# Preconditioning a mass-conserving DG discretization of the Stokes Equations

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October 27, 2015

# Big Picture

Efficient solvers for coupled multiphysics systems

# What multiphysics means to me

Thinking about coupled systems of

- Incompressible (nonlinear) fluid or solid dynamics
- Something else
  - Electromagnetics
  - Heat transfer

Consider linearization and finite-element discretization, resulting in linear systems of the form

$$\mathcal{A}x = \begin{bmatrix} F & Z & B \\ Y & D & 0 \\ B^T & 0 & 0 \end{bmatrix} \begin{bmatrix} x_{\mathbf{u}} \\ x_a \\ x_p \end{bmatrix} = \begin{bmatrix} r_{\mathbf{u}} \\ r_a \\ r_p \end{bmatrix} .$$

# Discretization and Conservation

All conservation laws are equal, but some are more equal than others

Strong emphasis on conservation of mass

- Assumed in standard conservation of momentum equation

$$
\begin{aligned}
\frac{\partial \rho u_i}{\partial t} + \nabla \cdot (\rho u_i \mathbf{u}) &= Q_i \\
\rho \frac{\partial u_i}{\partial t} + u_i \frac{\partial \rho}{\partial t} + u_i \nabla \cdot (\rho \mathbf{u}) + \rho \mathbf{u} \cdot \nabla u_i &= Q_i \\
\rho \left( \frac{\partial u_i}{\partial t} + \mathbf{u} \cdot \nabla u_i \right) &= Q_i .
\end{aligned}
$$

- Seen as major source of errors in coupled systems

# Finite Elements for (Navier-)Stokes

Discrete weak form: find $(\mathbf{u}_h, p_h) \in \mathbf{V}_h \times Q_h$ such that

$$
\begin{aligned}
a(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) &= (\mathbf{f}, \mathbf{v}_h) \qquad && \forall \mathbf{v}_h \in \mathbf{V}_h \\
b(\mathbf{u}_h, q_h) &= 0 && \forall q_h \in Q_h,
\end{aligned}
$$

Standard finite-element approach:

- $a(\mathbf{u}, \mathbf{v})$ is coercive and continuous in $H^1(\Omega)$
- $b(\mathbf{u}, p)$ satisfies inf-sup condition
- Need inf-sup stable finite-element pair $\mathbf{V}_h$, $Q_h$
  - Taylor-Hood, **Q**2-Q1 or **P**2-P1
  - MINI element, Crouzeix-Raviart
  - Stabilized equal order

Boffi, Brezzi, Fortin, Springer 2013

# Conservative Finite Elements

None of these choices guarantee **strong** satisfaction of incompressibility constraint

Seek to ensure that

$$\int q_h \nabla \cdot \mathbf{u}_h = 0 \quad \forall q_h \in Q_h \text{ implies } \nabla \cdot \mathbf{u}_h = 0 \text{ pointwise.}$$

One approach: ensure $\nabla \cdot \mathbf{u}_h \in Q_h$

- Get this with BDM1 elements for $\mathbf{u}_h$ and $P0$ for $Q_h$
- BDM1 gives piecewise linear approximation to $\mathbf{u}_h$ with continuous normal components across edges

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Wang, Ye, SINUM 2007
Cockburn, Kanschat, Schötzau, J. Sci. Comp. 2009
Greif, Li, Schötzau, Wei, CMAME 2010
Ayuso de Dios, Brezzi, Marini, Xu, Zikatanov, J. Sci. Comp. 2014

# Linear Solvers

Two intertwined issues:

1. Do solvers for Galerkin discretizations of (Navier-)Stokes extend to this DG discretization?
   - Block factorizations
   - Monolithic multigrid
2. How do we handle coupling with multiphysics?
   - MHD

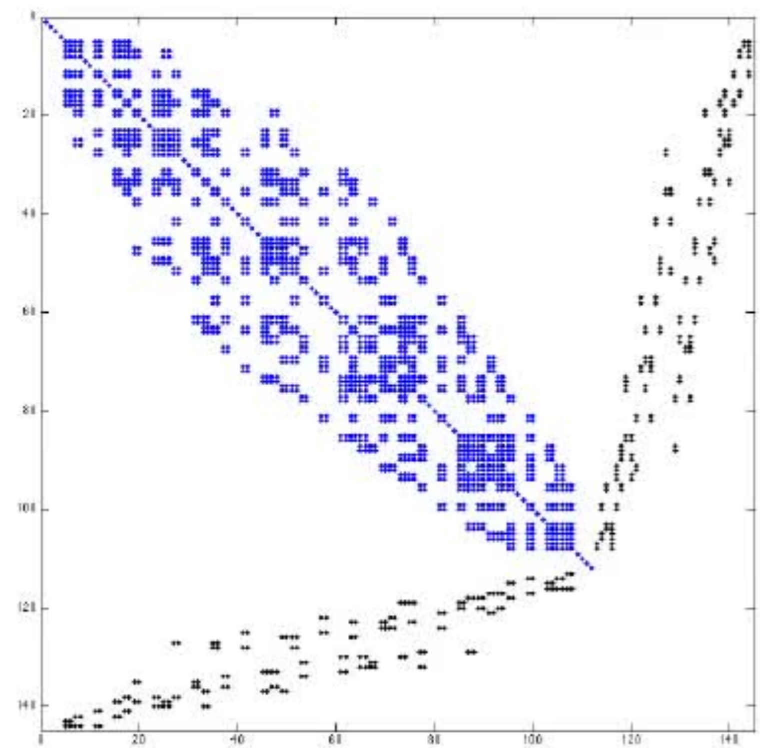
Work in progress.

