Synchronization over Cartan Motion Groups

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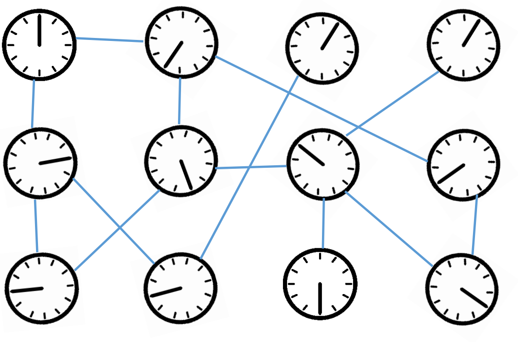
A joint work with Onur Özyesil and Amit Singer

Intro to Synchronization

The problem: define a unified clock given a set the time differences between locations.



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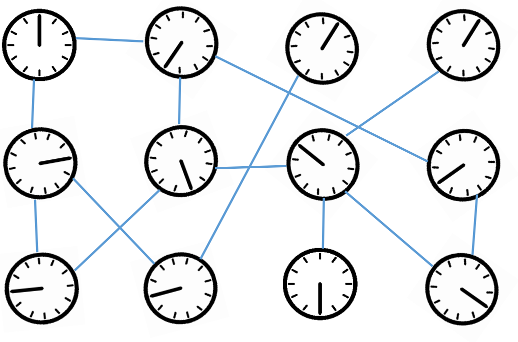

Mathematical interpretation – overdetermined system of linear equations, modulo 24 hours.

The Mathematical Problem of Synchronization

Problem formulation

Estimate n unknown group elements $\{g_i\}_{i=1}^n$ from a set of measurements g_{ij} of their ratios

$$g_{ij} \approx g_i g_j^{-1}, \quad 1 \leq i < j \leq n.$$

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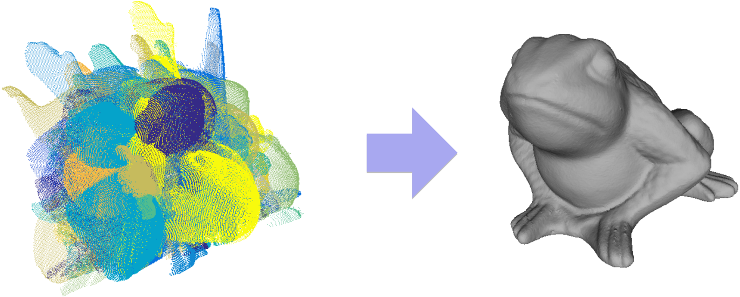
Any solution is up to a **global alignment**, as seen by

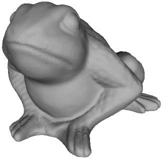
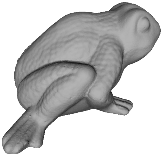
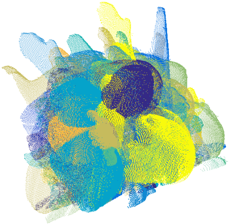
$$g_i g_j^{-1} = (g_i \mathbf{g})(g_j \mathbf{g})^{-1}.$$

3D Registration via Synchronization



Database available online, consists of 24 point clouds, each of between $24,000$ to $36,000$ points

State-of-the-art Registration

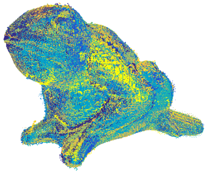




Numerical result

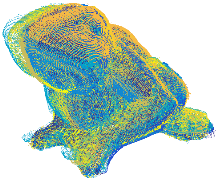
Available data	*Our method*	Separation	Spectral	Diffusion-based
29%	**.00175**	**.00176**	> 0.01	> 0.01
59%	**.00175**	**.00175**	> 0.01	> 0.01

