Phase approximation beyond the first order: The Kuramoto model with non-pairwise interactions

Iván León and Diego Pazó

Instituto de Física de Cantabria (IFCA), CSIC-University of Cantabria, Santander, Spain

Phase Reduction

P.R. is a simplified description of a weakly perturbed oscillator in terms of its phase (other degrees of freedom become enslaved).

However, implementation of P.R. for a generic oscillator is **not trivial**.

As a **perturbative technique**, P.R. can (in principle) be carried out up to any order, usually only the first order is calculated.

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Stuart-Landau oscillator (normal form of Hopf bifurcation) is a canonical model, in which first-order phase reduction is easily implemented analytically:

$$\dot{A} = A - (1 + i c_2)|A|^2 A \quad \overset{A = r e^{i\varphi}}{\longleftrightarrow} \quad \begin{cases} \dot{r} = r - r^3 \\ \dot{\varphi} = -c_2 r^2 \end{cases}$$

c_2 Nonisochronicity (shear)

Phase reduction: A success story (for an oscillatory field)

Complex Ginzburg-Landau Eq.: $\partial_t A(\vec{r},t) = A - (1+i c_2)|A|^2 A + (1+ic_1)\nabla^2 A$

c_1 coupling reactivity

phase reduction

$$\partial_t \theta(\vec{r},t) = -c_2 + \underbrace{(1+c_1 c_2)\nabla^2\theta + (c_2 - c_1)(\nabla\theta)^2}_{1^{st}\ \text{order}}$$

1st order

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phase reduction

$$\partial_t \theta(\vec{r},t) = -c_2 + \underbrace{(1+c_1c_2)\nabla^2\theta + (c_2-c_1)(\nabla\theta)^2}_{1^{st}\text{ order}} \underbrace{- \frac{c_1^2}{2}(1+c_2^2)\nabla^4\theta + \cdots}_{2^{nd}\text{ order}}$$

Phase reduction: A success story (for an oscillatory field)

Complex Ginzburg-Landau Eq.: $\partial_t A(\vec{r},t) = A - (1+i c_2)|A|^2 A + (1+ic_1)\nabla^2 A$

Kuramoto-Sivashinsky (phase-turbulence) Eq.

$$\partial_t \theta(\vec{r},t) = -c_2 + (1+c_1 c_2)\nabla^2\theta + (c_2 - c_1)(\nabla\theta)^2 - \frac{c_1^2}{2}(1+c_2^2)\nabla^4\theta + \cdots$$

1st order

2nd order

Mean-field complex Ginzburg-Landau Eq.

MF-CGLE or globally coupled Stuart-Landau oscillators:

$$\dot{A}_j = A_j - (1 + i c_2)|A_j|^2 A_j + \epsilon(1 + i c_1)(A_j - \bar{A})$$

(Hakim & Rappel, 1992)
(Nakagawa & Kuramoto, 1993)

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ϵ Coupling stregth

$$\bar{A} = \frac{1}{N}\sum_{k=1}^{N} A_k$$ mean field

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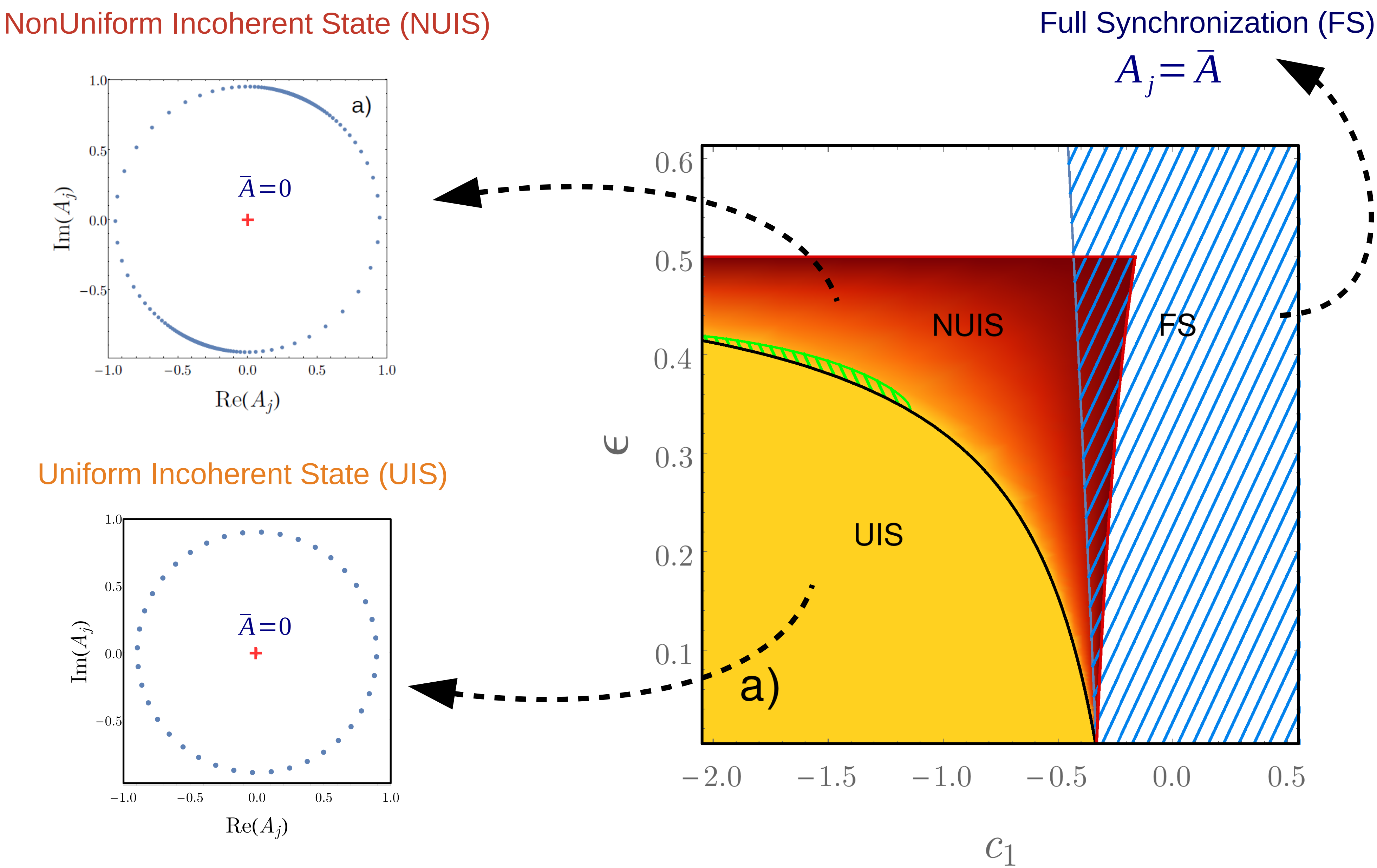
Canonical model of collective dynamics

Rich repertoire of dynamics: microscopic (extensive) chaos, macroscopic (collective) chaos, clustering, quasiperiodic partial synchronization, chimera, ...