For this talk, answer point 1 above, for Stokes.

$$\mathcal{A}x = \begin{bmatrix} A & B \\ B^T & 0 \end{bmatrix} \begin{bmatrix} x_{\mathbf{u}} \\ x_p \end{bmatrix} = \begin{bmatrix} r_{\mathbf{u}} \\ r_p \end{bmatrix}.$$

# What's So Hard?

- BDM1 discretization gives a fully coupled vector Laplacian
  - Taylor-Hood gives block diagonal structure with scalar Laplacians
- DG stabilization expands stencils



# Nonstandard Approaches

Two existing preconditioners

Auxiliary space preconditioner:

- Use projections onto curl space
- Solve auxiliary systems with scalar Laplacians
- Still need to deal with solver for velocity block (direct solver used)

Augmented Uzawa method:

- MG for velocity block using vertex-based block SOR relaxation

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Ayuso de Dios, Brezzi, Marini, Xu, Zikatanov, J. Sci. Comp. 2014
Hong, Kraus, Xu, Zikatanov, Numer. Math. 2015

# Lots of Questions

- Are these approaches necessary?
- What about "standard" approaches?
- Does "standard" multigrid work for velocity block?
- What are optimal parameters?

# Standard Approach

Consider "standard" block-factorization preconditioner

- Block diagonal and triangular form
- Multigrid V(1,1) cycle for velocity block
- BDM1 finite-element interpolation operator, $P$
- Restriction is $P^T$, Galerkin coarsening, $A_c = P^T AP$
- Pointwise SGS relaxation, under-relaxation parameter $\omega$
- Pressure mass matrix
  - P0 space gives diagonal matrix

Good signs:

- Know "exact" block preconditioners work well
- Two-level method for velocity block yields good results

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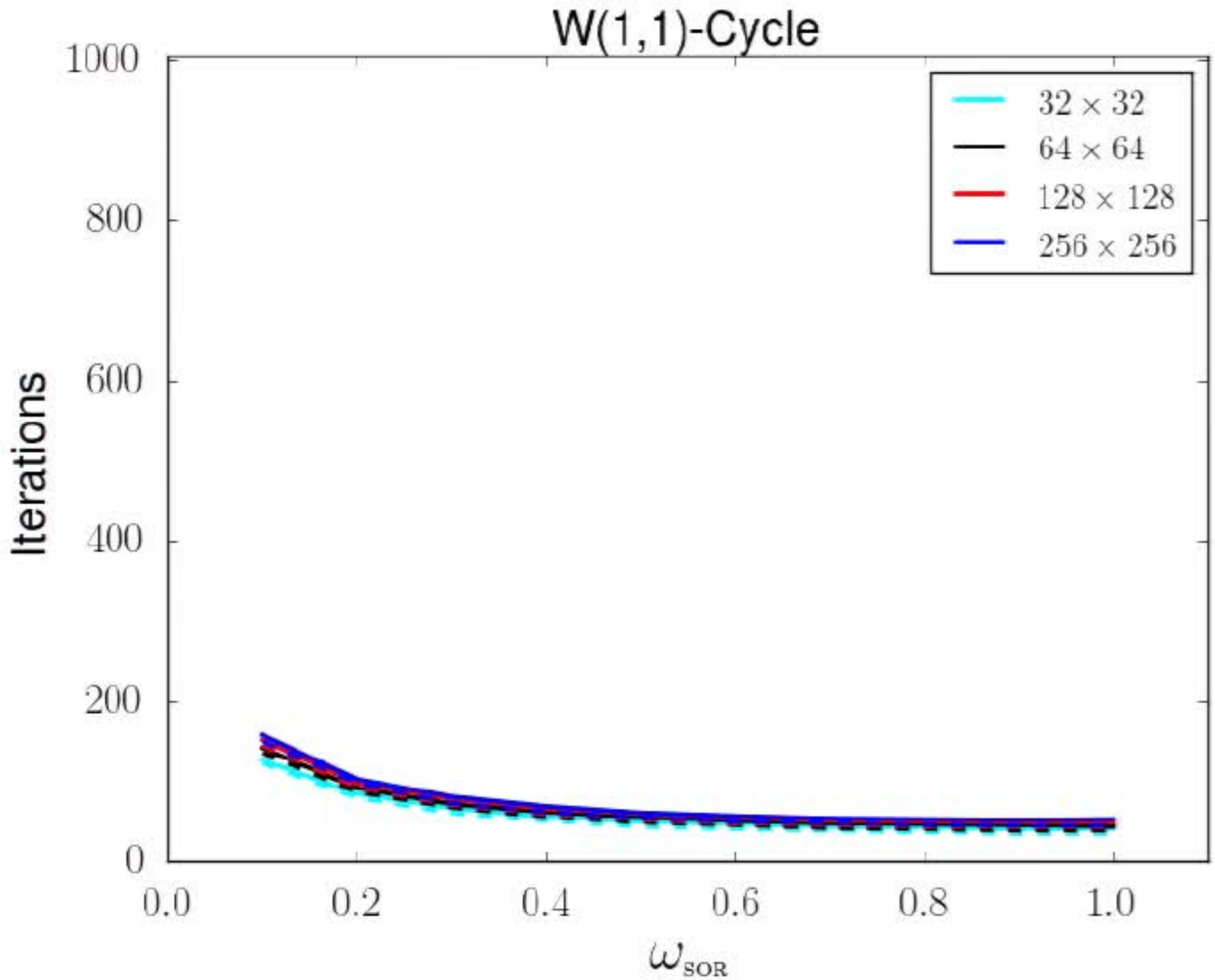
Elman, Silvester, Wathen, Oxford University Press 2014
Murphy, Golub, Wathen, SISC 2000
Verfürth, IMA JNA 1984