State-of-the-art Registration

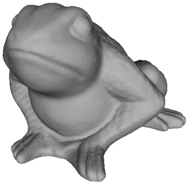
Visual result



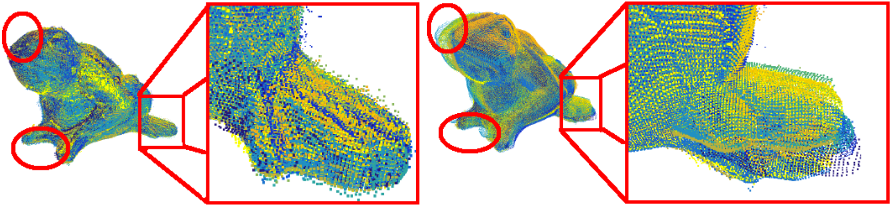
(a) Good registration by our method



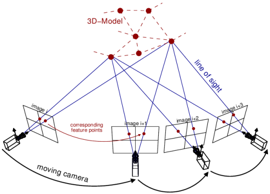
(b) Bad registration by spectral method



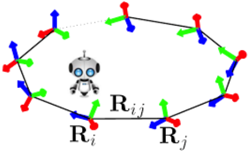
(c) Model



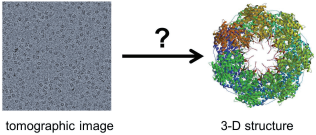
More Real World Applications of Synchronization

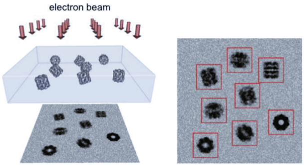


Structure from Motion (vision)



Pose graph optimization (robotics)





Estimate viewing directions (Cryo-EM)

Synchronization over Non-Compact Groups

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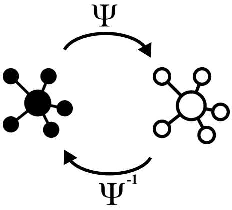

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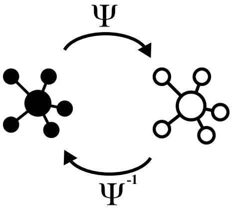
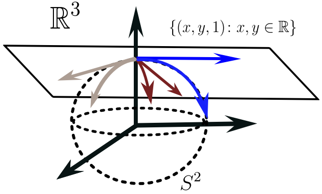


For Ψ we use the notion of **compactification**



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- The Cartan decomposition,

$$G_0 \xrightarrow{algebra} \mathfrak{g} = \mathfrak{t} \oplus \mathfrak{p},$$

defines

$$\boxed{K \xleftarrow{group} \mathfrak{t}}.$$

- Here $\mathfrak{g}$ is skew-symmetric matrices where

$$\mathfrak{t} = \left\{ \begin{bmatrix} Z_{d\times d} & 0 \\ 0 & 0 \end{bmatrix} : Z + Z^T = 0 \right\}$$

and

$$\mathfrak{p} = \left\{ \begin{bmatrix} \mathbf{0}_{d\times d} & b \\ -b^T & 0 \end{bmatrix} : b \in \mathbb{R}^d \right\}.$$

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- The associated Cartan motion group

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$$G = K \ltimes \mathfrak{p} \quad \text{with}$$

$(k_1, v_1)(k_2, v_2) =$
$\left(k_1 k_2, v_1 + \mathrm{Ad}_{k_1}(v_2)\right)$.

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We adopt one family of contractions, by Dooley and Rice (1983), $\{\Psi_\lambda\}_{\lambda \geq 1}$ defined based on the Cartan decomposition

$$\Psi_\lambda(k, v) = \exp\big(v/\lambda\big)k, \qquad (k, v) \in G.$$

The parameter λ induces the contraction.

The Fundamental Requirements

- **Invertibility**: guaranteed when v/λ is inside the injectivity radius of the exponential of G_0. For $G = \mathrm{SE}(d)$,

$$\boxed{\|b\|/\lambda < \pi}.$$

- **Approximated homomorphism**: conditions for admissible compactification,

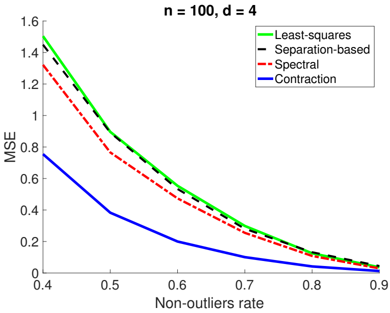
$$\Psi_\lambda\left(g^{-1}\right) = \left(\Psi_\lambda(g)\right)^{-1}, \qquad g \in G,$$

$$\left\|\Psi_\lambda(g_1 g_2) - \Psi_\lambda(g_1)\Psi_\lambda(g_2)\right\|_F = \mathcal{O}\left(\frac{1}{\lambda^2}\right), \quad g_1, g_2 \in G.$$