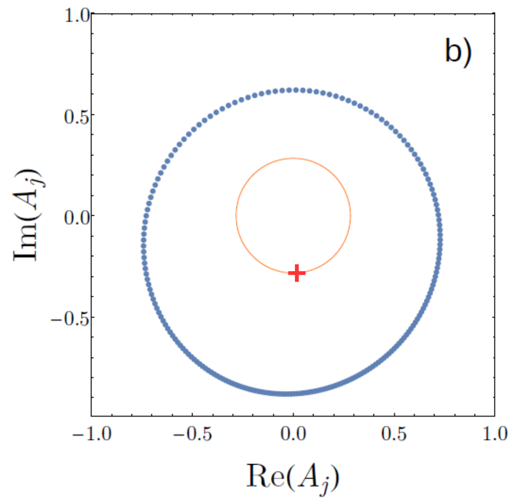
Which states can be described by phases alone? $\Longrightarrow$ Small coupling

MF-CGLE: Phase diagram (c_2=3)

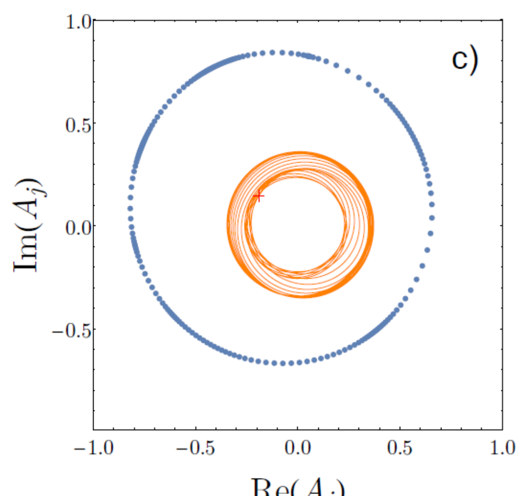


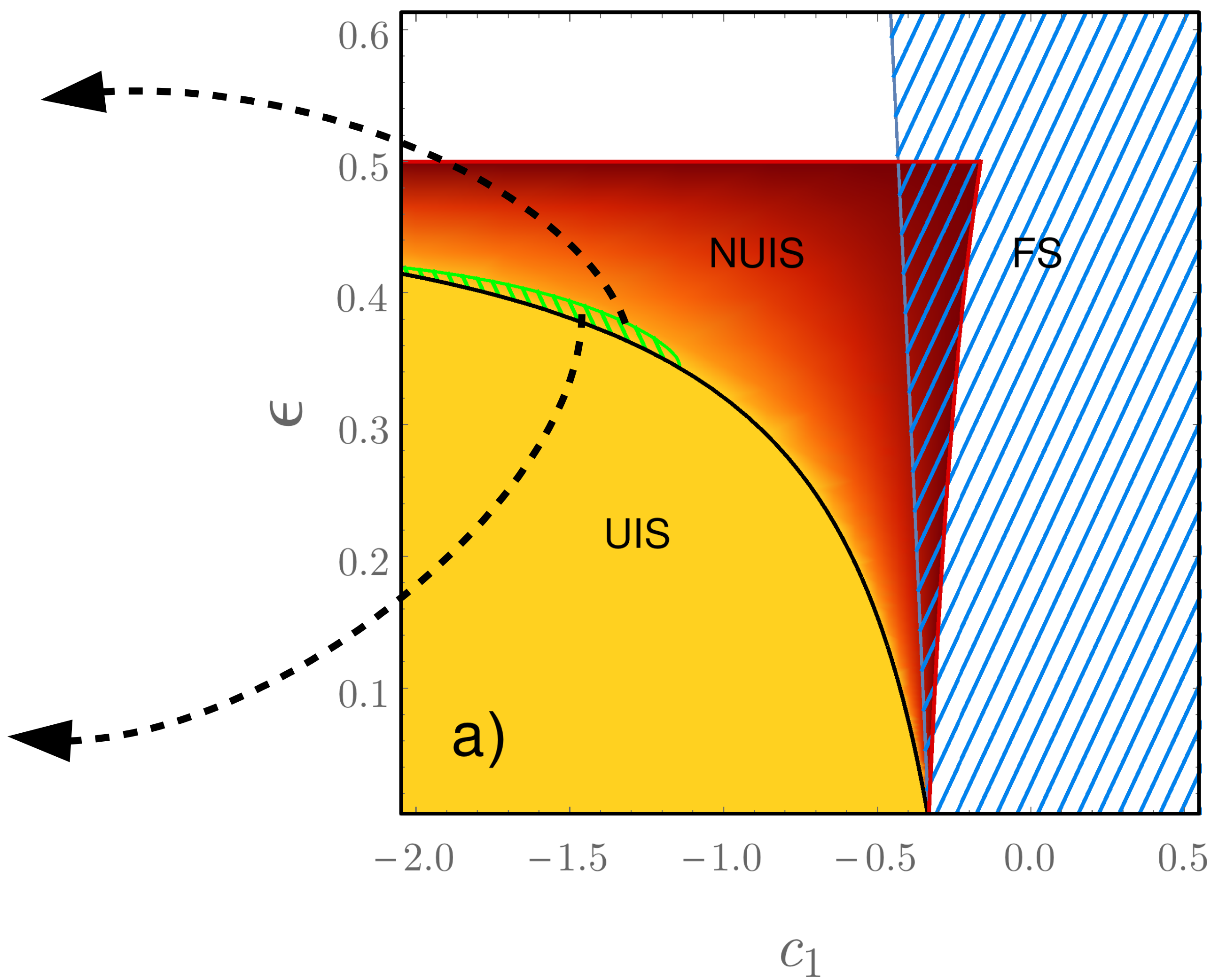
MF-CGLE: Phase diagram (c_2=3)

Quasiperiodic Partial Synchronization (QPS)