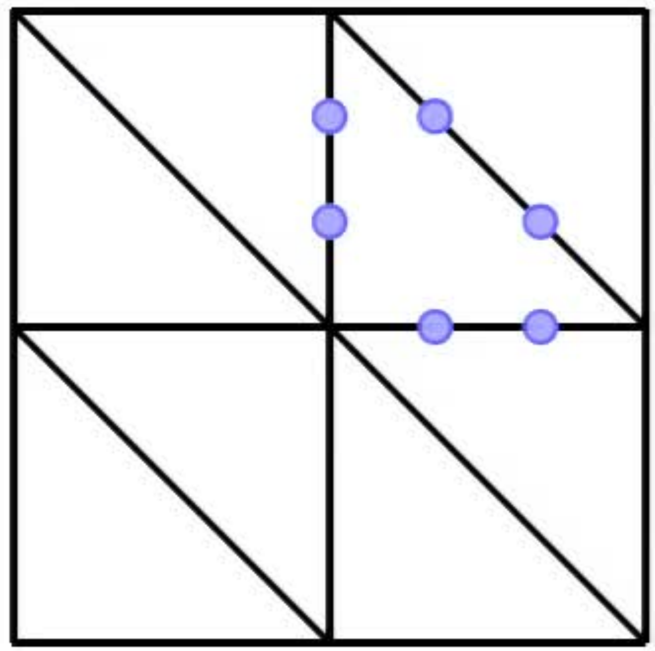
# W-Cycles



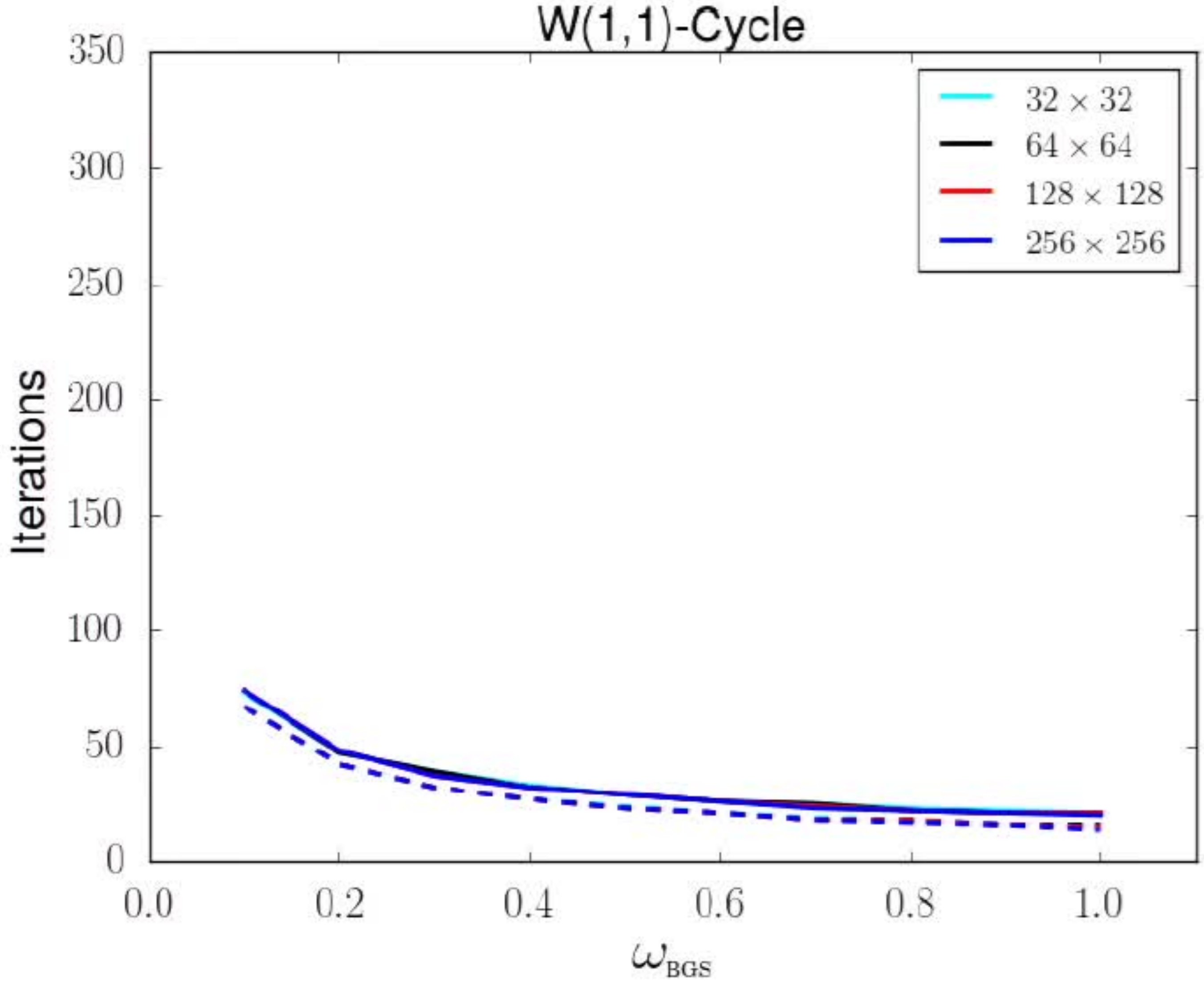

Manageable growth in iterations (41 $\rightarrow$ 53; 35 $\rightarrow$ 45)

# Block Relaxation

Replace pointwise Gauss-Seidel with element-wise overlapping block GS



# Block Relaxation Iterations




Improved and steady iteration counts (21 $\rightarrow$ 19; 16 $\rightarrow$ 14)

# CPU Times

Multigrid setup times

| | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|
| Point-SGS | 0.022 | 0.083 | 0.33 | 1.31 | 5.21 |
| Blk-GS | 0.033 | 0.13 | 0.50 | 1.99 | 8.04 |

Linear solve times

| | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|
| Blk-Diag Point-SGS* | 0.13 | 0.46 | 1.94 | 8.18 | 32.68 |
| Blk-Diag Blk-GS | 0.37 | 1.57 | 6.36 | 25.26 | 103.50 |
| Blk-Tri Point-SGS | 0.13 | 0.49 | 2.09 | 8.65 | 37.79 |
| Blk-Tri Blk-GS | 0.27 | 1.22 | 4.82 | 19.10 | 76.19 |

# Monolithic Multigrid

Natural question of comparison to monolithic approaches

From a multigrid viewpoint, choice of relaxation

- Distributed Relaxation
- Vanka Relaxation
- Braess-Sarazin Relaxation

plus finite-element interpolation and Galerkin(?) coarsening

Ignore distributed relaxation

- Relation between continuum and discrete commutators confused by DG