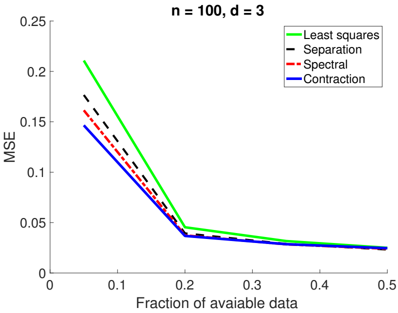
These conditions allow us to relate the metrics on G_0 and G.

Final Numerical Examples

Two Scenarios



(a) Outliers



(b) Missing data with noise

Th-th-th-that's all folks!

Thank you

Analysis Highlights – Synchronization via Contraction

- **Global alignment**: the choice in compact domain matters as

$$\Psi_\lambda^{-1}\left(Q_i\mathbf{Q}\right)\Psi_\lambda^{-1}\left(\left(Q_j\mathbf{Q}\right)^{-1}\right) \neq \Psi_\lambda^{-1}\left(Q_i\right)\left(\Psi_\lambda^{-1}(Q_j)\right)^{-1}.$$

- **Effect of parameter** λ: to retain the consistency of synchronization

$$\Psi_\lambda(g_{ij})\Psi_\lambda(g_{j\ell}) \approx \Psi_\lambda(g_{i\ell}).$$

- **Noise analysis**.

Additional example of Cartan motion group

Example (matrix motion group)

Let $G_0 = \mathrm{O}(d+\ell)$. Denote by $\mathrm{M}(d,\ell)$ the space of all real matrices of order $d \times \ell$.
Cartan decomposition of the Lie algebra:
$\mathfrak{t}$ is the Lie algebra of $\mathrm{O}(d) \times \mathrm{O}(\ell)$ and $\mathfrak{p} = \mathrm{M}(d,\ell)$.
This decomposition yields the so called **matrix motion group**,

$$G = \big(\mathrm{O}(d) \times \mathrm{O}(\ell)\big) \ltimes \mathrm{M}(d,\ell).$$

This Cartan motion group is associated with the quotient space G/K which is the Grassmannian manifold (all d dimensional linear subspaces of $\mathbb{R}^{d+\ell}$).

Additional example of Cartan motion group

Example

Let $G_0 = \mathrm{SU}(d)$, the group of all unitary matrices with determinants equal to one. One Cartan involution of $\{X \colon X + X^T = 0, \quad \mathrm{tr}(X) = 0\}$ is to a real part (same as the Lie algebra of $\mathrm{SO}(d)$) and it's orthogonal complement, denoted by $W = \mathrm{SO}(d)^{\perp}$.
Then, the Cartan Motion group in this case is

$$\mathrm{SO}(d) \ltimes W$$

.

Analysis of noisy synchronization over SE(d)

Assume a noise model

$$g_{ij} = g_i N_{ij} g_j^{-1}, \quad N_{ij} = (\upsilon_{ij}, a_{ij}) \in G.$$

Some algebraic simplification leads to en explicit form of the mapped synchronization problem

$$\Psi_\lambda(g_{ij}) = \Psi_\lambda\left(g_i\right) W_{ij} \Psi_\lambda\left(g_j\right)^T.$$

- If $\mathbb{E}\left[a_{ij}\right] = 0$ then $\Psi_\lambda\left(N_{ij}\right) = \exp\left(a_{ij}/\lambda\right)\upsilon_{ij}$ (given in Cartan form) is a good approximation to (the Cartan form) of $\mathbb{E}\left[W_{ij}\right]$.
- In the simplified model $W_{ij} = \Psi_\lambda\left(N_{ij}\right)$ we show how to "translate" the phase transition point of rotations synchronization to the setting of rigid motion synchronization.