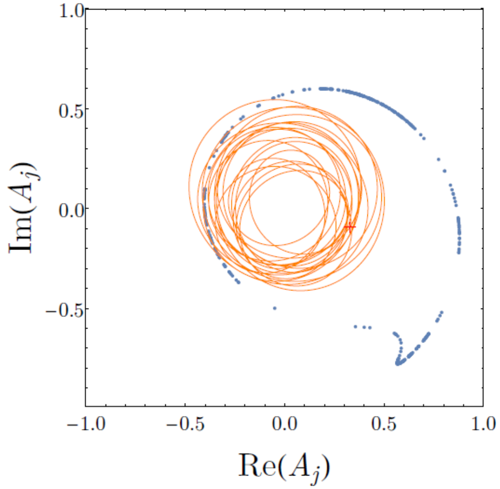
Pure collective chaos



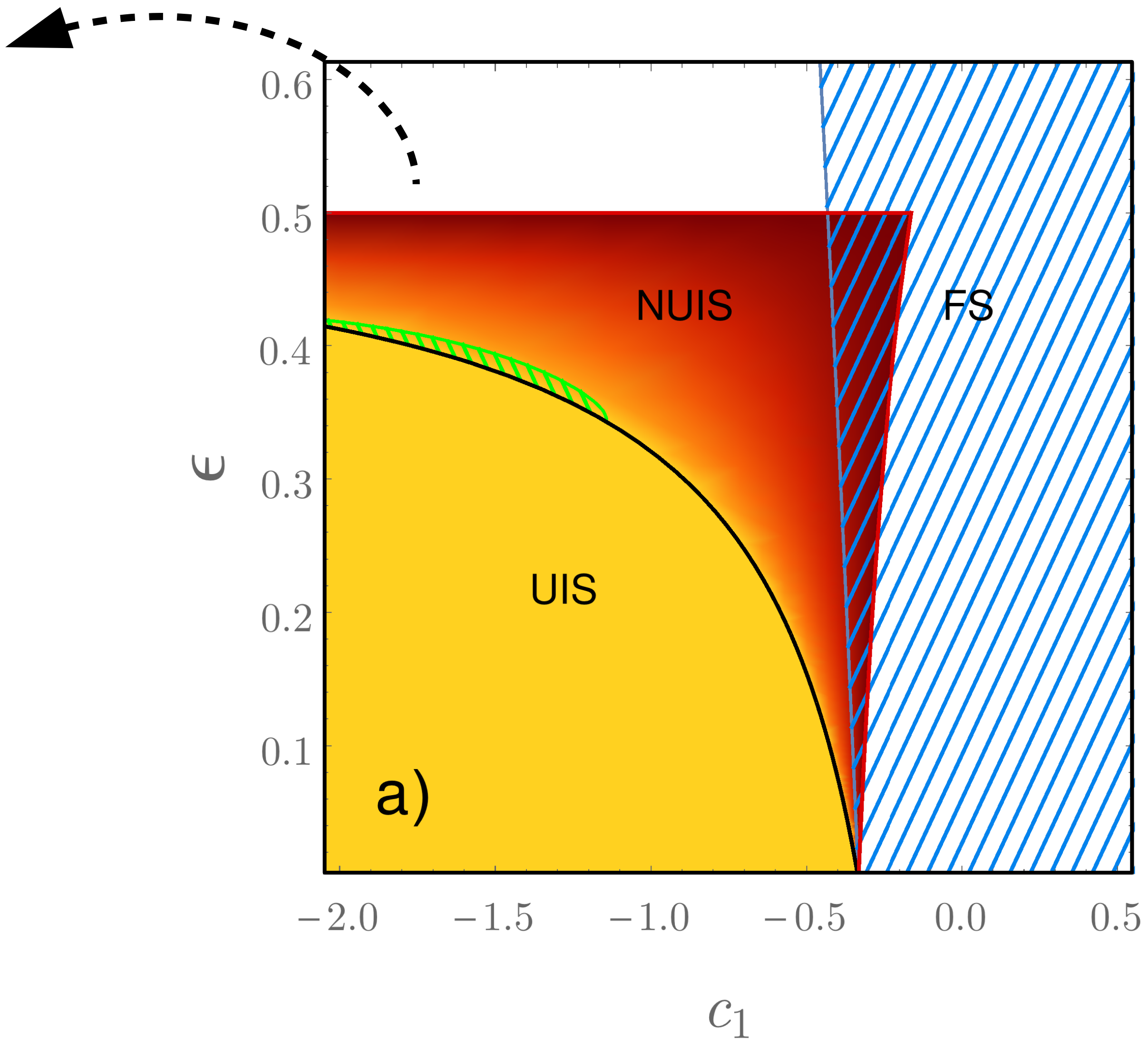


MF-CGLE: Phase diagram (c_2=3)

Microscopic and collective chaos

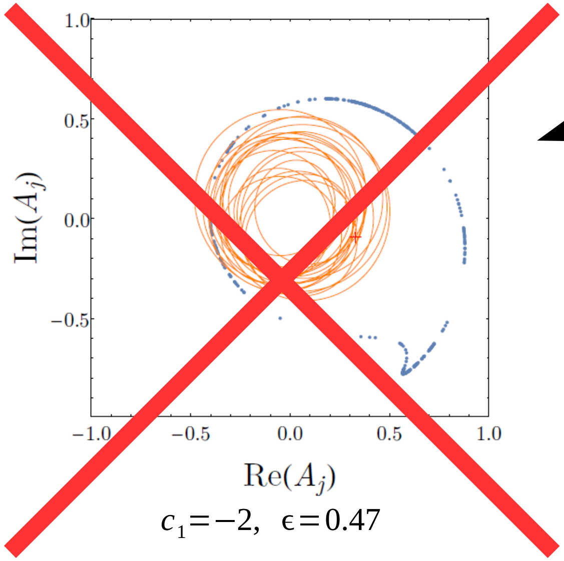


c_1=−2, ϵ=0.47



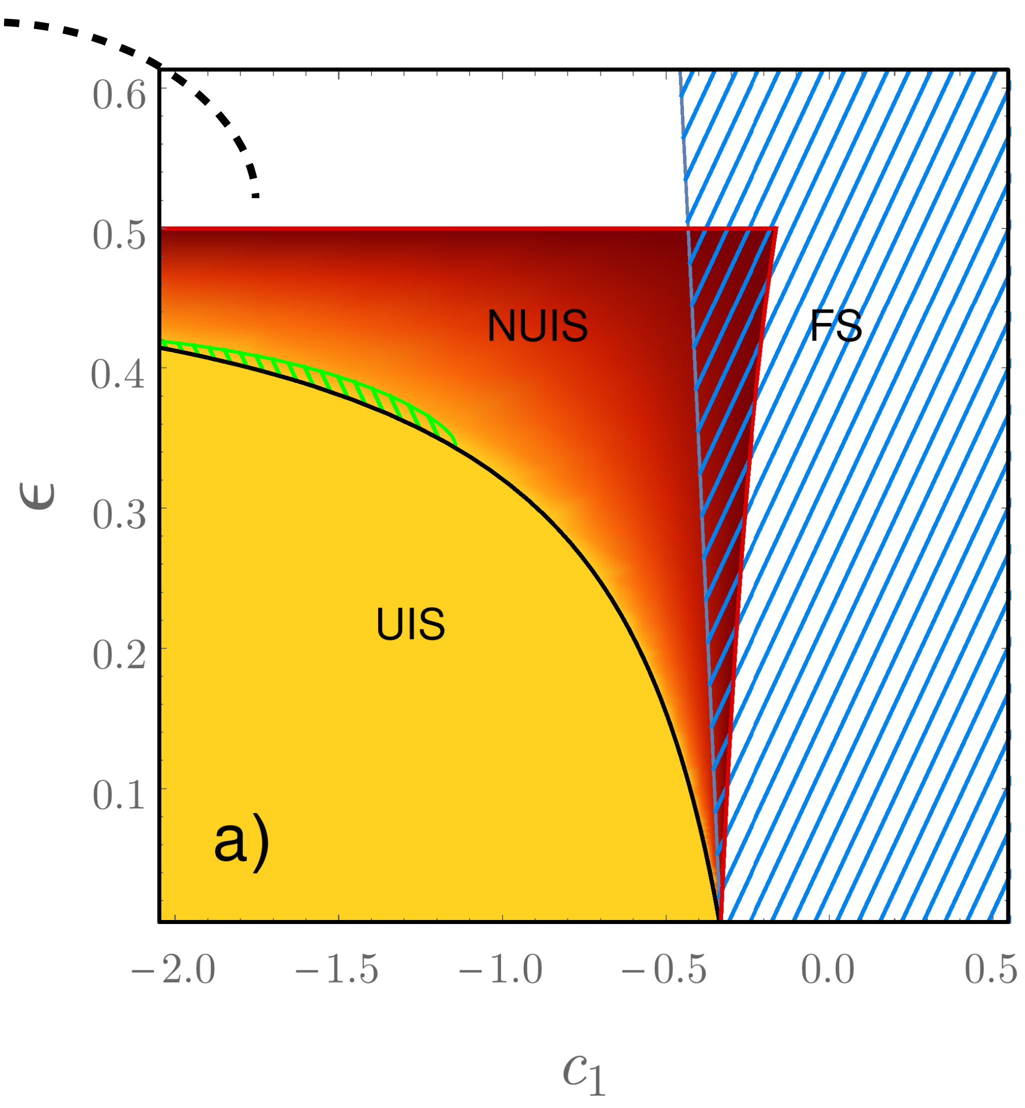
MF-CGLE: Phase diagram (c_2=3)