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Brandt, Dinar, in *Numerical methods for PDEs*, Acad. Press 1979
Vanka, CMAME 1986
Braess, Sarazin, App. Num. Math. 1997

# Vanka Options

Many possible variants for $\hat{M}_{\ell\ell}$

- Keep full constraint row/column
- Approximate velocity block
  - Diagonal
  - Block diagonal

Also need to choose parameter(s)

- Local Fourier Analysis
- Brute-force testing

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Sivaloganathan, CPC 1991
MacLachlan, Oosterlee, NLAA 2011

# Braess-Sarazin

Preserve saddle-point structure in relaxation

$$\begin{bmatrix} x_u \\ x_p \end{bmatrix} \leftarrow \begin{bmatrix} x_u \\ x_p \end{bmatrix} + \omega_{\text{BS}} \begin{bmatrix} \alpha_{\text{BS}} C & B^T \\ B & 0 \end{bmatrix}^{-1} \left( \begin{bmatrix} r_u \\ r_p \end{bmatrix} - \mathcal{A} \begin{bmatrix} x_u \\ x_p \end{bmatrix} \right)$$

Choose $C$ as preconditioner for velocity block that:

- Is easy to apply
- Gives sparse Schur complement

In practice, use "inexact" variant

- approximate solves with $BC^{-1}B^T$

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Braess, Sarazin, App. Num. Math. 1997
Zulehner, Computing 2000

