; H^{k_2+1}(\Omega_2))$.

Error Bounds for NSE Pressure

We first need to control the error in the discrete time derivative of velocity in Ω_1:

$$\|\delta_t(\mathbf{u} - \mathbf{u}_h)\|_{\ell^2(L^2)} \leq C(h^k + \Delta t)$$

This is obtained under the condition:

$$\frac{h^2 + \Delta t^2}{\min_{T \in \mathcal{T}_h^1} h_T} \leq \mathcal{C}$$

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Then we can show

$$\|p_1 - p_{1h}\|_{\ell^2(L^2)} \leq C(h^k + \Delta t)$$

Smooth Solutions: $k = 2$

$$\Omega = \Omega_1 \cup \Omega_2, \quad \Omega_1 = (0,1) \times (0,1), \quad \Omega_2 = (0,1) \times (-1,0)$$

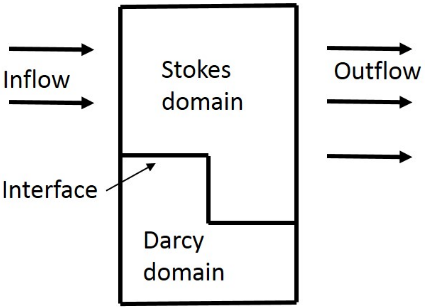
Velocity and pressure in NSE subdomain:

h	$\|\|\mathbf{u} - \mathbf{u}_h\|\|_{L^2(\Omega_1)}$	CR	$\|\|\nabla_h(\mathbf{u} - \mathbf{u}_h)\|\|_{L^2(\Omega_1)}$	CR	$\|\|p_1 - p_{1h}\|\|_{L^2(\Omega_1)}$	CR
1/2	2.694e-02		4.383e-01		4.974e-01	
1/4	4.300e-03	2.65	1.151e-01	1.93	1.608e-01	1.63
1/8	5.813e-04	2.89	2.871e-02	2.00	4.685e-02	1.78
1/16	7.385e-05	2.98	7.051e-03	2.03	1.263e-02	1.89
1/32	1.024e-05	2.85	1.738e-03	2.02	3.282e-03	1.94

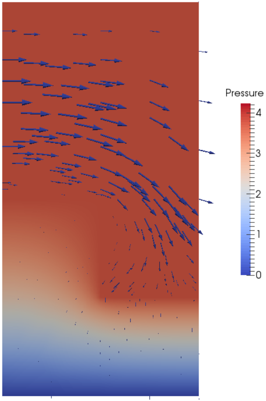
Pressure in Darcy subdomain:

h	$\|\|p_2 - p_{2h}\|\|_{L^2(\Omega_2)}$	Conv.	$\|\|\nabla_h(p_2 - p_{2h})\|\|_{L^2(\Omega_2)}$	Conv.
1/2	3.1803e-03		5.4874e-02	
1/4	4.9972e-04	2.67	1.4217e-02	1.95
1/8	6.8328e-05	2.87	3.6107e-03	1.98
1/16	8.8834e-06	2.94	9.0948e-04	1.99
1/32	1.1711e-06	2.92	2.2822e-04	1.99

Polygonal Interface: Set-up



Polygonal Interface: Pressure and Streamlines



Interface Conditions

Define errors in imposing interface conditions in L^2 norm:

$$
\begin{aligned}
E_7 &= \|\mathbf{u} \cdot \mathbf{n}_{12} + \mathbf{K}\nabla p_2 \cdot \mathbf{n}_{12}\|_{L^2(\Gamma_{12})} \\
E_8 &= \|p_2 - \left((-2\mu\mathbf{D}(\mathbf{u}) + p_1\mathbf{I})\mathbf{n}_{12}\right) \cdot \mathbf{n}_{12}\|_{L^2(\Gamma_{12})} \\
E_9 &= \|\mathbf{u} \cdot \boldsymbol{\tau}_{12} + 2\mu G^1 \left(\mathbf{D}(\mathbf{u})\mathbf{n}_{12}\right) \cdot \boldsymbol{\tau}_{12}\|_{L^2(\Gamma_{12})}
\end{aligned}
$$

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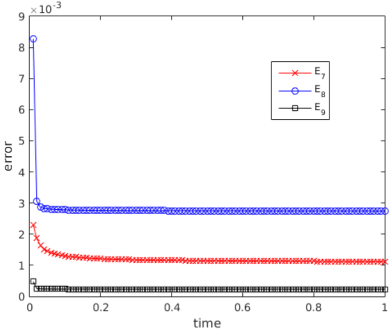

Conclusions

- DG scheme for coupling time-dependent NSE and Darcy
- Convergence analysis for any polynomial degree
- Interface curve or surface does not have to be smooth
- Acknowledgement: NSF

Chaabane, Girault, Puelz, Riviere. Journal of Computational and Applied Mathematics, 2017.

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