Microscopic and collective chaos



c_1=−2, ϵ=0.47

No phase describable!



Phase reduction of the MF-CGLE?

First-order phase reduction of MF-RGLE

SELF-ENTRAINMENT OF A POPULATION OF
COUPLED NON-LINEAR OSCILLATORS

Yoshiki Kuramoto
Department of Physics, Kyushu University, Fukuoka, Japan

Temporal organization of matter is a widespread phenomenon over a croscopic world in far from thermodynamic equilibrium. A previous udy on chemical instability[1] implies that a simplest nontrivial del for a temporally organized system may be represented by a macro- opic self-sustained oscillator Q obeying the equation of motion

$$\dot{Q} = (i\omega + \alpha)Q - \beta|Q|^2 Q \,, \qquad \alpha,\beta > 0. \qquad (1)$$

nsider a population of such oscillators $Q_1, Q_2, \cdots Q_N$ with various equencies, and introduce interactions between every pair as follows.

$$\dot{Q}_s = (i\omega_s + \alpha)Q_s + \sum_{r \neq s} V_{rs} Q_r - \beta|Q_s|^2 Q_s \qquad (2)$$

$$r,s = 1, 2, \cdots N \,.$$

found that it is possible to construct from (2) a soluble model for community exhibiting mutual synchronization or self-entrainment ove a certain threshold value of the coupling strength. Such a type phase transition has been considered by Winfree[2] without resorting specialized models but only phenomenologically.

Our simplifying assumptions are:

) $V_{rs} = V/N$ independently of r and s,

I) $\alpha,\beta \to \infty$ but α/β, ω_s, V = finite,

II) $N \to \infty$.

Let us put $Q_s = \rho_s e^{i\varphi_s}$. Owing to the assumption (II), the amplitude may be fixed at $\sqrt{\alpha/\beta}$. Thus we have only to consider the equation

$$\dot{\varphi}_s = \omega_s + \frac{V}{N} \sum_r \sin(\varphi_r - \varphi_s) \,. \qquad (3)$$

Kuramoto, 1975

1^{st} order P.R. of MF-CGLE: Kuramoto-Sakaguchi

First-order phase reduction yields the Kuramoto-Sakaguchi model (without heterogeneity):

$$\theta_j = \Omega + \frac{\epsilon}{N}\sum_{k=1}^{N}(1+c_1 c_2)\sin(\theta_k-\theta_j)+(c_1-c_2)\cos(\theta_k-\theta_j)+O(\epsilon^2)$$

In terms of the Kuramoto order parameter $R\,e^{i\Psi} \equiv \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}$

$$\dot{\theta}_j = \Omega + \epsilon\,\eta\,R\sin(\Psi-\theta_j+\alpha)+O(\epsilon^2)$$

$\Omega \equiv -c_2+\epsilon(c_2-c_1)$

$\eta \equiv \sqrt{(1+c_2^2)(1+c_1^2)}$

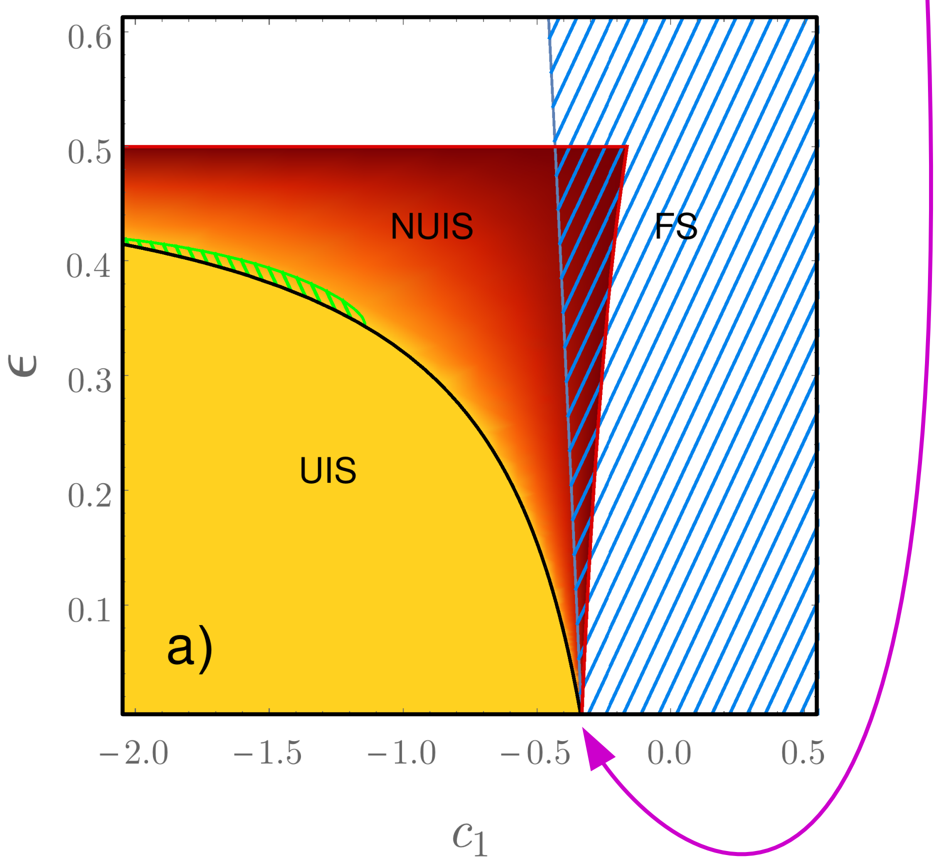
$\alpha \equiv \arctan((c_1-c_2)/(1+c_1 c_2))$