# Braess-Sarazin Variants

Consider two choices for $C$

- Diagonal of velocity block
- Block-diagonal, edge-wise

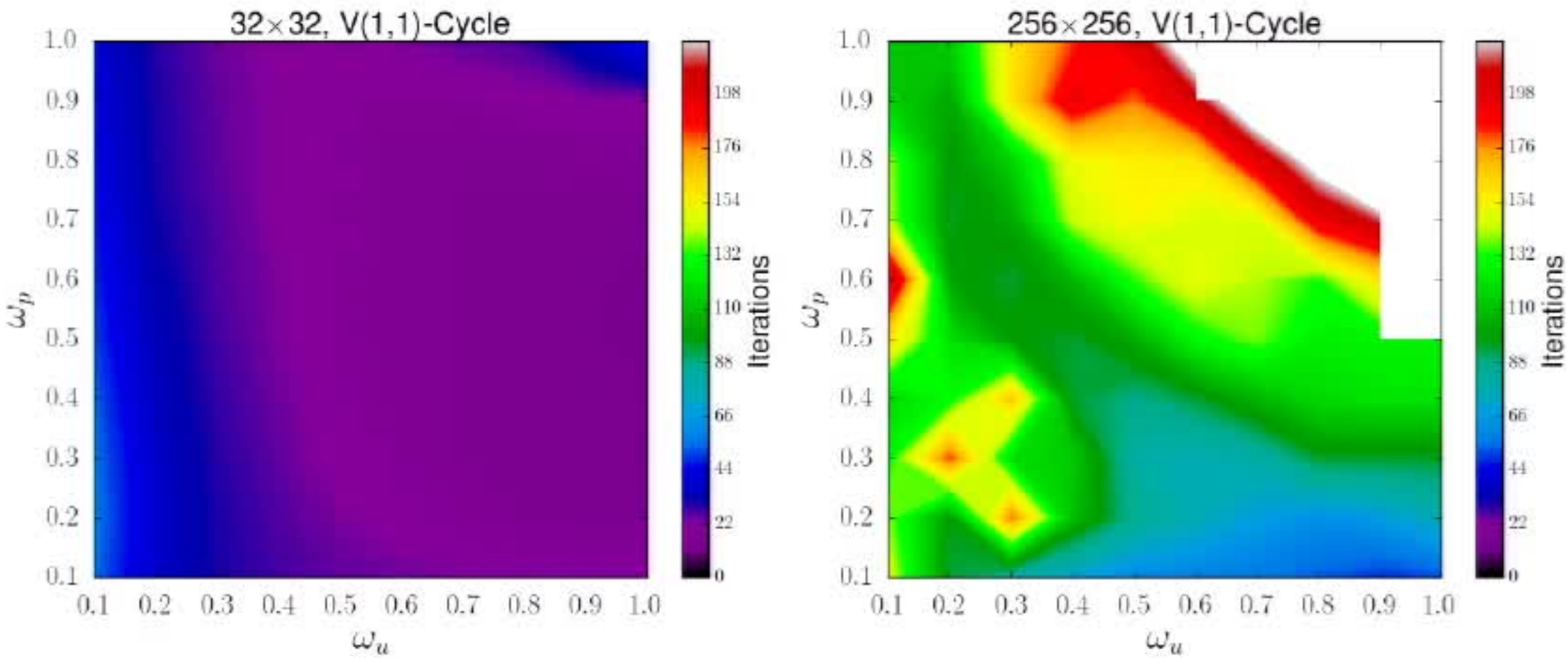
Also need to choose parameter(s)

- Local Fourier Analysis
- Brute-force testing

# Parameter Study - Bad

Element-wise Vanka

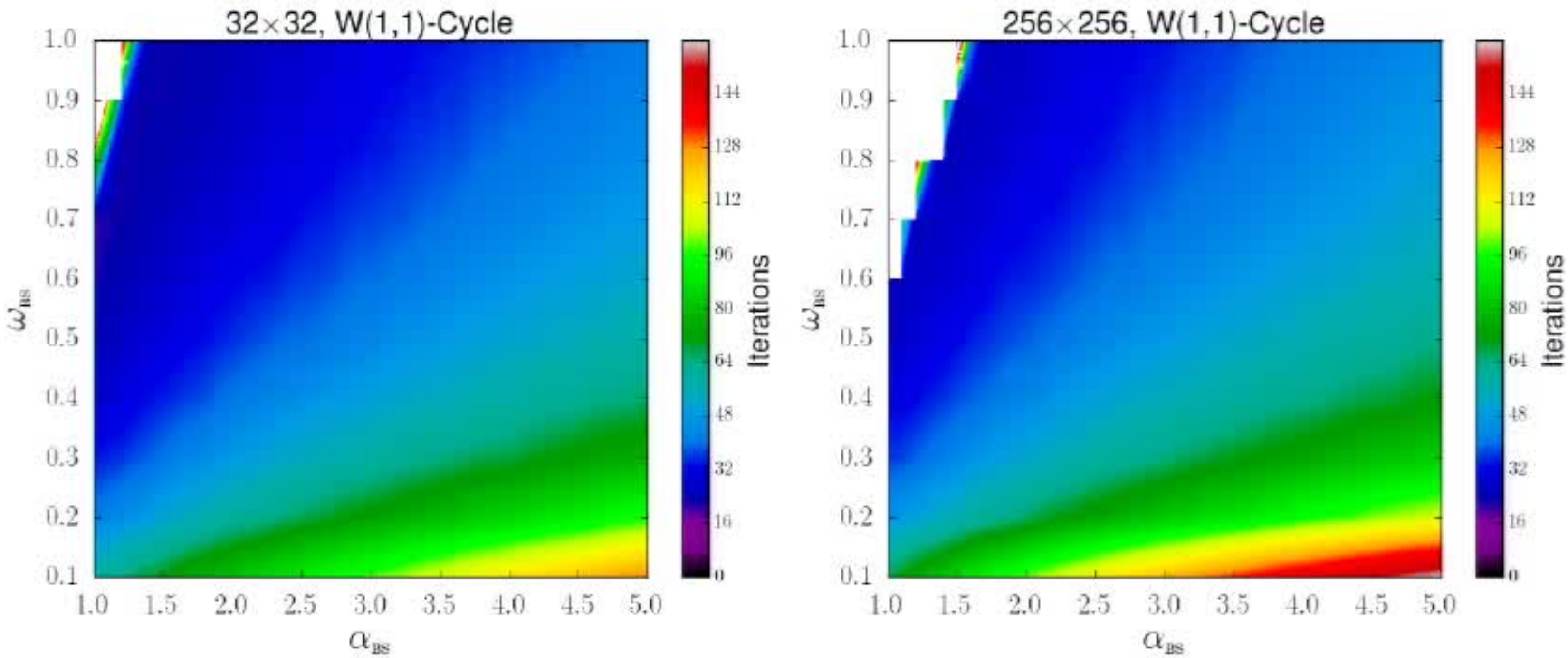
- Full Vanka submatrices
- V(1,1) cycles



# Parameter Study - Good

Block diagonal Braess Sarazin

- W(1,1) cycles



# CPU Times

Multigrid setup times

| | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|
| Diag. Vanka | 0.048 | 0.19 | 0.75 | 2.97 | 11.96 |
| Full Ext. Vanka | 0.083 | 0.29 | 1.18 | 4.76 | 19.23 |
| Diag. Ext. Vanka | 0.054 | 0.21 | 0.85 | 3.45 | 13.81 |
| Diag. Br-Sar | 0.026 | 0.10 | 0.41 | 1.65 | 6.77 |
| Blk-Diag. Br-Sar | 0.025 | 0.10 | 0.41 | 1.66 | 6.83 |

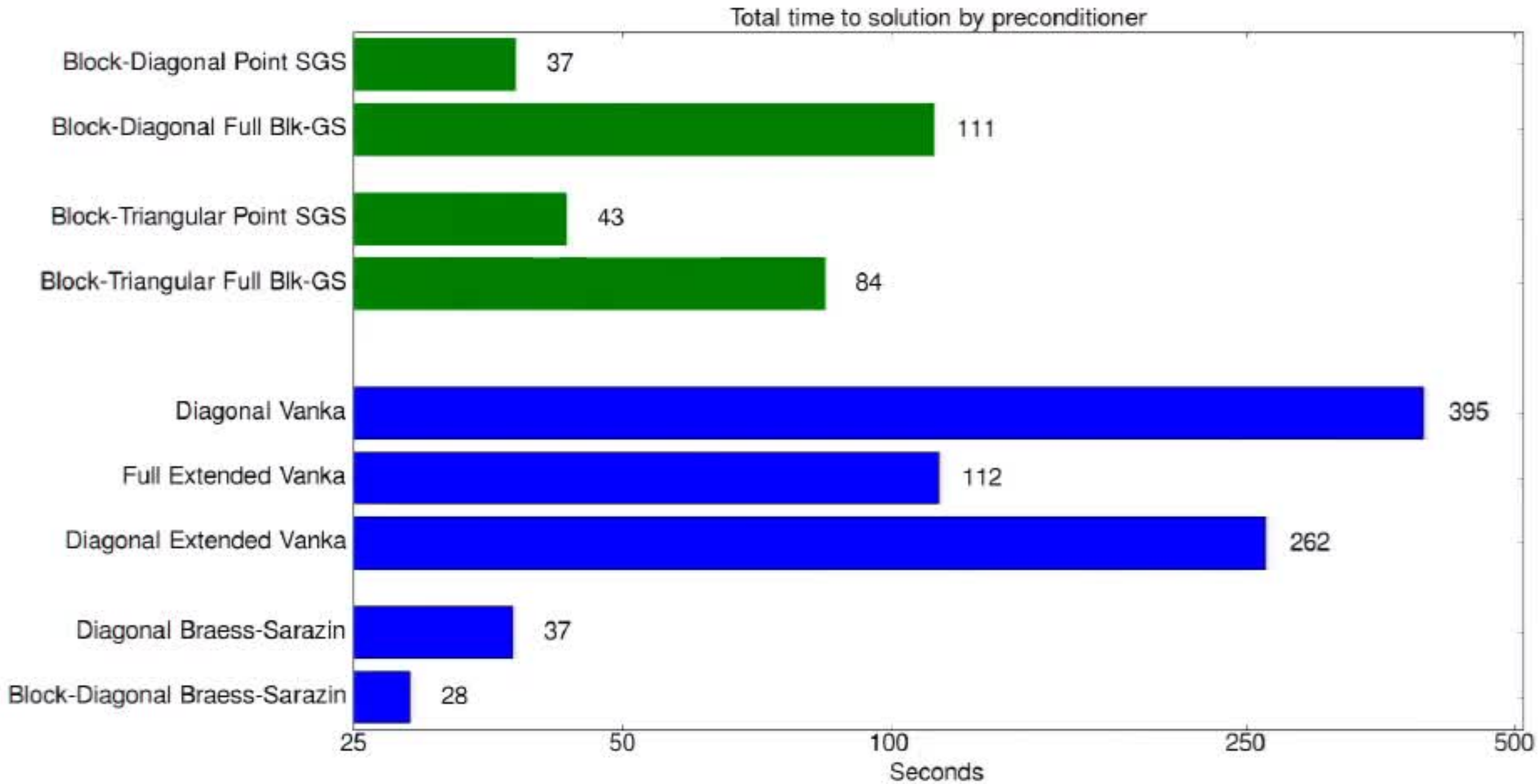
Linear solve times

| | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|
| Diag. Vanka | 1.01 | 4.68 | 20.86 | 91.39 | 383.13 |
| Full Ext. Vanka | 0.29 | 1.30 | 5.47 | 22.70 | 93.68 |
| Diag. Ext. Vanka | 0.73 | 3.32 | 14.78 | 60.94 | 248.47 |
| Diag. Br-Sar | 0.14 | 0.49 | 1.85 | 7.14 | 30.86 |
| Blk-Diag. Br-Sar | 0.11 | 0.38 | 1.34 | 5.21 | 22.04 |

# Iteration counts

| | $32^2$ | $64^2$ | $128^2$ | $256^2$ | $512^2$ |
|---|---|---|---|---|---|
| DOFs | 8K | 33K | 131K | 525K | 2,099K |
| Blk-Diag Point-SGS* | 41 | 44 | 49 | 53 | 54 |
| Blk-Diag Blk-GS | 21 | 21 | 21 | 20 | 19 |
| Blk-Tri Point-SGS | 35 | 39 | 43 | 45 | 48 |
| Blk-Tri Blk-GS | 16 | 16 | 15 | 14 | 14 |
| Diag. Vanka | 18 | 19 | 20 | 21 | 22 |
| Full Ext. Vanka | 6 | 6 | 6 | 6 | 6 |
| Diag. Ext. Vanka | 15 | 15 | 16 | 16 | 16 |
| Diag. Br-Sar | 28 | 30 | 32 | 33 | 35 |
| Blk-Diag. Br-Sar | 22 | 24 | 24 | 25 | 26 |