$1+c_1 c_2>0 \;\rightarrow\; FS$

$1+c_1 c_2<0 \;\rightarrow\; UIS$

1^{st} order P.R. of MF-CGLE: Kuramoto-Sakaguchi

Only FS or UIS (!). Stability boundary: $1+c_1 c_2=0$ (Benjamin-Feir-Newell criterion)



limit $\epsilon \to 0$

$1+c_1 c_2>0$: *FS*

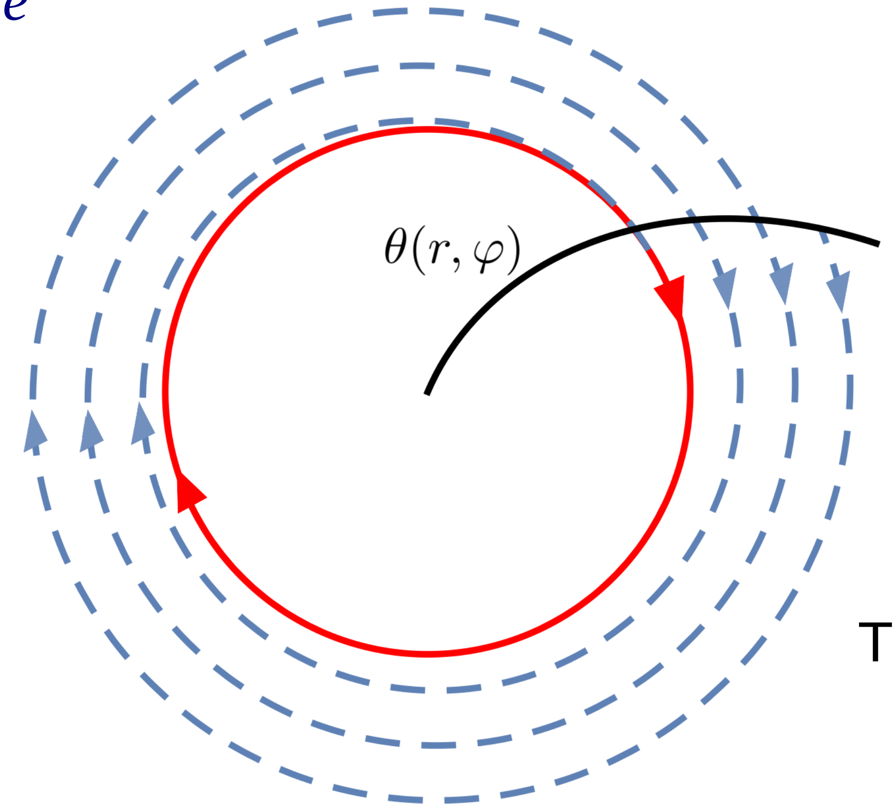
$1+c_1 c_2<0$: *UIS*

$O(\epsilon^2)$ terms are needed to remove degeneracies!

$$\dot{\theta}_j = \Omega + \epsilon \eta R \sin\left(\Psi - \theta_j + \alpha\right) + O\left(\epsilon^2\right)$$

Systematic isochron-based phase reduction

$A = r\,e^{i\varphi}$



Isochron

$$\theta(r,\varphi) = \varphi - c_2 \ln r$$

Systematic isochron-based phase reduction

Writing MF-CGLE in (r,θ) coordinates we get a system with two time scales (in a rotating frame):

$$\dot{r}_j = f(r_j) + \epsilon\, g_j(\boldsymbol{r}, \boldsymbol{\theta})$$
$$\dot{\theta}_j = \epsilon\, h_j(\boldsymbol{r}, \boldsymbol{\theta})$$

$$\boldsymbol{r} = \begin{pmatrix} r_1 \\ r_2 \\ \vdots \\ r_N \end{pmatrix} \; ; \quad \boldsymbol{\theta} = \begin{pmatrix} \theta_1 \\ \theta_2 \\ \vdots \\ \theta_N \end{pmatrix}$$

$$f(r) = r(1 - r^2)$$

$$g_j(\mathbf{r}, \boldsymbol{\theta}) = -r_j + \frac{1}{N} \sum_{k=1}^{N} \left\{ r_k \left[\cos\left(\theta_k - \theta_j + c_2 \ln \tfrac{r_k}{r_j}\right) - c_1 \sin\left(\theta_k - \theta_j + c_2 \ln \tfrac{r_k}{r_j}\right) \right] \right\}$$

$$h_j(\mathbf{r}, \boldsymbol{\theta}) = c_2 - c_1 + \frac{1}{N r_j} \sum_{k=1}^{N} \left\{ r_k \left[(c_1 - c_2) \cos\left(\theta_k - \theta_j + c_2 \ln \tfrac{r_k}{r_j}\right) + (1 + c_1 c_2) \sin\left(\theta_k - \theta_j + c_2 \ln \tfrac{r_k}{r_j}\right) \right] \right\}$$

Systematic isochron-based phase reduction

We assume amplitudes are enslaved to the phases $r_j = r_j(\boldsymbol{\theta})$

And taking an expansion in powers of ϵ : $\boldsymbol{r} = \boldsymbol{r}^{(0)} + \epsilon \boldsymbol{r}^{(1)} + \cdots$

$$\dot{\theta}_j = \epsilon h_j(\boldsymbol{r}, \boldsymbol{\theta}) \quad \Rightarrow \quad \dot{\theta}_j = \epsilon h_j(\boldsymbol{r}^{(0)}, \boldsymbol{\theta}) + \epsilon^2 \left(\nabla_r h_j(\boldsymbol{r}^{(0)}, \boldsymbol{\theta}) \right) \cdot \boldsymbol{r}^{(1)} + \cdots$$

$$\dot{r}_j = f(r_j) + \epsilon g_j(\boldsymbol{r}, \boldsymbol{\theta}) \quad \Rightarrow \begin{cases} 0 = f(r_j^{(0)}) \rightarrow r_j^{(0)} = 1 \\ (\nabla_\theta r_j^{(0)}) \cdot \boldsymbol{h} = f'(r_j^0) r_j^{(1)} + g_j(\boldsymbol{r}^{(0)}, \boldsymbol{\theta}) \rightarrow r_j^{(1)} = \dfrac{g_j(\boldsymbol{r}^{(0)}, \boldsymbol{\theta})}{2} \\ \vdots \end{cases}$$

2nd order P.R. of MF-CGLE

$$\dot{\theta}_j = \Omega + \epsilon \eta R \sin(\Psi - \theta_j + \alpha) + \frac{\epsilon^2 \eta^2}{4} \left[R \sin(\Psi - \theta_j + \beta) - R^2 \sin[2(\Psi - \theta_j) + \beta] + R Q \sin(\Phi - \Psi - \theta_j) \right]$$

2nd order P.R. of MF-CGLE

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$$+ \frac{\epsilon^2 \eta^2}{4} \left[R \sin(\Psi - \theta_j + \beta) - R^2 \sin[2(\Psi - \theta_j) + \beta] + R Q \sin(\Phi - \Psi - \theta_j) \right]$$

Kurmoto-Daido order parameter: $Q e^{i\Phi} \equiv \frac{1}{N} \sum_{k=1}^{N} e^{i 2 \theta_k}$

New phase lag: $\beta \equiv \arctan[(1 - c_1^2)/(2 c_1)]$

2^{nd} order P.R. of MF-CGLE

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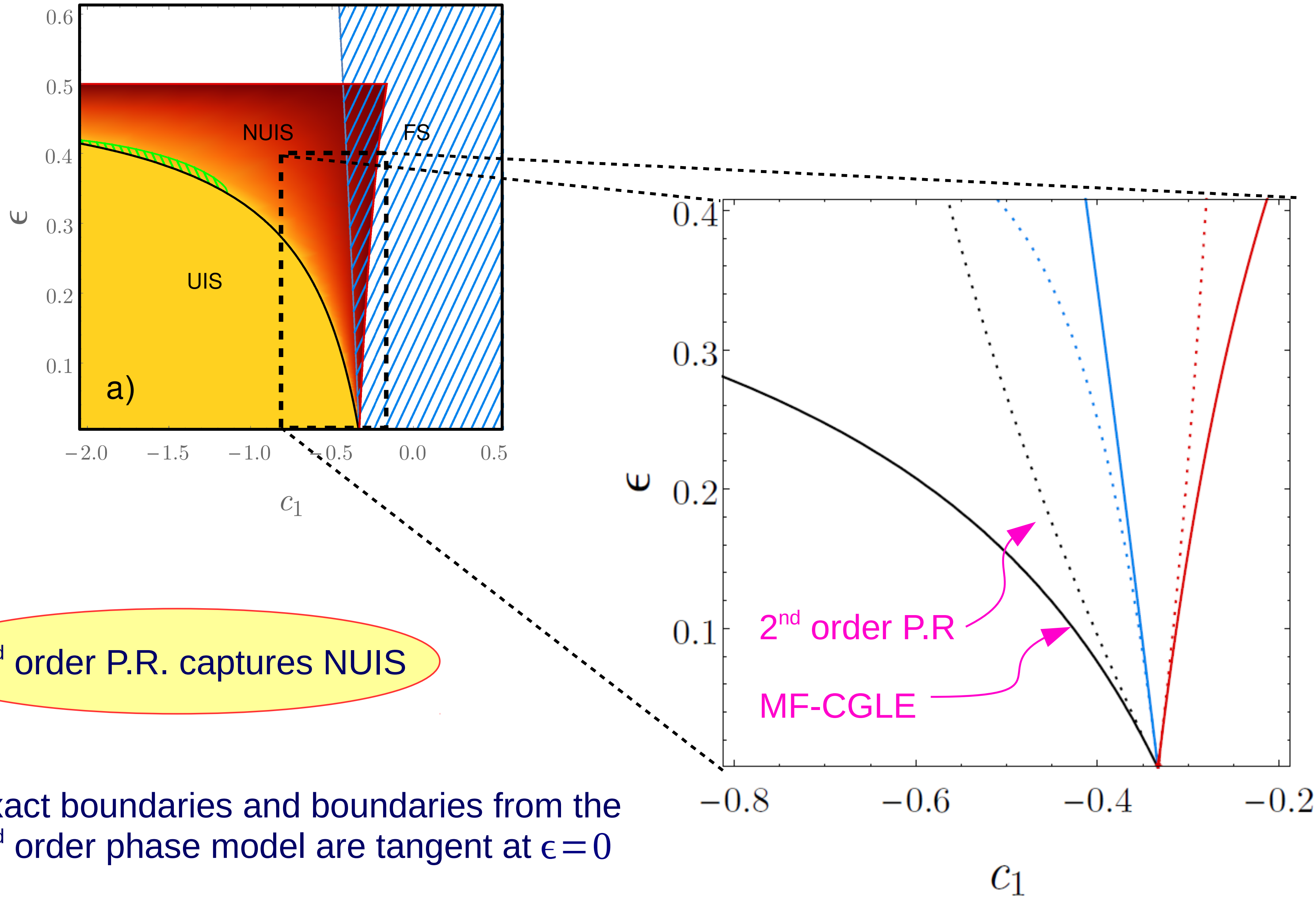
New phase lag: $\beta \equiv \arctan[(1-c_1^2)/(2c_1)]$

Three-body (non-pairwise) interactions!

$$R^2\sin[2(\Psi-\theta_j)+\beta] = \frac{1}{N^2}\sum_{k,l}^{N}\sin(\theta_k+\theta_l-2\theta_j+\beta)$$

$$RQ\sin(\Phi-\Psi-\theta_j) = \frac{1}{N^2}\sum_{k,l}^{N}\sin(2\theta_k+\theta_l-\theta_j)$$