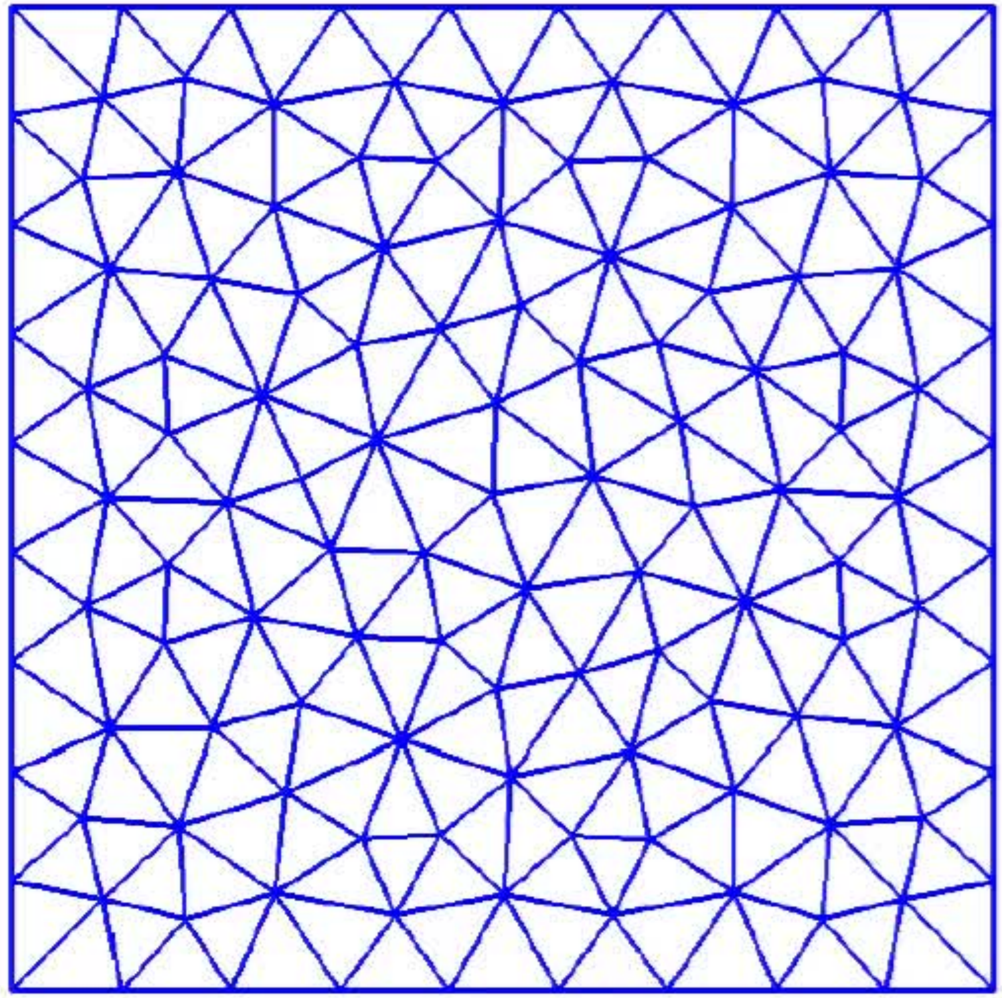
# The Bottom Line




Total time to solution, $512^2$ grid, 2.1M DoFs

# Unstructured Grid Problem

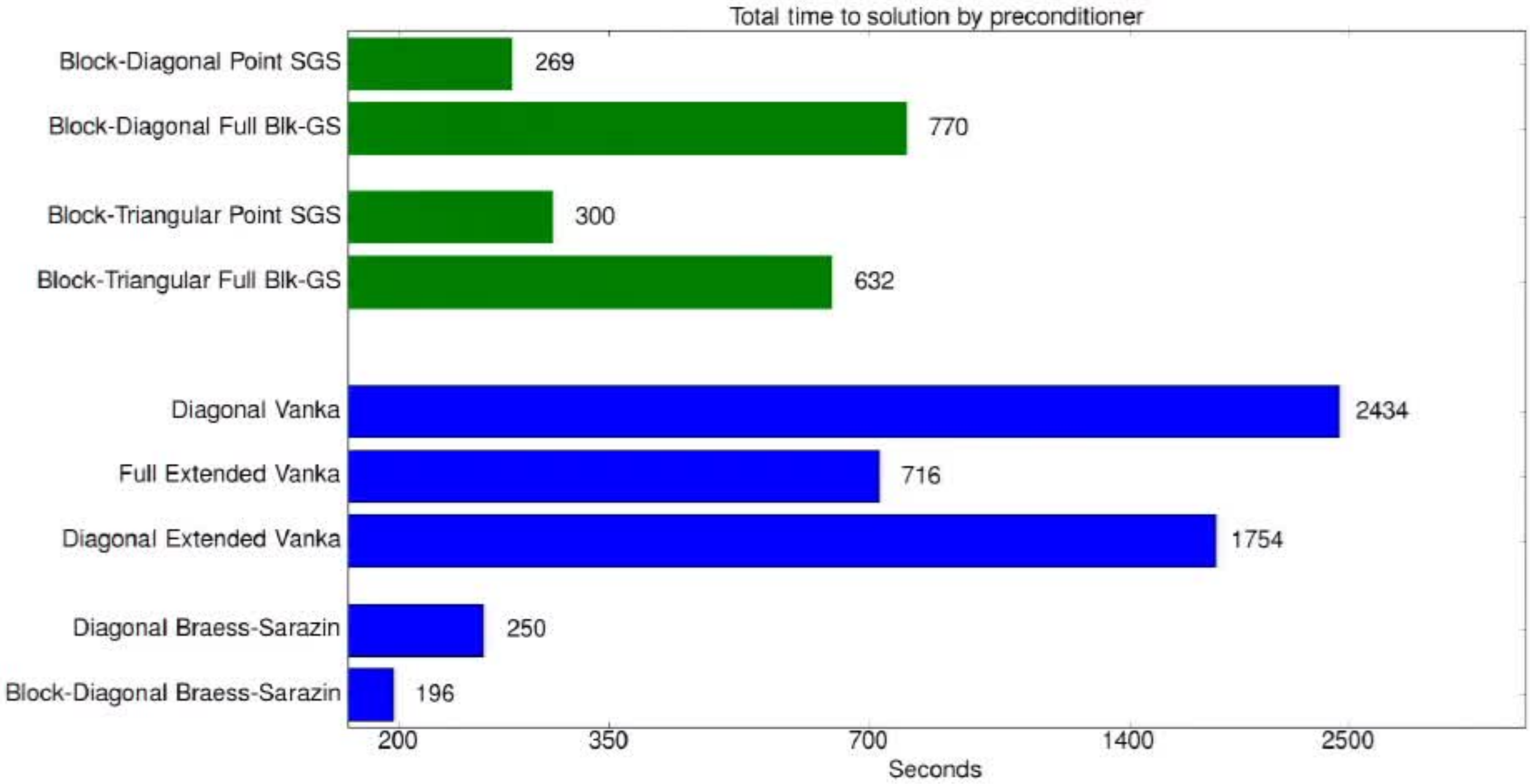
| Level | DOFs |
|---|---|
| 4 | 59K |
| 5 | 234K |
| 6 | 935K |
| 7 | 3,738K |
| 8 | 14,947K |



# Unstructured Grid Iterations

| Refinements | 4-level | 5-level | 6-level | 7-level | 8-level |
|---|---|---|---|---|---|
| Blk-Diag Point-SGS* | 38 | 43 | 47 | 50 | 51 |
| Blk-Diag Blk-GS | 19 | 19 | 18 | 18 | 18 |
| Blk-Tri Point-SGS | 33 | 37 | 41 | 43 | 45 |
| Blk-Tri Blk-GS | 14 | 14 | 14 | 14 | 14 |
| Diag. Vanka | 16 | 17 | 18 | 19 | 19 |
| Full Ext. Vanka | 5 | 5 | 5 | 5 | 5 |
| Diag. Ext. Vanka | 13 | 14 | 14 | 14 | 15 |
| Diag. Br-Sar | 25 | 27 | 29 | 31 | 32 |
| Blk-Diag. Br-Sar | 20 | 22 | 23 | 24 | 24 |

# Unstructured Grid Comparison




Total time to solution, 8-level refinement