2^{nd} order P.R.: (N)UIS and FS boundaries

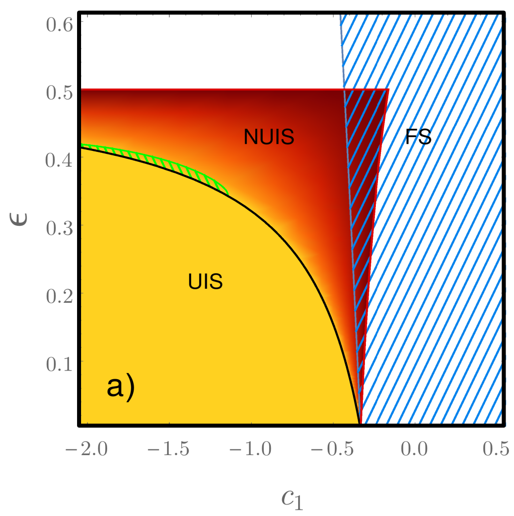


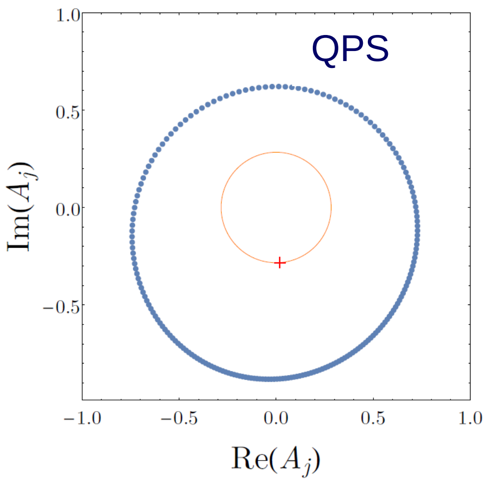
2nd order P.R. captures NUIS

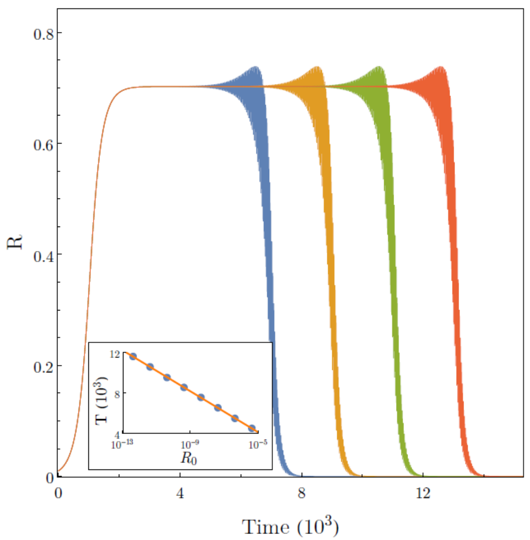
Exact boundaries and boundaries from the 2nd order phase model are tangent at $\epsilon = 0$

2nd order P.R.: Other dynamics

- Heteroclinic connection: UIS → **saddle QPS** → NUIS

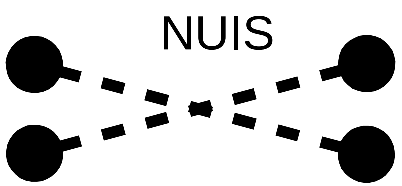


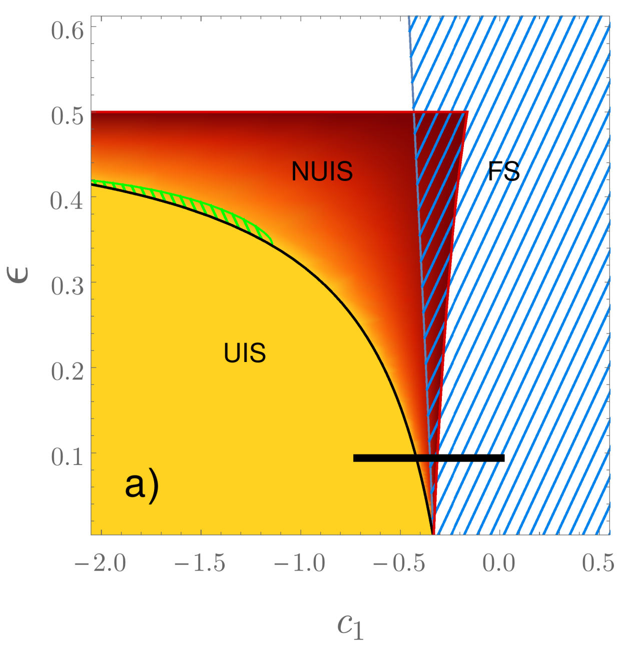


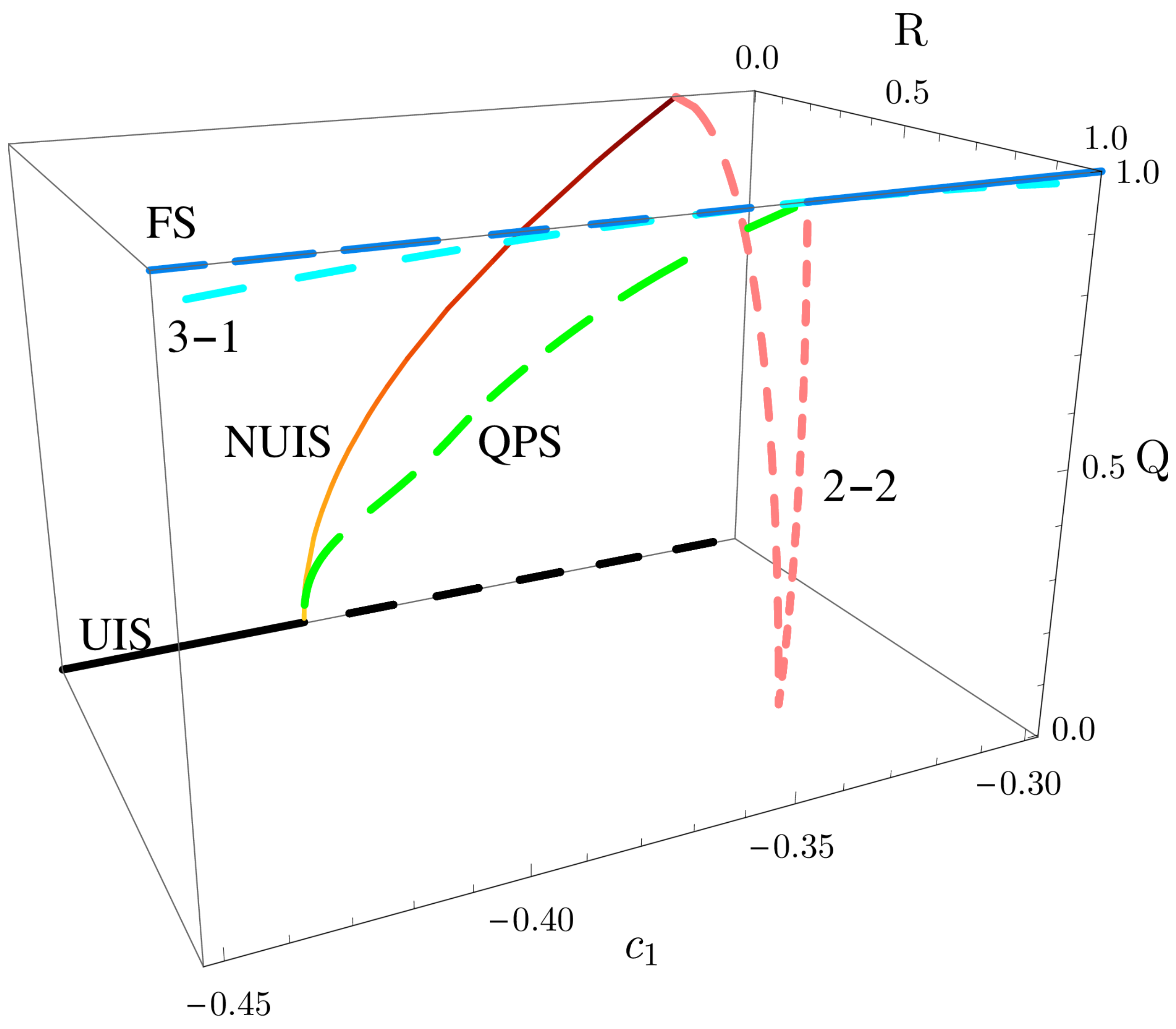


- **No** stable two-point **clustering**.
- **Slow switching** (heteroclinic conection between 2-cluster) is **impossible**.

2^{nd} order P.R.: N=4







- QPS connects UIS with FS

Is it possible to obtain third-order terms?

$$\dot{\theta}_j = \Omega + \epsilon \eta R \sin(\Psi - \theta_j + \alpha)$$
$$+ \frac{\epsilon^2 \eta^2}{4} \left[R \sin(\Psi - \theta_j + \beta) - R^2 \sin[2(\Psi - \theta_j) + \beta] + R Q \sin(\Phi - \Psi - \theta_j) \right]$$
$$+ O(\epsilon^3)$$

Yes!!

3^{rd} order P.R. of MF-CGLE

$$\dot{\theta}_j = \Omega + \epsilon\eta R\sin(\Psi - \theta_j + \alpha)$$

$$+\frac{\epsilon^2\eta^2}{4}\left[R\sin(\Psi-\theta_j+\beta) - R^2\sin[2(\Psi-\theta_j)+\beta] + RQ\sin(\Phi-\Psi-\theta_j)\right]$$

$$+\epsilon^3\frac{1+c_2^2}{16}\Big\{C_1Q^2R\sin(\Psi-\theta_j+\gamma_1) + C_2R^3\sin(\Psi-\theta_j+\gamma_2)$$

$$+C_3R\sin(\Psi-\theta_j+\gamma_3) + C_4QR^2\sin(\Phi-2\theta_j+\gamma_4)$$

$$+C_5QR\sin(\Phi-\Psi-\theta_j+\gamma_5) + C_6R^3\sin[3(\Psi-\theta_j+\gamma_6)]$$

$$+C_7R^2P\sin(\Xi-2\Psi-\theta_j+\gamma_7) + C_8R^2\sin[2(\Psi-\theta_j+\gamma_8)]$$

$$+C_9QR^2\sin(\Phi-2\Psi+\gamma_9) + DR^2\Big\}$$

3^{rd} order P.R.: corrections to 1^{st} and 2^{nd} orders

$$\epsilon^3 \frac{1+c_2^2}{16} \Big\{ C_1 Q^2 R \sin(\Psi - \theta_j + \gamma_1) + C_2 R^3 \sin(\Psi - \theta_j + \gamma_2)$$

$$+ C_3 R \sin(\Psi - \theta_j + \gamma_3) + C_4 Q R^2 \sin(\Phi - 2\theta_j + \gamma_4)$$

$$+ C_5 Q R \sin(\Phi - \Psi - \theta_j + \gamma_5) + C_6 R^3 \sin\left[3(\Psi - \theta_j + \gamma_6)\right]$$

$$+ C_7 R^2 P \sin(\Xi - 2\Psi - \theta_j + \gamma_7) + C_8 R^2 \sin\left[2(\Psi - \theta_j + \gamma_8)\right]$$

$$+ C_9 Q R^2 \sin(\Phi - 2\Psi + \gamma_9) + D R^2 \Big\}$$

3rd order P.R.: four-body interactions

$$\frac{1}{N^3}\sum_{k,l,n}\sin(\theta_k+\theta_l-\theta_n-\theta_j)$$

$$\epsilon^3\frac{1+c_2^2}{16}\Big\{C_1Q^2R\sin(\Psi-\theta_j+\gamma_1)+C_2R^3\sin(\Psi-\theta_j+\gamma_2)$$
$$+C_3R\sin(\Psi-\theta_j+\gamma_3)+C_4QR^2\sin(\Phi-2\theta_j+\gamma_4)$$
$$+C_5QR\sin(\Phi-\Psi-\theta_j+\gamma_5)+C_6R^3\sin\left[3(\Psi-\theta_j+\gamma_6)\right]$$
$$+C_7R^2P\sin(\Xi-2\Psi-\theta_j+\gamma_7)+C_8R^2\sin\left[2(\Psi-\theta_j+\gamma_8)\right]$$
$$+C_9QR^2\sin(\Phi-2\Psi+\gamma_9)+DR^2\Big\}$$

$$\frac{1}{N^3}\sum_{k,l,n}\sin(3\theta_k-\theta_l-\theta_n-\theta_j)$$

Kurmoto-Daido order parameter: $P e^{i\Xi}\equiv\frac{1}{N}\sum_{k=1}^{N}e^{i3\theta_k}$

Conclusions

- We present a systematic phase reduction in powers of the coupling (cf. Pikovsky & Rosenau, 2006; Matheny et al., 2019).
- Applicable to other geometries and oscillators.
- n-body interactions show up at order ϵ^{n-1} (for nonlinear coupling, see Ashwin & Rodrigues, 2016).
- Multi-body interactions appear to drive pure collective chaos (standard chaos with N=4, see Bick et al., 2016).
- Our model can be a benchmark for numerical phase reductions.

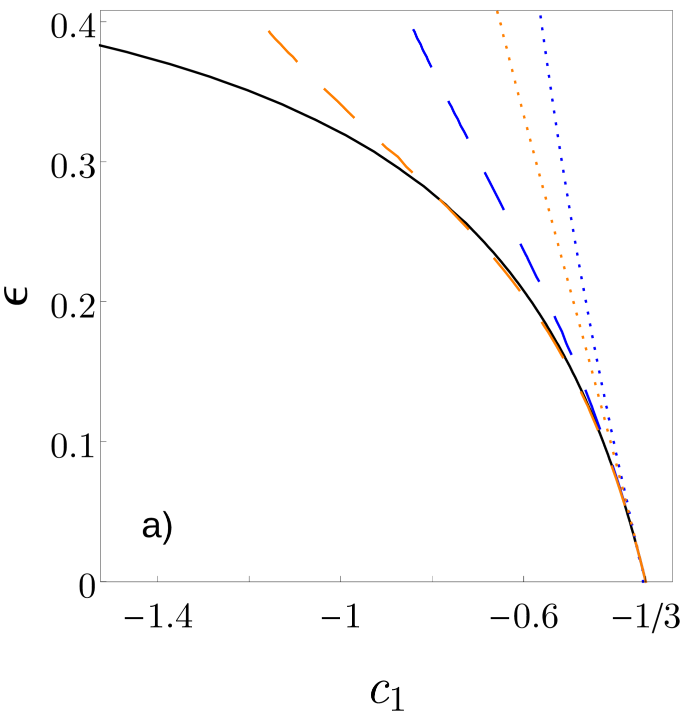
THE END
(THANK YOU!)

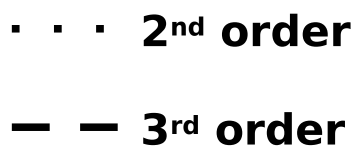
Stability boundaries 2nd and 3rd order

$$\frac{dA_j}{dt} = A_j - (1+ic_2)|A_j|^2 A_j + \epsilon(1+ic_1)(\bar{A} - A_j)$$

$$\frac{dB_j}{dt'} = B_j - (1+ic_2)|B_j|^2 B_j + \epsilon'(1+ic_1)\bar{B} \qquad \epsilon' = \frac{\epsilon}{1-\epsilon}$$

Uniform incoherent state





Synchronization